FIRST PRICE AUCTIONS

IN THE ASYMMETRIC N BIDDER CASE*

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* A shorter version of this paper is to appear in *International Economic Review*.

Abstract

The first price auction is the auction procedure awarding the item to the highest bidder at the price equal to his bid. Much attention has been devoted to the two bidder case or to the symmetric case where the bidders' valuations are identically and independently distributed. We consider the general case where the valuations' distributions may be different. Furthermore, we allow an arbitrary number of bidders as well as mixed strategies. We show that every Bayesian equilibrium is an “essentially" pure equilibrium formed by bid functions whose inverses are solutions of a system of differential equations with boundary conditions. We then prove the existence of a Bayesian equilibrium. We prove the uniqueness of the equilibrium when the valuation distributions have a mass point at the lower extremity of the support. When every bidder's valuation distribution is one of two atomless distributions, we give assumptions under which the equilibrium is unique. The n-tuples of distributions that result from symmetric settings after some bidders have colluded satisfy these assumptions. We establish inequalities between equilibrium strategies when relations of stochastic dominance exist between valuation distributions.

FIRST PRICE AUCTION

IN THE ASYMMETRIC N BIDDER CASE.

1.Introduction.

We study the first price auction in the general asymmetric framework with and without mandatory participation. An indivisible item is offered for sale to $n \geq 2$ bidders. We denote bidder 1's valuation by v_1, bidder 2's valuation by v_2, etc $\ldots$. We assume that the bidders' valuations $v_1, v_2, \ldots, v_n$ are chosen randomly by Nature according to commonly known independent probability measures $F_1, F_2, \ldots, F_n$ (respectively). Only bidder i is then informed of v_i. If at least one bidder takes part in the auction, the item goes to the highest bidder who has to pay the price equal to his bid.

Most of the literature in this "independent private value model" has dealt with the symmetric case where the measures $F_1, F_2, \ldots, F_n$ are equal to the same measure, or with the case where there are only two bidders. However, asymmetry arises naturally in many examples. Consider a first price auction with more than two bidders where bidder j is reputed to be very interested in the objects of the same style as the object being sold. The other bidders, on the other hand, are reputed to have only little interest in such objects. In this example, the measure F_j has to give more probability to high valuations than F_i does, for $i \neq j$.

Riley and Samuelson (1981) prove the existence of an equilibrium and give a mathematical expression for the equilibrium strategy in the symmetric case where the measures $F_1, \ldots, F_n$ are equal to the same absolutely continuous measure[2]. In the asymmetric case, Griesmer, Levitan and Shubik (1967) consider a first price auction with two bidders whose valuations are uniformly distributed over possibly different intervals. Already Vickrey (1961) analyzed the asymmetric two bidder case where one bidder knows the other bidder's valuation with certainty. Plum (1992) gives necessary and sufficient conditions of existence of a pure equilibrium in the two bidder case. Maskin and Riley (25 December 1996) examine several asymmetric two bidder examples.

The general model with mandatory bidding where the measures F_i are required to have compact supports, but are otherwise arbitrary, has been studied in Lebrun (1996). In this general case, a Nash equilibrium even in mixed strategies does not always exist[3]. However, it is shown that if the supports of the measures F_i have the same minimum and if this minimum is not a mass point of any of these measures, then there exists a Nash equilibrium. It is possible to obtain general positive results if the rules of the first price auction game are modified slightly. The existence theorems in Lebrun (1996) are proved in an indirect way by approximating the first price auction game by a sequence of games with a finite number of pure strategies. No characterization of the equilibria is given.

Asymmetric n bidder examples where all bidders except one have the same valuation probability measure have been numerically examined by Marshall, Meurer, Richard and Stromquist (1994). More numerical analysis can be found in Li and Riley (1997). Maskin and Riley (26 December 1996) consider the existence of an equilibrium in the asymmetric n bidder case by relying on discrete approximations and passing to the limit. They prove the existence in the cases of valuation measures absolutely continuous everywhere and of measures with finite supports. Maskin and Riley (November 1994) then study properties and descriptions of the equilibria when they exist. In Maskin and Riley (November 1994 and 26 December 1996), the uniqueness of the equilibrium is stated in the case of absolutely continuous measures with possible mass points at the lower extremities of the supports and density functions whose continuous extensions are striclty positive everywhere. Unfortunately, in the versions at my disposal the proof of even the case with common support is not complete[4].

In the present paper, we analyze the asymmetric n bidder case where the measures F_1, $F_2, \ldots, F_n$ have their supports equal to the same interval $[\underline{c}, \overline{c}]$ and are, either, absolutely continuous over the whole interval $[\underline{c}, \overline{c}]$ or, absolutely continuous over the interval $(\underline{c}, \overline{c}]$ with mass points at the lower extremity $\underline{c}$. This last case can be used to model situations where the reserve price set by the auctioneer is larger than the lower extremity of the valuation interval and is not covered by the existence results in Lebrun (1996) concerning the unaltered first price auction game nor by the results in Maskin and Riley (November 1994 and 26 December 1996). In the atomless case, the continuous extensions of the density functions are not required to exist at $\underline{c}$.

The approach we follow is, in a sense, reverse to and more direct than the approaches of Lebrun (1996) and Maskin and Riley (November 1994 and 26 December 1996). We first give a characterization of the equilibrium strategies as solutions of a system of differential equations with boundary conditions. We then proceed directly from this characterization in order to obtain the existence and other important properties of the equilibria, such as uniqueness.

We prove the existence of an equilibrium in the case of voluntary bidding when all valuation distributions have a mass point at $\underline{c}$ and in both cases with voluntary and mandatory bidding when the distributions are atomless. The difficulty of the proofs stems from the singularity of the differential system at $\underline{c}$. We circumvent it by considering the solution of the differential system as a function of the initial condition at the upper extremity $\overline{c}$.

When all distributions have a mass point at the lower extremity $\underline{c}$, we prove the uniqueness of the equilibrium. When every bidder's valuation distribution is one of two distributions, we give assumptions under which the equilibrium is unique in the atomless case. The distributions we obtain when we start from a symmetric setting and when several bidders collude into one cartel satisfy these assumptions. These results can be applied to all examples studied by Marshall, Meurer, Richard and Stromquist (1994). They can also be applied to situations where the bidders can be divided into two groups, the bidders of one group being reputed more interested in the object being auctioned as the bidders of the other. If all distributions except possibly one are identical, we prove the existence of an equilibrium with mandatory bidding when $\underline{c}$ is a mass point of the distributions[5].

We establish inequalities that hold between bidders' equilibrium strategies when relations of stochastic dominance exist between valuation probability distributions. As a consequence of these results, we show that if two bidders' valuation distributions are equal, then their equilibrium strategies are equal and the uniqueness of the equilibrium in the symmetric n bidder case follows from the known uniqueness of the symmetric equilibrium in this case.

In Section 2, we introduce the model and give necessary and sufficient conditions for a n-tuple of strategies to be an equilibrium. This characterization is proved in Section 3. We prove the existence of an equilibrium and investigate some of the properties of the equilibria in Section 4. Section 5 studies the case where $\{ F_1, \ldots, F_n \} = \{ G_1, G_2 \}$. Section 6 concludes. Details of the proofs and definitions can be found in Appendices 1 to 5.

2.The Model and the Characterization of the Equilibria.

The supports[6] of the probability measures $F_1, F_2, \ldots, F_n$ are equal to the same interval $[\underline{c}, \overline{c}]$, with $0 \leq \underline{c} < \overline{c}$. For the sake of convenience, we also denote by $F_1, F_2, \ldots, F_n$ the cumulative distribution functions continuous from the right. We assume that $F_1, F_2, \ldots, F_n$ are differentiable over $(\underline{c}, \overline{c}]$ and that their derivatives—the density functions $f_1, f_2, \ldots, f_n$—are locally bounded away from zero[7]. In the rest of the paper, this set of assumptions will be referred to as “the assumptions of Section 2".

In the case with a reserve price $r > \underline{c}$, the bidders with valuations not larger than r will bid as low as possible and will thus behave as if their valuations were equal to r. This case will then be equivalent to the case where the valuations are distributed over the interval $[r, \overline{c}]$, with the lower extremity r of this interval which is a mass point of the distributions $F_1, F_2, \ldots, F_n$.

After having observed his valuation v_i and if bidding is mandatory or if he has decided to bid, bidder i has to submit a bid $b_i \in \mathbb{R}$ at least as high as $\underline{c}$. We thus assume that $\underline{c}$ is a reserve price[8]. We denote the decision of bidder i of staying out[9] by $b_i = \text{OUT}$. Bidder i wins the auction if his bid b_i is strictly larger than the bids submitted by the other bidders and his payoff is equal to $(v_i - b_i)$. If bidder i stayed out of the auction or if at least one other bidder has submitted a bid strictly larger than b_i, he is not awarded the item and his payoff is equal to zero. If bidder i and at least one other bidder have submitted the highest bid ($\neq$ OUT), then there is a tie which is solved by a fair lottery. If $S(b_1, \ldots, b_n)$ is equal to the set of indices corresponding to the highest bidders, that is, $S(b_1, \ldots, b_n) = \{ j \mid 1 \leq j \leq n, b_j \neq \text{OUT} \text{ and } b_j \geq b_k, \text{ for all } 1 \leq k \leq n \text{ such that } b_k \neq \text{OUT}\}$ and if $i \in S(b_1, \ldots, b_n)$, the probability that bidder i wins the auction is $1/\#S(b_1, \ldots, b_n)$. If he wins, his payoff is again the difference between his valuation and his bid, and if he loses his payoff is equal to zero. We assume that the bidders are risk neutral. We denote by $p_i(v_i, b_1, \ldots, b_n)$ the expected payoff of bidder i if his valuation is equal to v_i and if $b_1, \ldots, b_n$ are the bids which have been submitted. Thus, we have

$$p_i(v_i, b_1, \ldots, b_n) = 0, \text{ if } i \notin S(b_1, \ldots, b_n)$$

$$p_i(v_i, b_1, \ldots, b_n) = (1/\#S(b_1, \ldots, b_n))\,(v_i - b_i), \text{ if } i \in S(b_1, \ldots, b_n).$$

The function $p_i(v_i, b_1, \ldots, b_n)$ is bounded from above. In fact, $p_i(v_i, b_1, \ldots, b_n) \leq (\overline{c} - \underline{c})$, for all v_i in $[\underline{c}, \overline{c}]$ and for all $b_1, \ldots, b_n$ in $\{OUT\} \cup [\underline{c}, +\infty)$.

A strategy of bidder i tells him what bid probability measure he should use as a function of his valuation. In Appendix 5, we formally define the strategies as "regular conditional probability distributions" ("stochastic kernels" or "transition probability distributions"). It enables us to consider the expected values of random variables of interest, such as the bidders' payoffs. For v in $[\underline{c}, \overline{c}]$, we denote by $\beta_i(v, .)$ the bid probability distribution, over $\{OUT\} \cup [\underline{c}, +\infty)$ in the case of voluntary bidding and $[\underline{c}, +\infty)$ in the case of mandatory bidding, bidder i uses if his valuation is equal to v and if he follows the strategy β_i. We say that a strategy β_i is pure if and only if $\beta_i(v, .)$ is concentrated at one point, that we denote by $\beta_i(v)$, for all v in $[\underline{c}, \overline{c}]$. In this case, we identify the strategy β_i with the bid function[10] from $[\underline{c}, \overline{c}]$ to $\{OUT\} \cup [\underline{c}, +\infty)$ or $[\underline{c}, +\infty)$, and whose value at v is equal to $\beta_i(v)$, for all v in $[\underline{c}, \overline{c}]$. A strategy β_i of bidder i and the valuation probability distribution F_i determine a probability measure $\beta_i * F_i$ over the product $[\underline{c}, \overline{c}] \times (\{OUT\} \cup [\underline{c}, +\infty))$ or $[\underline{c}, \overline{c}] \times [\underline{c}, +\infty)$ of the set of possible valuations with the set of possible actions (see Appendix 4). We denote by $[\beta_i * F_i]_2$ the marginal distribution of $\beta_i * F_i$ over the second component space. This marginal distribution should be interpreted as the "exante" probability distribution of the bid from bidder i prior to the choice by Nature of bidder i's valuation.

If bidder 1, …, bidder n follow the strategies $\beta_1, \ldots, \beta_n$, respectively, the expected payoff of bidder i conditional on his valuation being equal to v in $[\underline{c}, \overline{c}]$ is given by the following expression,

$$(1) \qquad \int p_i(v, b_1, \ldots, b_n)\, [\beta_i(v, .) \otimes \{\bigotimes_{j \neq i} (\beta_j * F_j)\}]\, (db_i, (dv_j, db_j)_{j \neq i}),$$

where $\otimes$ denotes the usual product between measures. Since the function $p_i(v, b_1, \ldots, b_n)$ is measurable and bounded from above, its integral above always exists in the weak sense. That is, the integral is equal to a finite number, when the function $p_i(v, b_1, \ldots, b_n)$ is integrable in the strong sense, or is equal to $-\infty$.

A n-tuple of strategies $(\beta_1, \ldots, \beta_n)$ form a Bayesian equilibrium if and only if,

$$\int p_i(v, b_1, \ldots, b_n)\, [\beta_i(v, .) \otimes \{\bigotimes_{j \neq i} (\beta_j * F_j)\}]\, (db_i, (dv_j, db_j)_{j \neq i}) \geq$$

$$\geq \int p_i(v, b_1, \ldots, b_n)\, [\mu_i \otimes \{\bigotimes_{j \neq i} (\beta_j * F_j)\}]\, (db_i, (dv_j, db_j)_{j \neq i}),$$

for all probability measures μ_i over the set of possible actions, that is, $[\underline{c}, +\infty)$ or $\{OUT\} \cup [\underline{c}, +\infty)$, for all valuation v in $[\underline{c}, \overline{c}]$ and for all $1 \leq i \leq n$, with the natural convention $x > -\infty$, for all $x \in \mathbb{R}$. The inequality above requires that the bid probability distribution $\beta_i(v, .)$ gives bidder i the highest possible payoff against the other bidders' strategies β_j, $j \neq i$, when his valuation is equal to v. We say that an equilibrium $(\beta_1, \ldots, \beta_n)$ is pure if and only if the strategies $\beta_1, \ldots, \beta_n$ are pure.

Theorems 1 and 2 below provide a characterization of all Bayesian equilibria.

Theorem 1 (mandatory bidding): Under the assumptions of Section 2, a n-tuple of strategies $(\beta_1, \ldots, \beta_n)$ *is a Bayesian equilibrium of the first price auction with mandatory bidding if and only if the strategies are pure, the bid functions are strictly increasing, and there exists* $\underline{c} < \eta < \overline{c}$ *such that the inverses* $\alpha_1 = \beta_1^{-1}, \ldots, \alpha_n = \beta_n^{-1}$ *form a solution over* $(\underline{c}, \eta]$ *of the system of differential equations (2) — considered in the domain* $D = \{ (b, \alpha_1, \ldots, \alpha_n) \in \mathbb{R}^{n+1} \mid \underline{c}, b < \alpha_i \leq \overline{c}$ *, for all* $1 \leq i \leq n \}$ *— and satisfy the boundary conditions (3),*

$$(2) \quad \frac{d}{db}\alpha_k(b) = \frac{F_k(\alpha_k(b))}{(n-1)\, f_k(\alpha_k(b))} \left\{ \frac{(-1)(n-2)}{\alpha_k(b)-b} + \sum_{\substack{l=1 \\ l\neq k}}^{n} \frac{1}{\alpha_l(b)-b} \right\}, 1 \leq k \leq n,$$

$$(3) \quad \alpha_i(\eta) = \overline{c} \text{ and } \alpha_i(\underline{c}) = \underline{c}, \text{ for all } 1 \leq i \leq n.$$

Theorem 2 (voluntary bidding): Under the assumptions of Section 2, a n-tuple of strategies $(\beta_1, \ldots, \beta_n)$ *is a Bayesian equilibrium of the first price auction with voluntary bidding if and only if the strategies* $\beta_1, \ldots, \beta_n$ *are equal to pure strategies over* $(\underline{c}, \overline{c}]$*, and there exists* $\underline{c} < \eta < \overline{c}$ *such that the inverses* $\alpha_1 = \beta_1^{-1}, \ldots, \alpha_n = \beta_n^{-1}$ *exist, are strictly increasing, and form a solution over* $(\underline{c}, \eta]$ *of the system of differential equations (2) — considered in the same domain D as in Theorem 1 — and satisfy the boundary conditions (4),*

(4) $\alpha_k(\eta) = \overline{c}$ *, for all* $1 \leq k \leq n$*, and* $\alpha_j(\underline{c}) = \underline{c}$ *, for all but at most one j between 1 and n,*

and the distributions $\beta_1(\underline{c}, .), \ldots, \beta_n(\underline{c}, .)$ *have their supports included in* $\{OUT, \underline{c}\}$ *and are such that (5) below holds true,*

(5) if there exists j such that $\alpha_j(\underline{c}) > \underline{c}$ *, then* $F_i(\underline{c}) > 0$ *and* $\beta_i(\underline{c}, .)$ *is concentrated at OUT, for all* $i \neq j$*, and* $\beta_j(v, .)$ *is concentrated at* $\underline{c}$ *, for all v in* $(\underline{c}, \alpha_j(\underline{c})]$.

In Theorems 1 and 2 above and in what follows, $\alpha_j(\underline{c})$ denotes the value of the continuous extension of α_j at $\underline{c}$, that is, $\alpha_j(\underline{c}) = \lim_{\substack{v \to \underline{c} \\ >}} \alpha_j(v)$. In using matrix notation, the system (1) can be rewritten as in (6) below,

$$(6) \qquad \frac{d}{db}\mathbb{LNF}(\alpha(b)) = \mathbb{M} \,.\, \mathbb{I}(\alpha(b), b),$$

where $\mathbb{LNF}(\alpha(b))$ and $\mathbb{I}(\alpha(b), b)$ are $n \times 1$ matrices and $\mathbb{M}$ is a $n \times n$ matrix defined as follows,

$$\mathbb{LNF}(\alpha(\mathrm{b})) \;=\; \begin{bmatrix} \ln F_1(\alpha_1(\mathrm{b})) \\ . \\ . \\ . \\ \ln F_n(\alpha_n(\mathrm{b})) \end{bmatrix} \qquad \mathbb{I}(\alpha(\mathrm{b}), \mathrm{b}) \;=\; \begin{bmatrix} \frac{1}{\alpha_1(\mathrm{b})-\mathrm{b}} \\ . \\ . \\ . \\ \frac{1}{\alpha_n(\mathrm{b})-\mathrm{b}} \end{bmatrix}$$

$$\mathbb{M} = \frac{1}{(\mathrm{n}-1)} \begin{bmatrix} (-1)(\mathrm{n}-2) & 1 & . & . & . & 1 \\ 1 & (-1)(\mathrm{n}-2) & 1 & . & . & . \\ . & 1 & (-1)(\mathrm{n}-2) & . & . & . \\ . & . & 1 & . & 1 & . \\ . & . & . & . & (-1)(\mathrm{n}-2) & 1 \\ 1 & 1 & . & . & 1 & (-1)(\mathrm{n}-2) \end{bmatrix},$$

with $\mathrm{b} \;=\; (\mathrm{b}_1, \ldots, \mathrm{b}_n)$ and $\alpha(\mathrm{b}) \;=\; (\alpha_1(\mathrm{b}_1), \ldots, \alpha_n(\mathrm{b}_n))$.

From Theorems 1 and 2, we see that all Bayesian equilibria can be obtained by taking the inverses of the solutions of the differential system (2). Remark also that the boundary conditions are different whether we consider the first price auction with or without mandatory bidding. In the case with mandatory bidding, the continuous extensions of the bid functions must all be equal to $\underline{\mathrm{c}}$ at $\underline{\mathrm{c}}$. In the case with voluntary bidding, there can be at most one bid function[11] such that the continuous extension of its inverse takes a value different from $\underline{\mathrm{c}}$ at $\underline{\mathrm{c}}$. However, when the distributions are atomless, (4) and (5) reduce to $\alpha_1(\underline{\mathrm{c}}) \;=\; \ldots \; \alpha_n(\underline{\mathrm{c}}) \;=\; \underline{\mathrm{c}}$. In all cases, all bid functions are strictly increasing over $(\max_{1 \leq k \leq n} \alpha_k(\underline{\mathrm{c}}), \overline{\mathrm{c}}\,]$ and there exists η in $(\underline{\mathrm{c}}, \overline{\mathrm{c}})$, such that $\alpha_k(\eta) = \overline{\mathrm{c}}$ and thus $\beta_k(\overline{\mathrm{c}}) = \eta$ for all $1 \leq \mathrm{k} \leq \mathrm{n}$, and η is the common value of the bid functions at the upper extremity $\overline{\mathrm{c}}$ of the valuation interval. We prove Theorems 1 and 2 in the next section.

3.Proof of the Characterization.

The proof that a n-tuple of strategies verifying the conditions given in Theorems 1 or 2 (Section 2) is an equilibrium is short enough to be kept in the main text.

<u>Proof of the "sufficiency parts" of Theorems 1 and 2:</u> We immediately see that if α_1, α_2, $\ldots$,α_n verify (2), then we have

(7) $$\frac{\mathrm{d}}{\mathrm{db}} \sum_{\substack{j=1 \\ j \neq i}}^{n} \ln F_j(\alpha_j(\mathrm{b})) \;=\; \frac{1}{\alpha_i(\mathrm{b})-\mathrm{b}},$$

for all b in $(\underline{c}\ ,\ \eta]$ and all $1 \leq i \leq n$. We have to prove that $\beta_i(v, .)$ maximizes bidder i's payoff when the other bidders bid according to β_j, $j \neq i$, for all v in $[\underline{c}\ , \overline{c}\ ]$ and all $1 \leq i \leq n$. It is easily seen that a bid larger than η is never a best response. It can also be checked that if $v = \underline{c}$, bidding in $\{\underline{c}\ \}$ or $\{\text{OUT}, \underline{c}\ \}$ is a best response. The probability distribution $\beta_i(\underline{c}\ ,$ $.)$ is thus a best response.

Suppose then that $v > \underline{c}$. Since $b = (v + \underline{c}\ )/2$, for example, gives a strictly positive expected payoff, bidding $b > v$ can never be a best response, and bidding $b = \underline{c}$ is not a best response when bidding is mandatory and when bidding is voluntary and $i \neq j$ where j is as in (5). Bidder i's expected payoff if he bids $b \in (\underline{c}\ , \eta]$ is equal to $(v - b) \prod_{\substack{j=1 \\ j \neq i}}^{n}$

$F_j(\alpha_j(b))$ and is strictly positive. When

bidding is voluntary and $i = j$ as in (5), this product is strictly positive and continuous at $b = \underline{c}$ and it again gives bidder i's expected payoff if $b = \underline{c}$. Since it is strictly positive, we can consider its logarithm. From (7), the derivative of this logarithm is equal[12] to $\frac{-1}{(v-b)} + \frac{1}{\alpha_i(b)-b}$, for $v > \underline{c}$ and $b < v$. Since α_i is strictly increasing over $[\underline{c}\ , \eta]$ and such that $\alpha_i(\beta_i(v)) = v > \beta_i(v)$ when $\beta_i(v) > \underline{c}$, we see that this derivative is strictly positive for $\underline{c} < b < \beta_i(v)$. Since $\alpha_i(\beta_i(v)) \geq v$ (it is equal except when $i = j$ as in (5) and $v < \alpha_i(\underline{c}\ )$), for all v in $(\underline{c}\ , \overline{c}\ ]$, and α_i is strictly increasing, we see that the derivative above is strictly negative for $v > b > \beta_i(v)$. Consequently, the global maximum of bidder i's expected payoff is obtained at $b = \beta_i(v)$ and the sufficiency parts of Theorems 1 and 2 are proved. ||

Next, we give the main steps of the proof of the necessity parts of Theorems 1 and 2 (Section 2). The complete proof can be found in Appendix 1. We use arguments which are now standard in the study of auctions (see Griesmer, Levitan and Shubik 1967) as well as of other games (see Baye, Kovenock and deVries 1992). We also use arguments from the theory of incentive compatible mechanisms (see Myerson 1981).

Let $(\beta_1, \ldots, \beta_n)$ be a Bayesian equilibrium. We denote by b_i the random variable, whose probability measure is $[\beta_i * F_i]_2$, and by $b_i(v_i)$ the random variable, whose probability measure is $\beta_i(v_i, .)$. Both random variables b_i and $b_i(v_i)$ should be interpreted as the bid from bidder i. However, the distribution of b_i is the exante distribution and the distribution of $b_i(v_i)$ is the distribution conditional on the choice by Nature of v_i as bidder i's valuation. When there cannot be any confusion about the strategies followed, we denote the expected payoff of bidder i when his valuation is equal to v by $P(i \mid v)$ instead of the long expression (1). Also for the sake of simplicity, we denote by $P(\ i \mid v, b\ )$ the payoff of bidder i if his valuation is equal to v and if his bid is equal to b; by Prob(i wins | v) the probability that bidder i wins if his valuation is equal to v; and by Prob(i wins | b) the probability that bidder i wins if his bid is equal to b. Thus, $P(\ i \mid v, b\ ) = (v - b)\text{Prob}(\ i \text{ wins} \mid v, b\ )$ when $b \neq \text{OUT}$.

We define the two functions b_{il} and b_{iu} as follows:

(8) $\quad b_{il}(v) = \inf\{ b \in [\underline{c}, +\infty) \mid P(i \mid v) = P(i \mid v, b) \}$,

(9) $\quad b_{iu}(v) = \sup\{ b \in [\underline{c}, +\infty) \mid P(i \mid v) = P(i \mid v, b) \}$.

Notice that since a bidder gets a zero payoff when he does not take part in the auction in the voluntary bidding case and when he submits the bid equal to his valuation, the sets in the definitions (8) and (9) are always nonempty. Since, from our definition of Bayesian equilibrium, $P(i \mid v)$ is the highest payoff bidder i can obtain when the other bidders follow the strategies β_j, $j \neq i$, we see that $b_{iu}(v)$ is the supremum of the set of "best bids" for bidder i if he takes part in the auction. Similarly, $b_{il}(v)$ is the infimum of the set of best allowable bids for bidder i. The random variable $b_i(v)$ may not be degenerate. However, what we know for sure is that $b_i(v)$ belongs with probability one to the interval[13] $[b_{il}(v), b_{iu}(v)]$ when bidding is mandatory and to $\{OUT\} \cup [b_{il}(v), b_{iu}(v)]$ otherwise.

We prove in Lemma A1-1 that every bidder's equilibrium payoff $P(i \mid v)$ is strictly positive and thus so is also his probability of winning Prob(i wins | v) when his valuation v is strictly larger than $\underline{c}$. It is then not difficult to prove that, at a Bayesian equilibrium, $b_{1l}(\underline{c}) = \ldots = b_{nl}(\underline{c}) = b_{1u}(\underline{c}) = \ldots = b_{nu}(\underline{c}) = \underline{c}$ (Lemma A1-3). Moreover, we show in Lemma A1-9 that the functions b_{il} and b_{iu} take strictly larger values than $\underline{c}$ over $(\underline{c}, \overline{c}]$, for all i in the mandatory bidding case and for all i except possibly one in the voluntary bidding case. In this latter case, if $b_{jl}(v) = \underline{c}$ for some $v > \underline{c}$ and j then $F_i(\underline{c}) > 0$, for all $i \neq j$, and there exists $w' > \underline{c}$ such that $b_{jl}(v) = \underline{c}$, for all v in $[\underline{c}, w']$, and $b_{jl}(v) > \underline{c}$, for all v in $(w', \overline{c}]$. We can easily understand why there cannot be such a w' for more than one bidder. If it was the case, then with strictly positive probability, there would be a tie at $\underline{c}$ and the bidders bidding $\underline{c}$ for valuations larger than $\underline{c}$ would be better off if they bid slightly higher instead.

It is rather straightforward to prove (see Lemma A1-8) the following "monotonicity" property of the two functions b_{il} and b_{iu}: $b_{iu}(v) \leq b_{il}(v')$, for all v, v' such that $\underline{c} \leq v < v' \leq \overline{c}$. This property implies, in particular, that both functions b_{il} and b_{iu} are nondecreasing (Lemma A1-11). A useful property of the equilibrium strategies $\beta_1, \ldots, \beta_n$, which is not much more difficult to establish in the n bidder case than in the two bidder case, is that the probability distributions $[\beta_1 * F_1]_2, \ldots, [\beta_n * F_n]_2$ have no mass point $b > \underline{c}$ (see Lemma A1-7). As a byproduct, we see that if $b > \underline{c}$, Prob(i wins | b) is equal to Prob($b_j = OUT$ or $b_j \leq b$, for all $j \neq i$) $= \prod_{j \neq i} [\beta_n * F_n]_2(\{OUT\} \cup [\underline{c}, b])$ and is a continuous function of $b > \underline{c}$, and thus $P(i \mid v, b) = (v - b)$ Prob(i wins | b) is a continuous function of v and $b > \underline{c}$. Consequently, in the definitions of b_{il} and b_{iu}, we can substitute "min" and "max for "inf" and "sup", respectively (see Lemma A1-10).

By comparing the bidders' behaviors at $\overline{c}$, we also prove that $b_{1l}(\overline{c}) = \ldots = b_{nl}(\overline{c}) = b_{1u}(\overline{c}) = \ldots = b_{nu}(\overline{c}) < \overline{c}$ (Lemma A1-12). This sequence of equalities imply that every bidder submits the same bid when his valuation is equal to the upper extremity $\overline{c}$ of the valuation interval. We denote this common bid by η.

We show (Lemma A1-4) that the expected payoff $P(i \mid v)$ of bidder i conditional on the valuation v, is a continuous function of v in $[\underline{c}, \overline{c}]$, for all $1 \leq i \leq n$. As a consequence (see Lemma A1-13), the functions b_{il} and b_{iu} are continuous from the left and from the right respectively, for all $1 \leq i \leq n$. Moreover, b_{iu} can be obtained from b_{il} by taking the limit from the right of b_{il}. Similarly, b_{il} is equal to the limit from the left of b_{iu}. We then see that the functions $b_{1l}, \ldots, b_{nl}, b_{1u}, \ldots, b_{nu}$ are strictly increasing when there are larger than $\underline{c}$ (see Lemma A1-14). For example, if b_{il} was equal to a constant larger than $\underline{c}$ on a nondegenerate interval, it would be continuous and thus equal to b_{iu} over this interval, and bidder i would bid the same bid when his valuation belongs to this interval. This bid would then be a mass point of the bid probability distribution, which is impossible at an equilibrium, as we saw earlier. We show in Figure 1 how these functions may look like according to what we know so far[14].

[FIGURE 1]

We now show the natural way discontinuities are ruled out in the two bidder case and give some intuition about how we proceed in the n bidder case. In the rest of this section we assume that bidding is mandatory. Considering the case with voluntary bidding would require only slight changes.

Imagine that $(\beta_1, \ldots, \beta_n)$ is a Bayesian equilibrium. We want to show that the equilibrium strategies are pure and that the bid functions are continuous. This will be done if we show that the functions $b_{1l}, \ldots, b_{nl}$ are continuous. Because $b_{1l}, \ldots, b_{nl}$ are strictly increasing, if one of them is discontinuous at a valuation v the discontinuity is of the “jump" kind.

Can there exist b_{il} discontinuous at v, that is, exhibiting a jump $(b_{il}(v), b_{iu}(v)) \neq \emptyset$ at v? In the two bidder case such a discontinuity is easily ruled out. Suppose $n = 2$ and $i = 1$. Bidder 1 bids within the jump with a probability zero, since he bids within this interval only if his valuation is equal to v. But then his opponent will not bid within this interval since bidding the lower extremity $b_{1l}(v)$ of the jump gives him the same probability of winning as any bid inside the jump and a lower payment in case of winning. Thus, b_{2l} also displays a discontinuity jump, which involves b_{1l}'s jump at v. In this case, the lower extremity $b_{1l}(v)$ of the discontinuity jump of b_{1l} would give bidder 1 a strictly higher payoff than the upper extremity $b_{1u}(v)$ of this jump since the probability of winning would not change while the payment in case of winning would strictly decrease. This, however, contradicts the fact we encountered earlier that $b_{1l}(v)$ and $b_{1u}(v)$ give bidder 1 with valuation v the same expected payoff ($b_{1l}(v)$ is bidder 1's lowest best bid and $b_{1u}(v)$ is bidder 1's highest best bid). Consequently, a discontinuity jump is impossible and the functions b_{1l} and b_{2l} are continuous.

In the case of n bidders, we cannot apply the same argument to prove the continuity of the equilibrium strategies. However, we can rule out discontinuities by looking more closely at the payoff function in the first price auction game. We first rule out situations like those in Figure 1, where bidder i's b_{il} jumps at v and where other bidders bid within the jump for valuations strictly smaller than v. If bidder $j \neq i$ submits b at w, the cost of any change from b, and in particular the change to $b_{il}(v)$, must outweigh its benefit. Note that maximizing the expected payoff $(w - b)$ Prob(j wins | b) (which, under our assumptions, is strictly positive) is equivalent to maximizing its logarithm $\ln(w - b) + \ln\text{Prob}($ j wins | b $)$. Thus the percentage decrease of the probability of winning, that is, the decrease of the term lnProb(j wins | b) due to a decrease of his bid to $b_{il}(v)$, must be at least as large as the percentage increase of payoff in case of winning; in other words the increase in the term $\ln(w - b)$, and we find (by using obvious notations):

(10) $\quad | \Delta \ln \text{Prob}(\text{ j wins }) | \;\; \geq \;\; | \Delta \ln (w - b) |$.

Bidder i's maximal expected payoff is reached at $b_{il}(v)$. Thus, if he increases his bid to b, the percentage decrease in his payoff in case of winning is not smaller than the percentage increase of his probability of winning; that is:

(11) $\quad | \Delta \ln (v - b) | \;\; \geq \;\; | \Delta \ln \text{Prob}(\text{ i wins }) |$.

However, the percentage change of bidder i's probability of winning is larger than the percentage change of bidder j's probability of winning; that is:

(12) $\quad | \Delta \ln \text{Prob}(\text{ i wins }) | \;\; \geq \;\; | \Delta \ln \text{Prob}(\text{ j wins }) |$.

In fact, bidder i has to take into account the increase in the probability of losing the auction to bidder j. On the other hand, the probability that bidder j looses the auction to bidder i does not change when bidder j decreases his bid to $b_{il}(v)$. As a consequence, we see from (10), (11) and (12) that the percentage change in bidder i's payoff in case of winning, when he increases his bid from $b_{il}(v)$ to b, must be at least as large as the percentage change in bidder j's payoff in case of winning when bidder j decreases his bid from b to $b_{il}(v)$; that is, $| \Delta \ln (v - b) | \;\; \geq \;\; | \Delta \ln (w - b) |$. Since the absolute changes in the payoffs are given by the difference between the two bids, and are thus equal, we can see that the only way this is possible is if $v \leq w$, and an example as that in Figure 1 is impossible.

Before ruling out the only remaining possible case of discontinuity, we need the following result. We prove in Lemma A1-18 that when b_{il} is continuous over a neighborhood of v, the probability Prob(i wins | b) is a differentiable function of b over a neighborhood of $b_{il}(v)$. Consequently, we can simply take the derivative with respect to b of the objective function (in its logarithmic form) $\ln(v - b) + \ln\text{Prob}(\text{ i wins} \mid \text{b })$ and set this derivative equal to zero at the best choice of bidder i. We find the equation:

(13) $$\frac{\frac{d}{db}\text{Prob}(\text{ i wins}|\text{b })}{\text{Prob}(\text{ i wins}|\text{b })} = \frac{1}{(v-b)},$$

which holds true at $b = b_{il}(v)$, and we obtain the mathematical expression of the equality of the “marginal benefit" of a change of the bid with its “marginal cost".

The only possible type of equilibria with discontinuities we still have to examine is the type of the example shown in Figure 2.

[FIGURE 2]

In this example, bidder i's b_{il} is discontinuous at v and all bidders k bidding within the discontinuity jump do so for valuations not smaller than v. Moreover, we have assumed in this example that these latter bidders k have their functions b_{kl} continuous over the ranges of valuations where they bid inside the jump. If this was not true, a function b_{kl}, $k \neq i$, would exhibit a jump included in the jump of b_{il}, and we would focus on b_{kl} instead of b_{il}. If necessary by repeating this argument, we see that our assumption does not imply any loss of generality. Assume that the bidders k bidding continuously within the jump are bidders $i + 1, \ldots, n$. The bidders $1, \ldots, i - 1$ have a discontinuity jump including b_{il}'s jump at v.

We know that when bidder k's b_{kl} is strictly increasing within a certain neighborhood of valuations, then bidder k's marginal cost of changing his bid is equal to his marginal benefit (see equation (13)). As a consequence, equation (13) holds for all b in $(b_{il}(v), b_{iu}(v))$ and for all bidders k, with $k \geq i + 1$. Taking the limit for b tending towards the lower extremity $b_{il}(v)$, we see that the same equation also holds at $b_{il}(v)$, if the derivative is interpreted as a right-hand derivative. Similarly, the equation holds at $b_{iu}(v)$ when the derivative is the left-hand derivative (see footnote 12 for a property similar to the one we use here).

For bids b from bidder k inside the jump of b_{il}, the probability that bidders $1, \ldots, i$ bid lower is constant since it is equal to the probability that all these bidders do not bid larger than $b_{il}(v)$. We can thus write $\ln \mathrm{Prob}(k \text{ wins} \mid b)$ as follows:

$$\ln \mathrm{Prob}(k \text{ wins} \mid b) = C + \sum_{\substack{j=i+1 \\ j \neq k}}^{n} \ln \mathrm{Prob}(b_j \leq b),$$

where C is a constant, for all $k \geq i + i$. Summing these equalities and dividing by $(n - i - 1)$, we find the following equality $\frac{1}{(n-i-1)} \sum_{k=i+1}^{n} \ln \mathrm{Prob}(k \text{ wins} \mid b) - K = \sum_{j=i+1}^{n} \ln \mathrm{Prob}(b_j \leq b)$, where K is also a constant. Reasoning as in the beginning of this paragraph, we see that up to an additive constant, the R.H.S. of the equality above is nothing but $\ln \mathrm{Prob}(i \text{ wins} \mid b)$. We thus obtain the equality $\ln \mathrm{Prob}(i \text{ wins} \mid b) = \frac{1}{(n-i-1)} \sum_{k=i+1}^{n} \ln \mathrm{Prob}(k \text{ wins} \mid b) + L$, where L is a constant, for all b in $[b_{il}(v), b_{iu}(v)]$. Taking the derivative of the equality above and using equation (13) which holds for bidders $i + 1, \ldots, n$, we see that $\frac{d}{db} \ln \mathrm{Prob}(i \text{ wins} \mid b)$ exists and we find:

$$(14) \qquad \frac{d}{db} \ln \mathrm{Prob}(i \text{ wins} \mid b) = \frac{1}{(n-i-1)} \sum_{k=i+1}^{n} \frac{1}{\alpha_k(b) - b},$$

for all b in $[b_{il}(v), b_{iu}(v)]$, where the derivative at $b_{il}(v)$ is a right-hand derivative, the derivative at $b_{iu}(v)$ is a left-hand derivative, and where α_k is the inverse of b_{kl}, or, $\alpha_k = b_{kl}^{-1}$.

Bidder i with valuation v reaches his maximum expected payoff when he bids $b_{il}(v)$. Consequently, the marginal percentage increase of probability $\frac{d}{db} \ln \mathrm{Prob}(i \text{ wins} \mid b_{il}(v))$ when he increases his bid, must be offset by the corresponding marginal percentage decrease $\frac{1}{v - b_{il}(v)}$ of his payoff if he wins. We thus obtain $\frac{d}{db} \ln \mathrm{Prob}(i \text{ wins} \mid b_{il}(v)) \leq \frac{1}{v - b_{il}(v)}$. Using equation (14) and rearranging, we find:

$$(15) \qquad \frac{1}{(\mathrm{n-i-1})} \sum_{k=i+1}^{n} \frac{1}{\alpha_k(\mathrm{b}_{il}(\mathrm{v}))-\mathrm{b}_{il}(\mathrm{v})} \leq \frac{1}{\mathrm{v}-\mathrm{b}_{il}(\mathrm{v})} .$$

Similarly, because the maximum expected payoff of bidder i with valuation v is also reached at $\mathrm{b}_{iu}(\mathrm{v})$, we find:

$$(16) \qquad \frac{1}{\mathrm{v}-\mathrm{b}_{iu}(\mathrm{v})} \leq \frac{1}{(\mathrm{n-i-1})} \sum_{k=i+1}^{n} \frac{1}{\alpha_k(\mathrm{b}_{iu}(\mathrm{v}))-\mathrm{b}_{iu}(\mathrm{v})} .$$

Equations (15) and (16) can be rewritten equivalently as follows:

$$(17) \qquad \sum_{k=i+1}^{n} \frac{\mathrm{v}-\mathrm{b}_{il}(\mathrm{v})}{\alpha_k(\mathrm{b}_{il}(\mathrm{v}))-\mathrm{b}_{il}(\mathrm{v})} \leq (\mathrm{n-i-1}) \leq \sum_{k=i+1}^{n} \frac{\mathrm{v}-\mathrm{b}_{iu}(\mathrm{v})}{\alpha_k(\mathrm{b}_{iu}(\mathrm{v}))-\mathrm{b}_{iu}(\mathrm{v})} .$$

However, as it can be easily checked (see the proof of Lemma A1-25), the functions $\frac{\mathrm{v-b}}{\alpha_k(\mathrm{b})-\mathrm{b}}$ are strictly decreasing functions of b over the domain $\mathrm{v} > \mathrm{b}$ and $\alpha_k(\mathrm{b}) \geq \mathrm{v}$. Since $\mathrm{b}_{il}(\mathrm{v}) < \mathrm{b}_{iu}(\mathrm{v})$ and $\alpha_k(\mathrm{b}_{il}(\mathrm{v})) \geq \mathrm{v}$, we have $\sum_{k=i+1}^{n} \frac{\mathrm{v}-\mathrm{b}_{il}(\mathrm{v})}{\alpha_k(\mathrm{b}_{il}(\mathrm{v}))-\mathrm{b}_{il}(\mathrm{v})} > \sum_{k=i+1}^{n} \frac{\mathrm{v}-\mathrm{b}_{iu}(\mathrm{v})}{\alpha_k(\mathrm{b}_{iu}(\mathrm{v}))-\mathrm{b}_{iu}(\mathrm{v})}$ which contradicts (17). We have ruled out the only possible type of discontinuities (see Figure 2) in the equilibrium strategies, and consequently, the equilibrium strategies have to be continuous bid functions.

Once we know that the equilibrium strategies are continuous, the differentiability over $(\underline{c}\ , \eta]$ of the inverses $\alpha_1 = \beta_1^{-1}, \ldots, \alpha_n = \beta_n^{-1}$ follows from the already mentioned Lemma A1-18. The system (2) (Section 2) is simply obtained by solving for $\frac{\mathrm{d}}{\mathrm{db}}\alpha_k(\mathrm{b})$, $1 \leq \mathrm{k} \leq \mathrm{n}$, the equations (13), with $1 \leq \mathrm{i} \leq \mathrm{n}$, where $\prod_{k \neq i} \mathrm{F}_k(\alpha_k(\mathrm{b}))$ has been substituted for $\mathrm{Prob}(\mathrm{i\ wins} \mid \mathrm{b})$ (see Lemmas A1-16 and A1-25).

4.Existence and Other Properties of the Equilibria.

We obtain the existence of a Bayesian equilibrium directly from the characterization given in Theorems 1 and 2 (Section 2). We prove this existence when $\underline{c}$ is not a mass point of any of the distributions $\mathrm{F}_1, \mathrm{F}_2, \ldots, \mathrm{F}_n$. In the voluntary bidding case, we also prove the existence when $\underline{c}$ is a mass point of all these distributions. In Section 5, we show a class of asymmetric n-tuples of distributions $(\mathrm{F}_1, \mathrm{F}_2, \ldots, \mathrm{F}_n)$ for which we prove the existence when bidding is mandatory even in the case of simultaneous mass point at $\underline{c}$. In Corollary 3 (v), our existence results in the symmetric case are extended somewhat (see also footnote 17).

Theorem 3: Let the assumptions of Section 2 be satisfied. If $F_1(\underline{c}\ ) = \ldots = F_n(\underline{c}\ ) = 0$, there exists a Bayesian equilibrium of the first price auction with or without mandatory bidding. If the right-hand derivatives of $F_1, \ldots, F_n$ at $\underline{c}$ exist, the density functions $\frac{d}{dv}F_1 = f_1, \ldots, \frac{d}{dv}F_n = f_n$ are bounded away from zero[15] over $[\underline{c}\ , \overline{c}\ ]$, and $F_1(\underline{c}\ ), \ldots, F_n(\underline{c}\ ) > 0$, there then exists a Bayesian equilibrium of the first price auction with voluntary bidding.

Proof: See Appendix 2.

The proof of Theorem 3 is long but straightforward. From Theorems 1 and 2 (Section 2), we know that the existence of a Bayesian equilibrium reduces to the existence of a parameter η for which there exists a solution $(\alpha_1, \ldots, \alpha_n)$ of (2, 3) or (2, 4, 5), depending on whether bidding is mandatory or not. The system (2) considered in the domain D is equivalent to the system (18)—considered in the domain $\mathcal{D} = \{ (\mathrm{b}, \psi_1, \ldots, \psi_n) \in \mathbb{R}^{n+1} \mid \mathrm{F}_i(\underline{\mathrm{c}}), \mathrm{F}_i(\mathrm{b}) < \psi_i \leq 1$, for all $1 \leq \mathrm{i} \leq \mathrm{n} \}$—in the unknown functions $\psi_1 = \mathrm{F}_1(\alpha_1), \ldots, \psi_n = \mathrm{F}_n(\alpha_n)$:

$$(18) \quad \frac{\mathrm{d}}{\mathrm{db}}\psi_k(\mathrm{b}) = \frac{\psi_k(\mathrm{b})}{(\mathrm{n}-1)} \left\{ \frac{(-1)(\mathrm{n}-2)}{\mathrm{F}_k^{-1}(\psi_k(\mathrm{b}))-\mathrm{b}} + \sum_{\substack{l=1 \\ l \neq k}}^{n} \frac{1}{\mathrm{F}_l^{-1}(\psi_l(\mathrm{b}))-\mathrm{b}} \right\}, 1 \leq \mathrm{k} \leq \mathrm{n}.$$

Under the assumptions of Section 2, F_k^{-1} is locally Lipschitz over $(\mathrm{F}_k(\underline{\mathrm{c}}), 1]$, for all $1 \leq \mathrm{k} \leq \mathrm{n}$, and the system (18) thus satisfies over $\mathcal{D}$ the standard requirements of the theory of ordinary differential equations.

Under the boundary conditions (2), the system (18) presents a singularity at $\underline{\mathrm{c}}$. In fact, $\mathrm{F}_i^{-1}(\psi_i(\underline{\mathrm{c}})) - \underline{\mathrm{c}} = \alpha_i(\underline{\mathrm{c}}) - \underline{\mathrm{c}} = 0$, for at least n – 1 values of the index i, and there may exist $1 \leq \mathrm{j} \leq \mathrm{n}$ such that F_j^{-1} is not locally Lipschitz at $\underline{\mathrm{c}}$. As a particular consequence, in the mandatory bidding case we cannot apply the classic theorems of the theory of ordinary differential equations to the system (18) and the initial condition $\alpha_i(\underline{\mathrm{c}}) = \underline{\mathrm{c}}$, for all $1 \leq \mathrm{i} \leq \mathrm{n}$. Furthermore, in the voluntary bidding case the boundary conditions (4, 5) do not provide us a complete initial condition at $\underline{\mathrm{c}}$. Rather we will consider the system (18) with the initial condition (19) below,

$$(19) \quad \psi_i(\eta) = 1, \text{ for all } 1 \leq \mathrm{i} \leq \mathrm{n}, \text{ or, equivalently, } \alpha_i(\eta) = \overline{\mathrm{c}}, \text{ for all } 1 \leq \mathrm{i} \leq \mathrm{n}.$$

For a parameter η such that $\underline{\mathrm{c}} < \eta < \overline{\mathrm{c}}$, the system (18) does not present a singularity at this initial condition. We can thus apply the theorems of the theory of ordinary differential equations to the problem (2, 19), through the system (18). We prove Theorem 3 by proving the existence of a parameter $\underline{\mathrm{c}} < \eta < \overline{\mathrm{c}}$ for which the solution of the problem (2, 19) consists of strictly increasing functions defined over $(\underline{\mathrm{c}}, \eta]$, such that the conditions (3) or (4, 5) are verified. To this end, we first study the system (2) when there is no mass point. When $\underline{\mathrm{c}}$ is a mass point of all distributions, we come back to the atomless case by extending all density functions to the left, and by considering a larger common support.

We first assume that $\mathrm{F}_1(\underline{\mathrm{c}}) = \ldots = \mathrm{F}_n(\underline{\mathrm{c}}) = 0$. We prove (Lemma A2-2) that for every $\underline{\mathrm{c}} < \eta < \overline{\mathrm{c}}$, the solution of (2, 19) in the domain D consists of strictly increasing functions. We then look at the maximal solution $(\alpha_1, \ldots, \alpha_n)$ of (2, 19) over $(\underline{\mathrm{c}}, \eta]$: that is, according to the terminology from Birkhoff and Rota (1978, p.162) the solution of (2, 19) that cannot be defined over a larger sub-interval of $(\underline{\mathrm{c}}, \eta]$ and still be a solution of (2, 19) in the domain D. Following Pontryagin (1962, p.21) we refer to the definition interval $(\underline{\gamma}, \eta] \subseteq (\underline{\mathrm{c}}, \eta]$ of the maximal solution as the maximal interval of existence, or simply as the maximal interval. We prove that only two cases are possible. In the first case, the maximal interval is equal to the whole $(\underline{\mathrm{c}}, \eta]$; in other words, $\underline{\gamma} = \underline{\mathrm{c}}$. In this case, we have (Lemma A2-4) either $\alpha_1(\underline{c}), \ldots, \alpha_n(\underline{c}) > \underline{c}$ or $\alpha_1(\underline{c}) = \ldots = \alpha_n(\underline{c}) = \underline{c}$ (see Figure 3). We say that such a solution is of type I.

[FIGURE 3]

In the second case, the definition interval of the maximum solution is a sub-interval $(\underline{\gamma}, \eta]$ strictly smaller than $(\underline{c}, \eta]$; or, $\underline{\gamma} > \underline{c}$. In this case, we show (Lemma A2-7) that all functions α_i, except possibly one, are such that $\alpha_i(\underline{\gamma}) = \underline{\gamma}$ (see Figure 4). We say that the solution is of type II.

[FIGURE 4]

An important property of the system (2) is that the solution $(\alpha_1, \ldots, \alpha_n)$ of the problem (2, 19) depends monotonically of η (Lemma A2-8); that is, if $\eta' > \eta$ and if $(\alpha_1, \ldots, \alpha_n)$ is the solution corresponding to η and $(\alpha'_1, \ldots, \alpha'_n)$, the solution corresponding to η', then $\alpha'_i \leq \alpha_i$, over the common definition domain of $(\alpha'_1, \ldots, \alpha'_n)$ and $(\alpha_1, \ldots, \alpha_n)$. Furthermore, we prove (in Lemma A2-13) that when η tends towards $\overline{c}$ the corresponding solution is of type II and $\underline{\gamma}$ tends towards $\overline{c}$, and that a solution corresponding to η close to $\underline{c}$ is of type I. By using continuity arguments (see Lemma A2-12 and the proof of Theorem 3 in Appendix 3), we then show that there exists η, such that the solution of (2, 19) is such that $\alpha_1(\underline{c}) = \ldots = \alpha_n(\underline{c}) = \underline{c}$. This solution is thus also a solution of the boundary value problem (2, 3); and, by Theorems 1 and 2 (Section 2), it corresponds to a Bayesian equilibrium and Theorem 3 is proved when $F_1(\underline{c}) = \ldots = F_n(\underline{c}) = 0$.

Assume now that the right-hand derivatives of $F_1, \ldots, F_n$ at $\underline{c}$ exist, $\frac{d}{dv}F_1 = f_1, \ldots, \frac{d}{dv}F_n = f_n$ are bounded away from zero over $[\underline{c}, \overline{c}]$, and $\underline{c}$ is a mass point of all distributions $F_1, \ldots, F_n$. The existence in the voluntary bidding case can now be proved simply by extending the density functions (for example in a piecewise linear way) to an interval $[\underline{c}_0, \overline{c}]$, with $\underline{c}_0 < \underline{c}$, in a such a way that they define new atomless probability distributions. From the continuity (proved in Lemma A2-13) of the lower extremity $\underline{\gamma}$ of the maximal interval with respect to η, we see that there exists η such that the corresponding $\underline{\gamma}$ is equal to $\underline{c}$. The solution of (2, 19) with this η is a type II solution, and therefore the initial condition (4) is immediate. We also prove (Lemma A2-7) that the condition (5) is satisfied. This value of η thus determines an equilibrium and the proof of Theorem 3 is complete.

The previous argument provides an interpretation to the type II solutions. Consider such a solution and the lower extremity $\underline{\gamma}$ of its definition interval. It defines a Bayesian equilibrium of the first price auction with voluntary bidding and with a reserve price equal to $\underline{\gamma}$. In our setting, this auction where the bidders' valuations are distributed over $[\underline{c}, \overline{c}]$ is equivalent to the first price auction with voluntary bidding where the valuations are distributed over $[\underline{\gamma}, \overline{c}]$ and where the probability weights previously spread over $[\underline{c}, \underline{\gamma}]$ by the distributions $F_1, \ldots, F_n$ are now concentrated at $\underline{\gamma}$. An immediate consequence of Theorem 3 is thus Corollary 1 (i) below. Corollary 1 (ii) follows from the property of monotonicity of the solutions of (2, 19) with respect to η (Lemma A2-8).

Corollary 1: Let the assumptions of Section 2 be satisfied.
(i). For all $r \in (\underline{c}, \overline{c})$, there exists a Bayesian equilibrium of the first price auction with voluntary bidding and with a reserve price equal to r.
(ii). If $(\beta_1, \ldots, \beta_n)$ is a Bayesian equilibrium when the reserve price is r and if $(\beta'_1, \ldots, \beta'_n)$ is a Bayesian equilibrium when the reserve price is r', with $\underline{c} \leq r < r' < \overline{c}$, then $\beta'_i(v) > \beta_i(v)$, for all v in $(r', \overline{c}]$.

Using the property of monotonicity (Lemma A3-8), we see that if $(\alpha_1, \ldots, \alpha_n)$ and $(\widetilde{\alpha}_1, \ldots, \widetilde{\alpha}_n)$ are two type II solutions with the same $\underline{\gamma}$, if $\alpha_i(\underline{\gamma}) > \underline{\gamma}$ and $\widetilde{\alpha}_j(\underline{\gamma}) > \underline{\gamma}$, for some i and j, then i = j. From this fact and the equation (7), we prove that under the conditions of Theorem 3 when $F_1(\underline{c}), \ldots, F_n(\underline{c}) > 0$ the equilibrium is uniquely determined over $(\underline{c}, \overline{c}]$. We refer to such an equilibrium as an “essentially" unique equilibrium. The Bayesian equilibrium of the first price auction with a reserve price in $(\underline{c}, \overline{c})$ as in Corollary 1 (i) is thus essentially unique.

Corollary 2: Let the assumptions of Section 2 be satisfied. If the right-hand derivatives of $F_1, \ldots, F_n$ at $\underline{c}$ exist, the density functions $\frac{d}{dv}F_1 = f_1, \ldots, \frac{d}{dv}F_n = f_n$ are bounded away from zero over $[\underline{c}, \overline{c}]$, and $F_1(\underline{c}), \ldots, F_n(\underline{c}) > 0$, there then exists an essentially unique Bayesian equilibirum $(\beta_1, \ldots, \beta_n)$ of the first price auction with voluntary bidding. Any other n-tuple of strategies which coincides with $(\beta_1, \ldots, \beta_n)$ over $(\underline{c}, \overline{c}]$ and which satisfies (5) is an equilibrium.

Proof: See Appendix 3.

Thanks again to the property of monotonicity, we see that the set of parameters η corresponding to solutions of (2, 3) or (2, 4, 5) and thus to Bayesian equilibria is an interval. Using continuity properties, we show that this interval (denoted Λ^* in the proof of Theorem 3 in Appendix 2) is closed and we prove Corollary 3 below[16]. From this corollary, we see that either there is a unique equilibrium or there exists a continuum of equilibria.

Corollary 3: Let the assumptions of Section 2 be satisfied. If $F_1(\underline{c}) = \ldots = F_n(\underline{c}) = 0$, there exist η^* and η^{**} in $(\underline{c}, \overline{c})$ such that $\eta^* \leq \eta^{**}$ and the solution $(\alpha_1, \ldots, \alpha_n)$ of (2)-(19) corresponds to a Bayesian equilibrium of the first price auction with mandatory bidding if and only if it is an equilibrium of the auction with voluntary bidding, and if and only if $\eta \in [\eta^*, \eta^{**}]$.

Proof: Immediate from Theorem 3, the monotonicity with respect to η (Lemma A2-8) and the observation made above that for a solution of type I and when there is no mass point at $\underline{c}$, we have (Lemma A2-4) either $\alpha_1(\underline{c}), \ldots, \alpha_n(\underline{c}) > \underline{c}$ or $\alpha_1(\underline{c}) = \ldots = \alpha_n(\underline{c}) = \underline{c}$. ||

We now give some properties the Bayesian equilibria display when there exists a relation of stochastic dominance between valuation distributions. Again, these properties mainly follow from results we have already proved in the course of the proof of Theorem 3 in Appendix 2.

Corollary 4: Let the assumptions of Section 2 be satisfied. Let $(\beta_1, \ldots, \beta_n)$ be a Bayesian equilibrium of the first price auction with or without mandatory bidding, and let i and j be two indices such that $1 \leq i, j \leq n$.
(i). If $F_j(v) \leq F_i(v)$, for all v in $[\underline{c}, \overline{c}]$, then we have $F_j(\alpha_j(b)) \leq F_i(\alpha_i(b))$, for all b in $[\underline{c}, \eta]$, with $\eta = \beta_1(\overline{c}) = \ldots = \beta_n(\overline{c})$; or, equivalently, $\beta_i(v) \leq \beta_j(F_j^{-1}(F_i(v)))$, for all v in $(\underline{c}, \overline{c}]$.
(ii). If F_i/F_j is nonincreasing over $(\underline{c}, \overline{c}]$, then we have $\beta_j(v) \leq \beta_i(v)$, for all v in $(\underline{c}, \overline{c}]$.
(iii). If $\frac{d}{dv} F_i/F_j\ (v) < 0$, for all v in $(\underline{c}, \overline{c}]$, then we have $\beta_j(v) < \beta_i(v)$, for all v in $(\underline{c}, \overline{c})$.
(iv). If $F_i(v) = F_j(v)$, for all v in $[\underline{c}, \overline{c}]$, then we have $\beta_j(v) = \beta_i(v)$, for all v in $(\underline{c}, \overline{c}]$.
(v). If $F_1 = \ldots = F_n = F$, then we have $\beta_1(v) = \ldots = \beta_n(v) = \beta(v) = v - \int_{\underline{c}}^{v} F^{n-1}(w)\, dw/F^{n-1}(v)$, for all v in $(\underline{c}, \overline{c}]$; these equalities define the unique equilibrium in the mandatory bidding case and the essentially unique equilibrium in the voluntary bidding case.

Proof: See Appendix 3.

Statement (i) of Corollary 4 tells us that the same relation of stochastic dominance passes from the valuation probability distributions to the bid probability distributions (for a related result in the case of two bidders, see Proposition 2.2 (ii) in Maskin and Riley, 25 December 1996). In fact, $F_j(\alpha_j(b))$, for example, is the cumulative distribution function of the probability measure of bidder j's bid b_j. As it can be easily seen, the assumption of (ii) is stronger than the assumption of (i). In addition to the competition from the other bidders, bidder j faces the competition from bidder i, who is more likely to have only little interest in the item. Since bidder i faces, likely, a more fierce competition from bidder j, and, under the assumption of (ii), bidder i bids higher. Statement (iii) is useful in the proof of the results in the next section. From (iv) two bidders whose valuations are identically distributed follow the same equilibrium strategy. Corollary 4 (v) thus extends this uniqueness in the set of symmetric n-tuples of (pure) strategies, proved by Riley and Samuelson (1981), to the set of all (symmetric and asymmetric) n-tuples of strategies[17].

5.A Special Class of Asymmetric Combinations of Distributions.

In this section, we obtain existence and uniqueness results for the class of asymmetric n-tuples of distributions $(F_1, \ldots, F_n)$ for which every bidder's probability distribution is one of two distributions. Without loss of generality for such a n-tuple we can assume that there exist $1 \leq m \leq n$, G_1 and G_2 such that:

(20) $F_i = G_1$, for all $1 \leq i \leq m$, and $F_i = G_2$, for all $m < i \leq n$.

Simple considerations of collusion, from a symmetric setting, lead to n-tuples in this class. Assume that the bidders' valuations are identically distributed according to F. Suppose that $m > 1$ bidders collude into one surplus maximizing cartel, with perfect information about its members' valuations, and perfect control over their actions. Since when it wins, the cartel will allocate the item to its member with the highest valuation, it is equivalent to a single bidder whose valuation is the maximum of m independent, random variables distributed according to F. We thus obtain an asymmetric situation where one bidder's valuation is distributed according to $G_2 = F^m$ and the other bidders' valuations are distributed according to $G_1 = F$. Notice that we would still obtain a n-tuple of distributions from the class we consider in this paragraph if several cartels of the same size m formed, or if all bidders colluded into cartels of two different sizes.

In the previous example, if the distribution F is absolutely continuous with a strictly positive, continuous density function over $(\underline{c}, \overline{c}]$, the assumption (21) below is satisfied,

(21) $$\frac{d}{dv} \frac{G_1}{G_2}(v) < 0, \text{ for all v in } (\underline{c}, \overline{c}].$$

In Corollary 5, we show that the equilibrium is unique under the assumption of the stochastic dominance relation (20) between atomless distributions G_1 and G_2. The examples studied by Marshall, Meurer, Richard and Stromquist (1994) satisfy this requirement and the equilibria obtained by these authors were thus the unique equilibria.

If (19) holds true, Corollary 4 (iv) (Section 4) implies that any equilibrium is determined by two bid functions β'_1 and β'_2 used by the bidders whose valuations are distributed according to G_1 and G_2 respectively. The system (2) thus reduces to a system of two equations in the unknown functions $\alpha'_1 = \beta'^{-1}_1$ and $\alpha'_2 = \beta'^{-1}_2$. If we divide these two equations by each other and simplify, we see that the differences $(\alpha'_1(b) - b)$ and $(\alpha'_2(b) - b)$ appear only in a quotient of two polynomials of degree one. For this reason, it is advantageous in our proofs to consider the differential system[18] the functions $\phi'_{21} = \alpha'_2\beta'_1$ and β'_1 form a solution of. Remark that Corollary 5 does not require any strengthening at $\underline{c}$ of the regularity conditions of Section2.

<u>Corollary</u> <u>5</u>: Let the assumptions of Section 2 be satisfied. Assume that there exist $1 \leq m \leq n$ and two distributions G_1, G_2 absolutely continuous over $[\underline{c}, \overline{c}]$ such that (20) and (21) hold true. There then exists a unique equilibrium in the mandatory bidding and an essentially unique equilibrium in the voluntary case.

<u>Proof</u>: See Appendix 3.

It turns out that if all distributions except at most one are identical, a type II-solution (see Section 4) of the differential system (2) with initial condition (19) is such that $\alpha_1(\underline{\gamma}) = \ldots = \alpha_n(\underline{\gamma}) = \underline{\gamma}$. For an arbitrary n-tuple of distributions, we were only able to prove that $\alpha_i(\underline{\gamma}) = \underline{\gamma}$, for all but at most one α_i (see Lemma A2-7). This property of the special class of n-tuples we study in this section implies the existence of a Bayesian equilibrium of the first price auction with mandatory bidding when there is a mass point at $\underline{c}$ (see also footnote 17).

<u>Corollary</u> <u>6</u>: Let the assumptions of Section 2 be satisfied. Assume that there exist $n - 1 \leq m \leq n$ and two distributions G_1, G_2 such that (20) holds true. Assume further that the right-hand derivatives of G_1, G_2 at $\underline{c}$ exist, the density functions $\frac{d}{dv}G_1 = g_1$, $\frac{d}{dv}G_2 = g_2$ are bounded away from zero over $[\underline{c}, \overline{c}]$, and $G_1(\underline{c})$, $G_2(\underline{c}) > 0$. There then exists a unique Bayesian equilibrium of the first price auction with mandatory bidding.

<u>Proof</u>: See Appendix 3.

6.Conclusion.

Without assumption of symmetry, and with an arbitrary number n of bidders, we obtained a characterization of the Bayesian equilibria of the first price auction game with or without mandatory bidding. Proceeding directly from this characterization, we proved the existence of a Bayesian equilibrium. We proved inequalities between equilibrium strategies when there exist relations of stochastic dominance between valuation distributions; as a consequence of these inequalities, two bidders have the same equilibrium strategy if their valuations are identically distributed. When the distributions have a mass point at the lower extremity of the support, we prove the uniqueness of the equilibrium. When there are no more than two different valuation distributions and when there exists a relation of stochastic dominance between them, we proved the uniqueness of the equilibrium in the atomless case. This result can be applied to the valuation distributions that result from a symmetric situation after some bidders have colluded.

Appendix 1.

<u>Lemma</u> <u>A1-1</u>: *If $(\beta_1, \ldots, \beta_n)$ is a Bayesian equilibrium of the first price auction with or without mandatory bidding, then P(1 | v) $>$ 0, …, P(n | v) $>$ 0, Prob(1 wins $|$ v) $>$ 0, …, Prob(n wins $|$ v) $>$ 0, and Prob($b_1 = OUT$ $|$ v) $=$ 0, …, Prob($b_n = OUT$ $|$ v) $=$ 0, for all v in $(\underline{c}\ ,\overline{c}\ ]$.*

<u>Proof</u>: Let v be a valuation in $(\underline{c}\ ,\overline{c}\ ]$. Suppose that there exists i such that P(i | v) = 0. We see that bidder i cannot win the auction with a strictly positive probability when he submits a bid strictly smaller than v. Otherwise he would submit such a bid and would obtain a strictly positive payoff. That is, Prob(i wins $|$ v, b) $=$ 0, for all b $<$ v. Consequently Prob ($\max_{j \neq i} b_j < b$) $=$ 0, for all b $<$ v, and thus Prob ($\max_{j \neq i} b_j < v$) $=$ 0 and Prob ($\max_{j \neq i} b_j \geq v$) $=$ 1. Since $\max_{j} b_j \geq \max_{j \neq i} b_j$, we also have Prob ($\max_{j} b_j \geq v$) $=$ 1. Since a winner is always declared as long as at least one bidder has bid, we see that there exists a winner with probability one. Consequently $\sum_{k=1}^{n}$ Prob (k wins $|$ $\underline{c} \leq v_j < v$, for all j) $=$ 1. We then see that there exists k such that Prob (k wins $|$ $\underline{c} \leq v_j < v$, for all j) $>$ 0 and thus Prob (k wins $|$ $\underline{c} \leq v_k < v$) $>$ 0. However this is impossible at the equilibrium because it would mean that for F_k-almost every v_k in $(\underline{c}\ , v)$, there is a strictly positive probability that bidder k wins the auction with a bid strictly larger than his valuation and would thus obtain a strictly negative payoff. Bidder k's payoff would be strictly higher if he bid his valuation instead.

We have proved that P(i | v) $>$ 0, for all i. If bidder i's probability of winning was equal to zero, his expected payoff would also be equal to zero and we have proved Prob(i wins $|$ v) $>$ 0, for all i and v $> \underline{c}$.

By reasoning as in the previous paragraph, we can prove that Prob($b_i = \text{OUT} \mid v$) $\neq 1$, for all i and all $v > \underline{c}$. If Prob($b_i = \text{OUT} \mid v$) > 0, bidder i would increase his expected payoff when his valuation is equal to v if he bid rather with probability one according to the conditional distribution $\beta_i(v, . \mid \{b_i \neq \text{OUT}\})$, and Lemma A1-1 is proved. ||

We need one more notation in addition to those introduced in Section 3. We denote by $\underline{b}_i$ the essential infimum of the random variable b_i conditional on $b_i \neq \text{OUT}$. That is, $\underline{b}_i$ is the highest number which is not larger than b_i with conditional probability one. From Lemma A1-1, we know that the event $b_i \neq \text{OUT}$ has a strictly positive probability (actually at least equal to $1 - F_i(\underline{c})$), for all i. Thus the definition of $\underline{b}_i$ is meaningful.

<u>Lemma</u> <u>A1-2</u>: *If $(\beta_1, \ldots, \beta_n)$ is a Bayesian equilibrium of the fist price auction with or without mandatory participation, then*

$$\underline{b}_1 = \ldots = \underline{b}_n = \underline{c} .$$

<u>Proof</u>: From the rules of the game and the definition of $\underline{b}_i$, we immediately obtain $\underline{b}_i \geq \underline{c}$. Denote $\max_{1 \leq i \leq n} \underline{b}_i$ by $\underline{b}$. Suppose that $\underline{b} > \underline{c}$. Let H be the set of indices i such that $\underline{b} = \underline{b}_i$. Let i be an element of H, that is, such that $\underline{b} = \underline{b}_i$. Then, for F_i-almost all v_i in $[\underline{c}, \underline{b}_i)$ we have Prob (i wins $\mid v_i$) $= 0$. In fact, from the definition of $\underline{b}_i$ we see that for F_i-almost all v_i in $[\underline{c}, \overline{c}]$, thus in $[\underline{c}, \underline{b}_i)$, if bidder i bids he bids at least $\underline{b}_i$ with probability one. If the probability Prob (i wins $\mid v_i$) was strictly positive over a Borel subset of $[\underline{c}, \underline{b}_i)$ of F_i-measure strictly positive, the expected payoff $P(i \mid v_i)$ would be strictly negative for F_i-almost all v_i in this subset. In fact, for F_i-almost all these v_i bidder i bids strictly more than v_i and the probability of winning is strictly positive. However, bidder i can guarantee a payoff equal to zero by simply submitting a bid equal to his valuation.

From the previous paragraph, we know that, for all i in H and for F_i-almost all v_i in $[\underline{c}, \underline{b}_i)$, Prob (i wins $\mid v_i$) $= 0$. However, this equality and $\underline{b}_i = \underline{b} > \underline{c}$ contradict Lemma A1-1 and Lemma A1-2 is proved. ||

<u>Lemma</u> <u>A1-3</u>: *If $(\beta_1, \ldots, \beta_n)$ is a Bayesian equilibrium of the first price auction with or without mandatory bidding, then $b_{1l}(\underline{c}) = \ldots = b_{nl}(\underline{c}) = b_{1u}(\underline{c}) = \ldots = b_{nu}(\underline{c}) = \underline{c}$ and the probability measures $\beta_1(\underline{c}, .), \ldots, \beta_n(\underline{c}, .)$ are concentrated at $\underline{c}$ in the case of mandatory bidding and have their supports included in $\{OUT, \underline{c}\}$ in the case of voluntary bidding.*

<u>Proof</u>: From Lemma A1-2 we see that all bids strictly larger than $\underline{c}$ have a strictly positive probability of winning and thus if bidder i with valuation $\underline{c}$ submits such bids, he will incur negative payoffs. However, he can obtain a payoff equal to zero if he submits $\underline{c}$ or, in the case with voluntary bidding, if he stays out. ||

Lemma A1-4: If $(\beta_1, \ldots, \beta_n)$ is a Bayesian equilibrium of the first price auction with or without mandatory bidding, then Prob(1 wins | v), ..., Prob(n wins | v) are nondecreasing functions of v and we have

$$P(1 \mid v) = \int_{\underline{c}}^{v} Prob(\text{ 1 wins } \mid v)\, dv, \ldots, P(n \mid v) = \int_{\underline{c}}^{v} Prob(\text{ n wins } \mid v)\, dv.$$

Proof: The proof proceeds as in Myerson (1981). Consider v' > v. When his valuation is equal to v, bidder i cannot obtain more than P(i | v), thus

$$P(i \mid v) \geq \int p_i(v, b_1, \ldots, b_n) \left[\beta_i(v', .) \otimes \left\{ \bigotimes_{j \neq i} (\beta_j * F_j) \right\} \right] (db_i, (dv_j, db_j)_{j\neq i}).$$

However, we know that $p_i(v, b_1, \ldots, b_n) = p_i(v', b_1, \ldots, b_n) + (v - v')\, I\{i \text{ wins} \mid b_1, \ldots, b_n\}$, where $I\{i \text{ wins} \mid b_1, \ldots, b_n\}$ is equal to zero if $i \notin S(b_1, \ldots, b_n) = \{ l \mid b_l \neq \text{OUT}$ and $b_l = \max_{1 \leq k \leq n} b_k \}$ and is equal to $1/\#S(b_1, \ldots, b_n)$ if $i \in S(b_1, \ldots, b_n)$. Substituting its value to $p_i(v, b_1, \ldots, b_n)$ and making use of the definitions of P(i | v') and Prob(i wins | v'), we obtain

$$P(i \mid v) - P(i \mid v') \geq (v - v')\int I\{i \text{ wins} \mid b_1, \ldots, b_n\}[\beta_i(v', .) \otimes \{ \bigotimes_{j \neq i} (\beta_j * F_j)\}](db_i, (dv_j, db_j)_{j\neq i})$$

$$= (v - v')\text{Prob(i wins } \mid v'),$$

and thus,

$$P(i \mid v) - P(i \mid v') \geq (v - v') \text{ Prob(i wins } \mid v').$$

Permuting v and v' gives the inequality

$$(v - v') \text{ Prob(i wins } \mid v) \geq P(i \mid v) - P(i \mid v').$$

Regrouping the two last inequalities yields

$$(v - v') \text{ Prob(i wins } \mid v) \geq P(i \mid v) - P(i \mid v') \geq (v - v') \text{ Prob(i wins } \mid v'),$$

which implies that Prob(i wins | v) is nondecreasing in v and that P(i | v) − P(i | v') is equal to $\int_{v'}^{v}$ Prob(i wins | w) dw. Lemma A1-4 then follows by taking v' = $\underline{c}$ and by using the fact implied by Lemma A1-3 that P(i | $\underline{c}$) = 0. ||

Lemma A1-5: Assume that $(\beta_1, \ldots, \beta_n)$ is a Bayesian equilibrium of the first price auction with or without mandatory bidding. For all $1 \leq i \leq n$, if $b > \underline{c}$ is a mass point of $[\beta_i * F_i]_2$, that is, $\beta_i * F_i([\underline{c}, \overline{c}] \times \{b\}) > 0$, then there exists $j \neq i$ such that b is a point of increase to the left of $[\beta_j * F_j]_2$, that is, $\beta_j * F_j([\underline{c}, \overline{c}] \times (b - \epsilon, b]) > 0$, for all $\epsilon > 0$.

Proof: Assume that b $> \underline{c}$ is a mass point of $[\beta_i * F_i]_2$ and that b is not a point of increase to the left of $[\beta_j * F_j]_2$, for all j $\neq$ i. Thus, for all j $\neq$ i, there exists $\epsilon_j > 0$ such that $\beta_j * F_j$ $([\underline{c}, \overline{c}] \times (b - \epsilon_j, b]) = 0$. Consider $\epsilon > 0$ the minimum of these ϵ_j, that is, $\epsilon = \min_{j \neq i} \epsilon_j$. When bidder i submits b with a strictly positive probability, his valuation must be strictly larger than $\underline{c}$ (see Lemma A1-3). From Lemma A1-1, we know that his expected payoff and his probability of winning are strictly positive. Thus bidder i would strictly increase his payoff if, instead of bidding b, he submitted $\max(\frac{b+\underline{c}}{2}, b - \epsilon/2)$. This is impossible at an equilibrium and Lemma A1-5 is proved. ||

Lemma A1-6: If $(\beta_1, \ldots, \beta_n)$ is a Bayesian equilibrium of the first price auction with or without mandatory bidding, then, for all $1 \leq i \leq n$, if $b > \underline{c}$ is a point of increase to the left of $[\beta_i * F_i]_2$, that is, $\beta_i * F_i$ $([\underline{c}, \overline{c}] \times (b - \epsilon, b]) > 0$, for all $\epsilon > 0$, then

$$\beta_j * F_j([\underline{c}, \overline{c}] \times \{b\}) = 0,$$

for all j $\neq$ i.

Proof: Take b $> \underline{c}$ a point of increase to the left of $[\beta_i * F_i]_2$ and assume that there exists j $\neq$ i such that b is a mass point of $[\beta_j * F_j]_2$, that is, $\beta_j * F_j([\underline{c}, \overline{c}] \times \{b\}) > 0$. For all $\epsilon > 0$, denote by $B(\epsilon)$ the Borel subset of $[\underline{c}, \overline{c}]$ such that $\text{Prob}(b_i(v) \in (b - \epsilon, b]) > 0$, for all v in $B(\epsilon)$. If $\epsilon < \epsilon'$, then $B(\epsilon) \subseteq B(\epsilon')$, that is, $B(\epsilon)$ is nondecreasing in ϵ. By assumption, we have $F_i(B(\epsilon)) > 0$, for all $\epsilon > 0$. There exists η such that $b - \underline{c} > \eta > 0$ and $v > b + \eta$, for all $v \in B(\eta)$. Otherwise, for all m $\geq$ 1 such that $b - \underline{c} > \frac{1}{m}$ there would exist $v_m \in B(\frac{1}{m})$ such that $v_m \leq b + \frac{1}{m}$. Lemma 4 implies that $v \geq b - \frac{1}{m}$, for all v in $B(\frac{1}{m})$, and thus $v_m \geq b - \frac{1}{m}$. Since $P(i \mid v) = P(i \mid v \text{ and } b_i(v) \in (b - \frac{1}{m}, b])$, for all v in $B(\frac{1}{m})$, we see that $P(i \mid v_m) \leq \frac{2}{m}$. From $b - \frac{1}{m} \leq v_m \leq b + \frac{1}{m}$, we see that $\lim_{m \to +\infty} v_m$ exists and is equal to b. From Lemma A1-4 and $P(i \mid v_m) \leq \frac{2}{m}$, we see that $P(i \mid b) = 0$. This contradicts Lemma A1-1 and there exists such a η.

We now see that for ϵ small enough, bidder i with valuation in $B(\epsilon)$ would obtain a strictly higher payoff by bidding slightly above b. Take $\epsilon > 0$ and $\delta > 0$ such that[19] $\epsilon < \eta$ and $\eta \frac{[\beta_j * F_j]_2(\{b\})}{2[\beta_j * F_j]_2([\underline{c}, b])} - \epsilon \frac{[\beta_j * F_j]_2([\underline{c}, b)) + [\beta_j * F_j]_2(\{b\})/2}{[\beta_j * F_j]_2([\underline{c}, b])} > \delta$. For all v in $B(\epsilon) \subseteq B(\eta)$, we have

$$P(i \mid v \text{ and } b_i(v) \in (b - \epsilon, b]) \leq (v - b + \epsilon) \left\{[\beta_j * F_j]_2([\underline{c}, b)) + \frac{1}{2} [\beta_j * F_j]_2(\{b\})\right\} \prod_{k \neq i,j} [\beta_k * F_k]_2([\underline{c}, b])$$

$$< (v - b - \delta) \prod_{k \neq i} [\beta_k * F_k]_2([\underline{c}, b])$$

$$\leq P(i \mid v, b + \delta).$$

The first inequality above is obtained by using upper bounds of the probability of winning and of the payoff in case of winning. The upper bound of the probability of winning was derived by assuming that all ties involving only bidder i and bidders k $\neq$ j are solved in favor of bidder i and that all ties involving bidder i, bidder j and bidders k $\neq$ i, j, are solved between bidders i and j. Remark that v $\geq$ b $-$ ϵ, for all v in B (ϵ). The second inequality is obtained from the definition of η, ϵ and δ and from the fact that v $-$ b $\geq$ η, for all v in B (ϵ). The third inequality is immediate. From this chain of inequalities, we find that P(i | v and b_i(v) $\in$ (b $-$ ϵ, b]) $<$ P(i | v, b $+$ δ), for all v in B (ϵ), which is impossible at an equilibrium and Lemma A1-6 is proved. ||

Lemma A1-7: If $(\beta_1, \ldots, \beta_n)$ is a Bayesian equilibrium of the first price auction with or without mandatory bidding and if b $\in$ $(\underline{c}, +\infty)$, then

$$[\beta_1{*}F_1]_2\ (\{b\})\ =\ \ldots\ =\ [\beta_n{*}F_n]_2\ (\{b\})\ =\ 0.$$

Proof: Assume that there exists b $>$ $\underline{c}$ and 1 $\leq$ i $\leq$ n such that b is a mass point of $[\beta_i{*}F_i]_2$. Then Lemma A1-5 implies that there exists j $\neq$ i such that b is a point of increase to the left of $[\beta_j{*}F_j]_2$. However, from Lemma A1-6 we know that, since b is a point of increase to the left of $[\beta_j{*}F_j]_2$, b is not a mass point of $[\beta_k{*}F_k]_2$, for all k $\neq$ j, and in particular of $[\beta_i{*}F_i]_2$. We thus have a contradiction and Lemma A1-7 is proved for b $>$ $\underline{c}$. ||

Lemma A1-8: If $(\beta_1, \ldots, \beta_n)$ is a Bayesian equilibrium of the first price auction with or without mandatory bidding, then

$$b_{iu}(v)\ \leq\ b_{il}(v'),$$

for all 1 $\leq$ i $\leq$ n and for all $\underline{c}$ $\leq v < v' \leq \overline{c}$.

Proof: Suppose that there exist $1 \leq i \leq n$ and $\underline{c}$ $\leq v < v' \leq \overline{c}$ such that $b_{iu}(v) > b_{il}(v')$. From the definitions of $b_{1u}(v)$ and $b_{1l}(v')$, there exist two sequences $d_m \underset{\leq}{\to} b_{1u}(v)$ and $d'_m \underset{\geq}{\to} b_{1l}(v')$, such that P(i | v) $=$ P(i | v, d_m) and P(i | v') $=$ P(i | v', d'_m), for all m $\geq$ 1. Without loss of generality, we can assume that $d_m > d'_m$ and thus that $d_m > \underline{c}$, for all m $\geq$ 1. From Lemma A1-2, Prob(i wins | d_m) $>$ 0, for all m $\geq$ 1.

From the definition of an equilibrium, bidder i with his valuation equal to v cannot obtain an expected payoff larger than P(i | v), we see that (v $-$ d_m) Prob(i wins | d_m) $\geq$ P(i | v, d'_m) $=$ (v $-$ d'_m) Prob(i wins | d'_m). Similarly, if his valuation is equal to v' he cannot obtain an expected payoff larger than P(i | v'), and thus (v' $-$ d'_m) Prob(i wins | d'_m) $\geq$ P(i | v', d_m) $=$ (v' $-$ d_m) Prob(i wins | d_m). Combining these two inequalities together, we find (v' $-$ v) [Prob(i wins | d'_m) $-$ Prob(i wins | d_m)] $\geq$ 0. Since v' $>$ v, we obtain Prob(i wins | d'_m) $\geq$ Prob(i wins | d_m). However, from our initial assumption we have $d_m > d'_m$ and thus Prob(i wins | d_m) $\geq$ Prob(i wins | d'_m), for all m $\geq$ 1. Consequently, Prob(i wins | d_m) $=$ Prob(i wins | d'_m), for all m $\geq$ 1. But this implies that P(i | v) $=$ (v $-$ d_m) Prob(i wins | d_m) $<$ P(i | v, d'_m) $=$ (v $-$ d'_m) Prob(i wins | d'_m), which is impossible at an equilibrium and Lemma A1-8 is proved. ||

Lemma A1-9: If $(\beta_1, \ldots, \beta_n)$ is a Bayesian equilibrium of the first price auction with mandatory bidding, then $b_{il}(v) > \underline{c}$, for all v in $(\underline{c}, \overline{c}]$ and all $1 \le i \le n$. If $(\beta_1, \ldots, \beta_n)$ is a Bayesian equilibrium of the first price auction with voluntary bidding, then $b_{il}(v) > \underline{c}$, for all v in $(\underline{c}, \overline{c}]$ and all i between 1 and n except possibly one. Moreover, in this latter case if there exists i and $v > \underline{c}$ such that $b_{il}(v) = \underline{c}$ then $F_k(\underline{c}) > 0$ and $[\beta_k * F_k]_2(\{\underline{c}\}) = 0$, for all $k \neq i$.

Proof: Let $(\beta_1, \ldots, \beta_n)$ be a Bayesian equilibrium of the first price auction with mandatory bidding. Consider $v > \underline{c}$. If $b_{il}(v) = \underline{c}$, there exists $d'_m \underset{\geq}{\to} \underline{c}$ such that $P(i \mid v) = (v - d'_m)$ Prob(i wins $\mid d'_m$), for all $m \geq 1$. If Prob(i wins $\mid \underline{c}$) $= 0$, the expression $(v - b)$ Prob(i wins $\mid$ b) would be continuous at $b = \underline{c}$ and thus $P(i \mid v)$ would be equal to zero which contradicts Lemma A1-4. Thus Prob(i wins $\mid \underline{c}$) > 0. Since $b_{il}(v) = \underline{c}$, we obtain from Lemmas A1-7 and A1-2, $b_{il}(w) = b_{iu}(w) = \underline{c}$, for all w in $[\underline{c}, v)$. Take w in $(\underline{c}, v)$. Since Prob(i wins $\mid \underline{c}$) > 0, we see that in the first price auction with mandatory bidding there is a strictly positive probability of a tie at $\underline{c}$ and bidder i with valuation w would be better off bidding slightly more than $\underline{c}$. This is impossible at an equilibrium and we have proved that $b_{il}(v) > \underline{c}$. The same reasoning shows that there cannot be a tie at $\underline{c}$ in the case of voluntary bidding. Consequently, bidder i must be the only one to bid $\underline{c}$ with a strictly positive probability and thus $[\beta_k * F_k]_2(\{\underline{c}\}) = 0$ and $b_{kl}(u) > \underline{c}$, for all $k \neq i$, and all $u > \underline{c}$. Moreover, Prob(i wins $\mid \underline{c}$) > 0 implies that $F_k(\underline{c}) > 0$, for all $k \neq i$. ||

Lemma A1-10: If $(\beta_1, \ldots, \beta_n)$ is a Bayesian equilibrium of the first price auction with or without mandatory bidding, then Prob(i wins $\mid$ b) is a continuous function of b in $(\underline{c}, +\infty)$ and is equal to $\prod_{j \neq i} [\beta_k * F_k]_2 (\{OUT\} \cup [\underline{c}, b])$. These properties of Prob(i wins $\mid$ b) hold true also at $b = \underline{c}$ when bidding is voluntary and there exists $u > \underline{c}$ such that $b_{il}(u) = \underline{c}$. Moreover, in both auctions $P(i \mid v) = (v - b_{il}(v))$ Prob(i wins $\mid b_{il}(v)) = (v - b_{iu}(v))$ Prob(i wins $\mid b_{iu}(v))$ and we can substitute min and max to inf and sup in the definitions (8), (9) (Section 3) of $b_{il}(v)$ and $b_{iu}(v)$, respectively, for all $1 \le i \le n$ and all v in $[\underline{c}, \overline{c}]$.

Proof: From Lemma A1-7, no bidder bids $b > \underline{c}$ with a strictly positive probability and thus Prob(i wins $\mid$ b) $= \prod_{j \neq i} [\beta_k * F_k]_2 (\{OUT\} \cup [\underline{c}, b])$, for all b in $(\underline{c}, +\infty)$. From Lemma A1-9, the same is true when bidding is voluntary and there exists $u > \underline{c}$ such that $b_{il}(u) = \underline{c}$. Every factor $[\beta_k * F_k]_2 (\{OUT\} \cup [\underline{c}, b])$ and thus also Prob(i wins $\mid$ b) are continuous at b and the first part of the lemma is proved.

From Lemma A1-7 and A1-9 we see that $(v - b)$ Prob(i wins $\mid$ b) is continuous over $(\underline{c}, +\infty)$ in both auctions and over $[\underline{c}, +\infty)$ in the case of voluntary bidding when there exists $u > \underline{c}$ such that $b_{il}(u) = \underline{c}$. From Lemma A1-9 we also see that, for $v > \underline{c}$, $b_{il}(v)$ and thus $b_{iu}(v)$ belong to the interval of continuity of $(v - b)$ Prob(i wins $\mid$ b) and the second part of Lemma A1-10 follows.

The second part of Lemma A1-10 for $v = \underline{c}$ follows immediately from Lemma A1-3. ||

Lemma *A1-11*: *If* $(\beta_1, \ldots, \beta_n)$ *is a Bayesian equilibrium of the first price auction with or without mandatory bidding, then the functions* b_{il} *and* b_{iu} *are nondecreasing over* $[\underline{c}, \overline{c}]$, *for all* $1 \leq i \leq n$.

Proof: Let v, v' such that $\underline{c} \leq v < v' \leq \overline{c}$. From Lemma A1-8, we have $b_{iu}(v) \leq b_{il}(v')$. Since $b_{il}(v) \leq b_{iu}(v)$ and $b_{il}(v') \leq b_{iu}(v')$, we obtain $b_{il}(v) \leq b_{il}(v')$ and $b_{iu}(v) \leq b_{iu}(v')$ and Lemma A1-11 is proved. ||

Lemma *A1-12*: *Let* $(\beta_1, \ldots, \beta_n)$ *be a Bayesian equilibrium with or without mandatory bidding. Then there exists* $\underline{c} < \eta < \overline{c}$ *such that* $b_{1u}(\overline{c}) = \ldots = b_{nu}(\overline{c}) = b_{1l}(\overline{c}) = \ldots = b_{nl}(\overline{c}) = \eta$.

Proof: Denote $\max_{1 \leq i \leq n} b_{iu}(\overline{c})$ by η and $\min_{1 \leq i \leq n} b_{il}(\overline{c})$ by m. Assume that $\eta >$ m. We first prove that for all i such that $b_{iu}(\overline{c}) = \eta$, we have $[\beta_i * F_i]_2\,([m, \eta]) = 0$. Let i and j be such that $b_{iu}(\overline{c}) = \eta$ and $b_{jl}(\overline{c}) =$ m. If i = j, the result $[\beta_i * F_i]_2\,([m, \eta]) = 0$ follows immediately from Lemmas A1-7 and A1-8. We can thus assume that $i \neq j$. From Lemma A1-10, we have $P(i \mid \overline{c}) = (\overline{c} - b_{iu}(\overline{c}))$ Prob (i wins $\mid b_{iu}(\overline{c})$) and $P(j \mid \overline{c}) = (\overline{c} - b_{jl}(\overline{c}))$ Prob (i wins $\mid b_{jl}(\overline{c})$). Because $P(i \mid \overline{c})$ is the largest payoff bidder i can obtained and $b_{iu}(\overline{c}) = \eta$, we have

(A1.1) $\quad P(i \mid \overline{c}) = (\overline{c} - \eta)$ Prob (i wins $\mid \eta) \geq (\overline{c} - b_{jl}(\overline{c}))$ Prob (i wins $\mid b_{jl}(\overline{c})$).

From Lemmas A1-7 and A1-8, we see that Prob (i wins $\mid \eta) = 1$. Moreover, from Lemma A1-10 we have Prob (i wins $\mid b_{jl}(\overline{c})) = [\beta_j * F_j]_2\,([\underline{c}, b_{jl}(\overline{c})]) \prod_{k \neq i,j} [\beta_k * F_k]_2\,([\underline{c}, b_{jl}(\overline{c})]) = \prod_{k \neq i,j} [\beta_k * F_k]_2\,([\underline{c}, b_{jl}(\overline{c})])$. We also have Prob (j wins $\mid b_{jl}(\overline{c})) = [\beta_i * F_i]_2\,([\underline{c}, b_{jl}(\overline{c})]) \prod_{k \neq i,j} [\beta_k * F_k]_2\,([\underline{c}, b_{jl}(\overline{c})])$. From Lemma A1-1, we have $(\overline{c} - b_{jl}(\overline{c})) > 0$, $(\overline{c} - \eta) = (\overline{c} - b_{iu}(\overline{c})) > 0$ and $[\beta_i * F_i]_2\,([\underline{c}, b_{jl}(\overline{c})]) \prod_{k \neq i,j} [\beta_k * F_k]_2\,([\underline{c}, b_{jl}(\overline{c})]) > 0$. If $[\beta_i * F_i]_2\,([\underline{c}, b_{jl}(\overline{c})]) < 1$, the inequality (A1.1) would then imply (using Prob(j wins $\mid \eta) = 1$)

$$(\overline{c} - \eta) \text{ Prob (j wins} \mid \eta) > (\overline{c} - b_{jl}(\overline{c})) \text{ Prob (j wins} \mid b_{jl}(\overline{c})) = P(j \mid \overline{c}, b_{jl}(\overline{c})),$$

which is impossible and thus $[\beta_i * F_i]_2\,([\underline{c}, b_{jl}(\overline{c})]) = 1$, that is, $[\beta_i * F_i]_2\,([b_{jl}(\overline{c}), \eta]) = [\beta_i * F_i]_2\,([m, \eta]) = 0$.

In the previous paragraph, we showed that $[\beta_i * F_i]_2\,([m, \eta]) = 0$, for all $1 \leq i \leq n$ such that $b_{iu}(\overline{c}) = \eta$. If $b_{iu}(\overline{c}) = \eta$, for all $1 \leq i \leq n$, the expected payoff of any bidder i is strictly larger if he submits m than if he submits $\eta = b_{iu}(\overline{c})$, since his probability of winning does not change (and is strictly positive) and his payment in case of winning decreases. If there exists j such that $b_{ju}(\overline{c}) \neq \eta$, any bidder i such that $b_{iu}(\overline{c}) = \eta$ sees his payoff increase if he submits max$\{\ b_{ju}(\overline{c}\ )\ \mid\ 1 \leq j \leq n$ and $b_{ju}(\overline{c}\ ) < \eta\ \}$ instead of $b_{iu}(\overline{c}\ ) = \eta$. This contradicts the definition of an equilibrium and Lemma A1-12 is proved. ||

Lemma _A1-13_: *Let $(\beta_1, \ldots, \beta_n)$ be a Bayesian equilibrium of the first price auction with or without mandatory bidding. Then we have,*

$$b_{il}(v) = \lim_{w \underset{<}{\to} v} b_{iu}(w), \text{ for all } v \text{ in } (\underline{c}\ , \overline{c}\ ],$$

and

$$b_{iu}(v) = \lim_{w \underset{>}{\to} v} b_{il}(w), \text{ for all } v \text{ in } [\underline{c}\ , \overline{c}\ ),$$

for all $1 \leq i \leq n$. Consequently, b_{iu} is continuous from the right and b_{il} is continuous from the left. Moreover, if $v \in [\underline{c}\ , \overline{c}\ ]$, $1 \leq i \leq n$ and if $b_{il}(v)$ is continuous at v, then $b_{il}(v) = b_{iu}(v)$. Similarly, if $b_{iu}(v)$ is continuous at v, then $b_{il}(v) = b_{iu}(v)$.

Proof: Let us prove that $b_{il}(v) = \lim_{w \underset{<}{\to} v} b_{iu}(w)$, for all v in $(\underline{c}\ , \overline{c}\ ]$. Since, from Lemma A1-11 the function b_{iu} is nondecreasing, we see that $\lim_{w \underset{<}{\to} v} b_{iu}(w) = \sup_{w \in [\underline{c}\ , v)} b_{iu}(w)$. From Lemma A2-8, we know that $b_{iu}(w) \leq b_{il}(v)$, for all w in $[\underline{c}\ , v)$, and thus $\lim_{w \underset{<}{\to} v} b_{iu}(w) \leq b_{il}(v)$. Take a sequence $w_1, w_2, \ldots$ in $[\underline{c}\ , v)$ which converges towards v and such that $\lim_{m \to +\infty} b_{iu}(w_m) = \sup_{w \in [\underline{c}\ , v)} b_{iu}(w)$. From Lemma A1-10, we have $P(\ i \mid w_m, b_{iu}(w_m)\ ) = P(\ i \mid w_m\ )$, that is, $P(\ i \mid w_m\ ) = (w_m - b_{iu}(w_m))$ Prob(i wins $\mid b_{iu}(w_m))$, for all $m \geq 1$. Taking the limit of the previous equality for $m \to +\infty$ and using the continuity with respect to w of $P(\ i \mid w\ )$ (see Lemma A1-10), we find $P(\ i \mid v\ ) = (v - \sup_{w \in [\underline{c}\ , v)} b_{iu}(w))$ Prob(i wins $\mid \sup_{w \in [\underline{c}\ , v)} b_{iu}(w))$. From the definition of $b_{il}(v)$ (see (8), Section 3), it follows immediately that $\sup_{w \in [\underline{c}\ , v)} b_{iu}(w) \geq b_{il}(v)$. Since we have proved the two inequalities, we obtain the equality $\sup_{w \in [\underline{c}\ , v)} b_{iu}(w) = b_{il}(v)$. The equality $b_{iu}(v) = \lim_{w \underset{>}{\to} v} b_{il}(w)$, for all v in $[\underline{c}\ , \overline{c}\ )$, can be proved similarly.

The function b_{il} is continuous from the left since $\lim_{w \underset{<}{\to} v} b_{il}(w) = \sup_{w \in [\underline{c}\ , v)} b_{il}(w) = \sup_{w \in [\underline{c}\ , v)} \sup_{x \in [\underline{c}\ , w)} b_{iu}(x) = \sup_{x \in [\underline{c}\ , v)} b_{iu}(x) = b_{il}(v)$, for all v in $[\underline{c}\ , \overline{c}\ )$. Similarly, b_{iu} is continuous from the right.

The second part of Lemma A1-13 follows immediately from the first part and the facts $b_{il}(\underline{c}\ ) = b_{iu}(\underline{c}\ )$ (from Lemma A1-3) and $b_{il}(\overline{c}\ ) = b_{iu}(\overline{c}\ )$ (from Lemma A1-12). ||

Lemma A1-14: If $(\beta_1, \ldots, \beta_n)$ is a Bayesian equilibrium of the first price auction with mandatory bidding, then the functions b_{il} and b_{iu} are strictly increasing over $[\underline{c}, \overline{c}]$, for all $1 \leq i \leq n$. If $(\beta_1, \ldots, \beta_n)$ is a Bayesian equilibrium of the first price auction with voluntary bidding, then either the functions b_{il} and b_{iu} are strictly increasing over $[\underline{c}, \overline{c}]$, for all $1 \leq i \leq n$, or there exists j such that the functions b_{il} and b_{iu} are strictly increasing over $[\underline{c}, \overline{c}]$, for all $i \neq j$, and there exists w' in $(\underline{c}, \overline{c})$ such that b_{jl} and b_{ju} are equal to $\underline{c}$ over $[\underline{c}, w')$ and are strictly increasing over $(w', \overline{c}]$.

Proof: From Lemma A1-11, the functions b_{il} and b_{iu} are nondecreasing over $[\underline{c}, \overline{c}]$. Suppose that b_{il} is constant over an interval $(u, w) \subseteq [\underline{c}, \overline{c}]$, with $u < w$. Then, from Lemma A1-13 $b_{il}(v) = b_{iu}(v)$, for all v in (u, w), and $b_{il}(v')$, with $v' \in (u, w)$, is a mass point of the distribution of b_i. If this mass point is strictly larger than $\underline{c}$, it contradicts Lemma A1-7 and Lemma A1-14 is proved. We can thus assume that $b_{il}(v) = b_{iu}(v) = \underline{c}$, for all v in (u, w). From Lemma A1-8 and the obvious inequality $b_{il}(x) \geq \underline{c}$, for all x in $(\underline{c}, \overline{c}]$, we have $b_{il}(v) = b_{iu}(v) = \underline{c}$, for all v in $[\underline{c}, w)$.

In the case with voluntary bidding, Lemma A1-9 implies that there can be only one such i. The proof is complete in this case when we define w' as follows, $w' = \sup \{ w \in [\underline{c}, \overline{c}] \mid b_{il}(v) = b_{iu}(v) = \underline{c}$, for all v in $[\underline{c}, w) \}$.

In the case of mandatory bidding, it contradicts Lemma A1-9 and Lemma A1-13 is proved. ||

For all $1 \leq i \leq n$, we define the function α_i as follows,

$$\text{(A1.2)} \qquad \alpha_i(b) = \sup \{ v \in [\underline{c}, \overline{c}] \mid b_{il}(v) \leq b\},$$

for all b in $[\underline{c}, +\infty)$. Remark that since $b_{il}(\underline{c}) = \underline{c}$, the set $\{v \mid b_{il}(v) \leq b\}$ is not empty. We have gathered in Lemma A1-15 below some useful properties of the functions $\alpha_1, \ldots, \alpha_n$.

Lemma A1-15: If $(\beta_1, \ldots, \beta_n)$ is a Bayesian equilibrium of the first price auction with or without mandatory bidding and the functions $\alpha_1, \ldots, \alpha_n$ are defined by (A1.2), then we can substitute max to sup in the definition of $\alpha_i(b)$, that is, $\alpha_i(b) = \max \{ v \mid b_{il}(v) \leq b\}$, for all $1 \leq i \leq n$ and all b in $[\underline{c}, +\infty)$. The functions $\alpha_1, \ldots, \alpha_n$ are continuous and nondecreasing. Moreover, we have $\alpha_i(b_{il}(\alpha_i(b))) = \alpha_i(b)$, for all $1 \leq i \leq n$ and all b in $[\underline{c}, +\infty)$. If $v < \alpha_i(b)$, then $b_{iu}(v) \leq b$. If $\alpha_i(b) < v$, then $b < b_{il}(v)$. When b_{il} is continuous at $\alpha_i(b)$, we have $b = b_{il}(\alpha_i(b))$, for all $1 \leq i \leq n$ and all b in $[\underline{c}, +\infty)$. If b_{il} is strictly increasing over $[v, v+\delta]$, where $\delta > 0$, then $\alpha_i(b_{il}(v)) = \alpha_i(b_{iu}(v)) = v$. If b_{il} is continuous and strictly increasing over an interval (v, w), then α_i is strictly increasing and is equal to b_{il}^{-1} over the interval $(b_{iu}(v), b_{il}(w))$, for all $1 \leq i \leq n$. If the function b_{il} is discontinuous at v, then $\alpha_i(b) = v$, for all b in $[b_{il}(v), b_{iu}(v))$.

Proof: Immediate from the definition of α_i and Lemma A1-13. ||

Lemma A1-16: Let $(\beta_1, \ldots, \beta_n)$ be a Bayesian equilibrium of the first price auction with or without mandatory bidding. Then $b_{il}(\alpha_i(b)) \leq b \leq b_{iu}(\alpha_i(b))$, for all $1 \leq i \leq n$ and all

$\underline{c} \leq b \leq \eta$, with η defined in Lemma A1-12. Moreover, $[\beta_i * F_i]_2$ $(\{OUT\} \cup [\underline{c}\ ,\ b])$ $=$ $F_i(\alpha_i(b))$ and Prob(i wins | b) $= \prod_{\substack{j=1\\ j\neq i}}^{n} F_j(\alpha_j(b))$, for all $b > \underline{c}$ and all $1 \leq i \leq n$ and also in the voluntary bidding case for $b = \underline{c}$

and $i = j$ as in Lemma A1-14.

Proof: As an immediate consequence of the definition of α_i and from the remark before the statement of the lemma, we see that $b_{il}(\alpha_i(b)) \leq b$, for all $b \geq \underline{c}$ and for all $1 \leq i \leq n$. Assume that $b \leq \eta$ and $b_{iu}(\alpha_i(b)) < b$. Since $b_{iu}(\overline{c}) = \eta$ and b_{iu} is non decreasing, we see that $\alpha_i(b) < \overline{c}$. From Lemma A1-12, we have $b_{iu}(\alpha_i(b)) = \lim_{w \underset{>}{\rightarrow} \alpha_i(b)} b_{il}(w)$. Since $b_{iu}(\alpha_i(b)) < b$, there exists $w > \alpha_i(b)$ such that $b_{il}(w) < b$, which contradicts the definition of $\alpha_i(b)$ and we have proved the first part of Lemma A1-15.

The probability $[\beta_i * F_i]_2$ $(\{OUT\} \cup [\underline{c}\ ,\ b])$ can be written equivalently as Prob$(\{b_i = OUT\} \cup \{b_i \leq b\})$, for $b > \underline{c}$. From Lemmas A1-3 ad A1-13, we see that if $b > \underline{c}$, then $\alpha_i(b) > \underline{c}$. For all $v < \alpha_i(b)$, we have $b_i(v) = OUT$ or $b_i(v) \leq b$ with probability one. In fact, $b_i(v) = OUT$ or $b_i(v) \leq b_{iu}(v)$, with probability one, and $b_{iu}(v) \leq b_{il}(\alpha_i(b)) \leq b$. Moreover, for all $v > \alpha_i(b)$ we have $b_i(v) \geq b$ with probability one. In fact, $b_i(v) \neq OUT$ and $b_i(v) \geq b_{il}(v)$, with probability one, and $b_{il}(v) \geq b_{iu}(\alpha_i(b)) \geq b$. We do not have to worry about the case $b_i = b$, since from Lemma A1-7 the probability of this event is equal to zero. Moreover, $F_i(\{\alpha_i(b)\}) = 0$. Consequently, we obtain $[\beta_i * F_i]_2$ $(\{OUT\} \cup [\underline{c}\ ,\ b])$ $=$ $F_i(\alpha_i(b))$. The rest of Lemma A1-16 when $b > \underline{c}$ is an immediate consequence of the previous result and of Lemma A1-10.

When bidding is voluntary, $b = \underline{c}$ and $i = j$ as in Lemma A1-14, $[\beta_k * F_k]_2$ $(\{OUT\} \cup [\underline{c}\ ,\ b])$ $=$ $[\beta_k * F_k]_2$ $(\{OUT\})$ $=$ $F_k(\alpha_k(b))$ $=$ $F_k(\underline{c})$, for all $k \neq i$, and thus Prob(i wins | b) $= \prod_{\substack{k=1\\ k\neq i}}^{n} F_k(\alpha_k(b))$. ||

<u>Lemma</u> <u>A1-17</u>: Let $(\beta_1, \ldots, \beta_n)$ be a Bayesian equilibrium of the first price auction with or without mandatory bidding. If $b \geq \underline{c}$ is a point of increase (to the left or to the right) of α_i, that is, if α_i is not constant on any neighborhood of b, then $b = b_{il}(\alpha_i(b))$ or $b = b_{iu}(\alpha_i(b))$.

Proof: Let i and b be such that α_i is not constant on any neighborhood of b. If η is defined in Lemma A1-12, α_i is equal to $\overline{c}$ over $[\eta, +\infty)$ and thus $b \leq \eta$. Assume that $b \neq b_{il}(\alpha_i(b))$ and $b \neq b_{iu}(\alpha_i(b))$. Lemma A1-16 then implies $b_{il}(\alpha_i(b)) < b < b_{iu}(\alpha_i(b))$. The interval $(b_{il}(\alpha_i(b)), b_{iu}(\alpha_i(b)))$ is a neighborhood of b over which α_i is equal to the constant $\alpha_i(b)$. We thus obtain a contradiction and Lemma A1-17 is proved. ||

<u>Lemma</u> <u>A1-18</u>: Let $(\beta_1, \ldots, \beta_n)$ be a Bayesian equilibrium of the first price auction with or without mandatory bidding and let v be an element of $(\underline{c}\ ,\ \overline{c})$. If i is such that b_{il} is

continuous over a neighborhood of v and $b_{il}(v) > \underline{c}$ *, then Prob(i wins* $|$ *b) is differentiable at b* $= b_{il}(v)$ *and*

$$\frac{d}{db}\, Prob(i\ wins \mid b) \;=\; \frac{Prob(i\ wins \mid b)}{v-b},$$

if b $= b_{il}(v)$*. When v* $= \overline{c}$ *, this derivative is a left-hand derivative. Moreover, when bidding is voluntary, when v = w′ as in Lemma A1-14 for i = j, that is, when* b_{il} *is strictly increasing over* $(w', \overline{c}\,]$ *and equal to* $\underline{c}$ *over* $[\underline{c}\,, w']$*, and when* b_{il} *is continuous over a neighborhood of v, then the equality above also holds if the derivative is a right-hand derivative.*

<u>Proof</u>: The proof is similar to the proof of Lemma 3.6 in Griesmer, Levitan and Shubik (1967). Let i and v be as in the statement of the lemma. There exists $\epsilon > v - \underline{c} > 0$ such that b_{il} is continuous over $(v - \epsilon, v]$. From Lemma A1-13, $b_{il} = b_{iu}$ over this interval. From Lemmas A1-9 and A1-1, we thus see that β_i is pure and the bid function is equal to b_{il}. We also denote this bid function by β_i. From Lemma A1-14, β_i is strictly increasing over $(v - \epsilon, v]$. The function α_i defined in (A1.2) is equal to the inverse of β_i over the interval $(\beta_i(v - \epsilon), \beta_i(v)]$ (see Lemma A1-15).

Let $d_1, d_2, \ldots$ be a sequence in $(\beta_i(v - \epsilon), \beta_i(v))$ such that $d_k < \beta_i(v)$, for all $k \geq 1$, and $d_k \rightarrow \beta_i(v)$, as $k \rightarrow +\infty$. From the definition of an equilibrium, we have

$$(v - \beta_i(v))\ \mathrm{Prob}(\text{ i wins } \mid \beta_i(v)) \;\geq\; (v - d_k)\ \mathrm{Prob}(\text{ i wins } \mid d_k),$$

for all $k \geq 1$. Substituting $\{\, (v - \beta_i(v)) + (\beta_i(v) - d_k) \,\}$ to $(v - d_k)$ in the above expression, regrouping and using the fact that $d_k < \beta_i(v)$, we obtain

$$\frac{\mathrm{Prob}(\text{ i wins} \mid \beta_i(v)) - \mathrm{Prob}(\text{ i wins} \mid d_k)}{\beta_i(v) - d_k} \;\geq\; \frac{\mathrm{Prob}(\text{ i wins} \mid d_k)}{v - \beta_i(v)},$$

for all $k \geq 1$. From Lemma A1-10, $\mathrm{Prob}(\text{ i wins } \mid d_k)$ is a continuous function of d_k. Consequently, the above inequality implies

$$\text{(A1.3)}\qquad \liminf_{k \rightarrow +\infty} \frac{\mathrm{Prob}(\text{ i wins} \mid \beta_i(v)) - \mathrm{Prob}(\text{ i wins} \mid d_k)}{\beta_i(v) - d_k} \;\geq\; \frac{\mathrm{Prob}(\text{ i wins} \mid \beta_i(v))}{v - \beta_i(v)}.$$

For all $k \geq 1$, $\alpha_i(d_k)$ is in $(v - \epsilon, v)$. Again from the definition of a Bayesian equilibrium, we have

$$(\alpha_i(d_k) - d_k)\ \mathrm{Prob}(\text{ i wins } \mid d_k) \;\geq\; (\alpha_i(d_k) - \beta_i(v))\ \mathrm{Prob}(\text{ i wins } \mid \beta_i(v)),$$

for all $k \geq 1$. Substituting $\{\, (\alpha_i(d_k) - \beta_i(v)) + (\beta_i(v) - d_k) \,\}$ to $(\alpha_i(d_k) - d_k)$ in the above expression, regrouping and using the fact that $d_k < \beta_i(v)$, we obtain

$$\frac{\mathrm{Prob}(\text{ i wins} \mid \beta_i(v)) - \mathrm{Prob}(\text{ i wins} \mid d_k)}{\beta_i(v) - d_k} \;\leq\; \frac{\mathrm{Prob}(\text{ i wins} \mid d_k)}{\alpha_i(d_k) - \beta_i(v)},$$

for all k ≥ 1. Using the continuity of $\alpha_i(\mathrm{d}_k)$ and Prob(i wins $\mid$ d_k) with respect to d_k, we find

$$\text{(A1.4)} \qquad \limsup_{k \to +\infty} \frac{\text{Prob(i wins} \mid \beta_i(\mathrm{v})) - \text{Prob(i wins} \mid \mathrm{d}_k)}{\beta_i(\mathrm{v}) - \mathrm{d}_k} \leq \frac{\text{Prob(i wins} \mid \beta_i(\mathrm{v}))}{\mathrm{v} - \beta_i(\mathrm{v})}.$$

The inequalities (A1.3) and (A1.4) imply the equality $\lim_{k \to +\infty} \frac{\text{Prob(i wins} \mid \beta_i(\mathrm{v})) - \text{Prob(i wins} \mid \mathrm{d}_k)}{\beta_i(\mathrm{v}) - \mathrm{d}_k}$ $= \frac{\text{Prob(i wins} \mid \beta_i(\mathrm{v}))}{\mathrm{v} - \beta_i(\mathrm{v})}$. Since d_1, d_2, $\ldots$ is an arbitrary sequence converging to $\beta_i(\mathrm{v})$ from below, we find that the left-hand derivative of Prob(i wins $\mid$ b) exists at b $= \beta_i(\mathrm{v})$ and is equal to $\frac{\text{Prob(i wins} \mid \beta_i(\mathrm{v}))}{\mathrm{v} - \beta_i(\mathrm{v})}$. Proceeding similarly, it is possible to prove that the right-hand derivative exists as well, if v $< \overline{\mathrm{c}}$, and is equal to the same value. Lemma A1-18 is thus proved when β_i is strictly increasing over $(\mathrm{v} - \epsilon, \mathrm{v}]$.

The two other results of Lemma A1-18 are proved by taking the limit for v $\to \overline{\mathrm{c}}$ and v $\to$ w' (we use properties similar to the one described in footnote 11). ||

<u>Lemma</u> <u>A1-19</u>: *Let $(\beta_1, \ldots, \beta_n)$ be a Bayesian equilibrium of the first price auction with or without mandatory bidding. Assume that there exist $0 \leq j < n$, $b \in (\underline{c}, \overline{c})$ and $\epsilon > 0$ such that $b - \epsilon > \underline{c}$ and α_i is constant over $(b - \epsilon, b + \epsilon)$, for all $1 \leq i \leq j$, and α_k is strictly increasing over $(b - \epsilon, b + \epsilon)$, for all $j < k \leq n$. Then the functions $\alpha_{j+1}, \ldots, \alpha_n$ are differentiable over $(b - \epsilon, b + \epsilon)$, differentiable on the right at $b - \epsilon$, differentiable on the left at $b + \epsilon$ and form a solution over the interval $[b - \epsilon, b + \epsilon]$ of the following differential system considered on the domain $D_j = \{ (b, \alpha_{j+1}, \ldots, \alpha_n) \in \mathbb{R}^{(n-j+1)} \mid \underline{c}, b < \alpha_k$, for all $j < k \leq n \}$,*

$$\text{(A1. 5)} \qquad \frac{d}{db} lnF_k(\alpha_k(b)) = \frac{1}{(n-j-1)} \left\{ \frac{(-1)(n-j-2)}{\alpha_k(b)-b} + \sum_{\substack{l=j+1 \\ l \neq k}}^{n} \frac{1}{\alpha_l(b)-b} \right\}, j < k \leq n,$$

where the derivative at $b - \epsilon$ is a right-hand derivative and the derivative at $b + \epsilon$ is a left-hand derivative or, in using matrix notation,

$$\text{(A1.6)} \qquad \frac{d}{db} \mathbb{LNF}(\alpha(b)) = \mathbb{M} \,.\, \mathbb{I}(\alpha(b), b),$$

where $\mathbb{LNF}(\alpha(b))$ and $\mathbb{I}(\alpha(b), b)$ are $(n-j) \times 1$ matrices and $\mathbb{M}$ is a $(n-j) \times (n-j)$ matrix defined as follows,

$$\mathbb{LNF}(\alpha(b)) = \begin{bmatrix} lnF_{j+1}(\alpha_{j+1}(b)) \\ . \\ . \\ . \\ lnF_n(\alpha_n(b)) \end{bmatrix} \qquad \mathbb{I}(\alpha(b), b) = \begin{bmatrix} \frac{1}{\alpha_{j+1}(b)-b} \\ . \\ . \\ . \\ \frac{1}{\alpha_n(b)-b} \end{bmatrix}$$

$$\mathbb{M} = \frac{1}{(n-j-1)}\begin{bmatrix} (-1)(n-j-2) & 1 & . & . & . & 1 \\ 1 & (-1)(n-j-2) & 1 & . & . & . \\ . & 1 & (-1)(n-j-2) & . & . & . \\ . & . & 1 & . & 1 & . \\ . & . & . & . & (-1)(n-j-2) & 1 \\ 1 & 1 & . & . & 1 & (-1)(n-j-2) \end{bmatrix}$$

with $\alpha(b) = (\alpha_{j+1}(b), \ldots, \alpha_n(b))$.

<u>Proof</u>: We know that α_i is strictly increasing over $(b - \epsilon, b + \epsilon)$, for all $j < i \leq n$, and thus that b_{il} is continuous over $(\alpha_i(b - \epsilon), \alpha_i(b + \epsilon))$. From Lemma A1-18, Prob(i wins | b) is differentiable and $\frac{d}{db}$ Prob(i wins | b) $= \frac{\text{Prob(i wins | b)}}{v-b}$ or, equivalently,

(A1.7) $$\frac{d}{db} \ln\text{Prob(i wins} \mid \text{b)} = \frac{1}{\alpha_i(b)-b},$$

over $(b - \epsilon, b + \epsilon)$, for all $j < i \leq n$. Remark that from Lemma A1-2 and $b > \underline{c}$, we have Prob(i wins | b) > 0. From Lemma A1-16, we know that Prob(i wins | b) $= \prod_{\substack{k=1 \\ k\neq i}}^{n} F_k(\alpha_k(b))$, for all $b > \underline{c}$ and all $1 \leq i \leq n$. Here $\prod_{k=1}^{j} F_k(\alpha_k(b))$ is constant over $(b - \epsilon, b + \epsilon)$. Denote by K the value of this constant. We have $K > 0$. We obtain Prob(i wins | b) $= K \prod_{\substack{k=j+1 \\ k\neq i}}^{n} F_k(\alpha_k(b))$, for all b in

$(b - \epsilon, b + \epsilon)$ and all $j < i \leq n$. Substituting its value to Prob(i wins | b) in (A1.7), we obtain

$$\frac{d}{db} \sum_{\substack{k=j+1 \\ k\neq i}}^{n} \ln F_k(\alpha_k(b)) = \frac{1}{\alpha_i(b)-b},$$

for all $j < i \leq n$ and for all b in $(b - \epsilon, b + \epsilon)$, or, in matrix notation,

(A1.8) $$\frac{d}{db} \mathbb{A} \,.\, \mathbb{LNF}(\alpha(b)) = \mathbb{I}(\alpha(b), b),$$

for all b in $(b - \epsilon, b + \epsilon)$, where $\mathbb{LNF}(\alpha(b))$ and $\mathbb{I}(\alpha(b), b)$ are defined as in the statement of the lemma and where $\mathbb{A}$ is the $(n - j) \times (n - j)$ matrix defined as follows,

$$\mathbb{A} \; = \; \begin{bmatrix} 0 & 1 & . & . & . & 1 \\ 1 & 0 & 1 & . & . & . \\ . & 1 & 0 & . & . & . \\ . & . & 1 & . & 1 & . \\ . & . & . & . & 0 & 1 \\ 1 & 1 & . & . & 1 & 0 \end{bmatrix} .$$

It is not complicated to verify that $\mathbb{A}$ is regular and that $\mathbb{A}^{-1} \; = \; \mathbb{M}$, with $\mathbb{M}$ defined as in the statement of the lemma. From (A1.8) we see that $\mathbb{A}$. $\mathbb{LNF}(\alpha(\mathrm{b}))$ is differentiable over (b $- \, \epsilon$, b $+ \, \epsilon$). Since $\mathbb{LNF}(\alpha(\mathrm{b})) \; = \; \mathbb{M}$. $(\mathbb{A}$. $\mathbb{LNF}(\alpha(\mathrm{b})))$, $\mathbb{LNF}(\alpha(\mathrm{b}))$ itself is differentiable over $(\mathrm{b} - \epsilon$, $\mathrm{b} + \epsilon)$ and from (A1.8) again we obtain (A1.6).

The proof of Lemma A1-19 is over (see footnote 12) if we take the limits of (A1.5) for b $\underset{>}{\rightarrow}$ b $-$ ϵ and for b $\underset{<}{\rightarrow}$ b $+$ ϵ and we make use of the continuity of α_k and the fact α_k(d) $>$ d, for all d in ($\underline{\mathrm{c}}$, $\overline{\mathrm{c}}$]. ||

<u>Lemma</u> <u>A1-20</u>: Under the assumptions of Lemma A1-19, we have

$$\sum_{k=j+1}^{n} \frac{d}{db} \; lnF_k(\alpha_k(b)) \;\; = \;\; \frac{1}{(n-j-1)} \sum_{k=j+1}^{n} \frac{1}{\alpha_k(b)-b},$$

for all b in $[b - \epsilon,\, b + \epsilon]$, where the derivative at $b - \epsilon$ is a right-hand derivative and the derivative at $b + \epsilon$ is a left-hand derivative.

<u>Proof</u>: Immediate from (A1.5). ||

<u>Lemma</u> <u>A1-21</u>: Let $(\beta_1, \ldots, \beta_n)$ be a Bayesian equilibrium of the first price auction with or without mandatory bidding and η be as defined in Lemma A1-12. Then, for all b in ($\underline{c}$, η) there exist at least two indices j and j', with j $\neq$ j', such that b is a point of increase of α_j and $\alpha_{j'}$.

<u>Proof</u>: Let b be an element of ($\underline{\mathrm{c}}$, η). We first prove that, for all b in ($\underline{\mathrm{c}}$, η), there exist at least two indices j and j', with j $\neq$ j', such that b is a point of increase of α_j and $\alpha_{j'}$. Suppose that this is not the case. Then there exists i such that, for all j $\neq$ i, b is not a point of increase of α_j. From Lemmas A1-16 and A1-17, we know that $\mathrm{b}_{jl}(\alpha_j(\mathrm{b})) \; < \; \mathrm{b} \; < \; \mathrm{b}_{ju}(\alpha_j(\mathrm{b}))$, for all j $\neq$ i. Let h such that $\mathrm{b}_{hu}(\alpha_h(\mathrm{b})) \; = \; \min_{1 \leq k \leq n} \; \mathrm{b}_{ku}(\alpha_k(\mathrm{b}))$. From Lemma A1-16, $\mathrm{b}_{hu}(\alpha_h(\mathrm{b}))$ $\geq$ b. If $\mathrm{b}_{hu}(\alpha_h(\mathrm{b})) \; = \;$ b, then h $=$ i and $\max_{j \neq h} \; \mathrm{b}_{jl}(\alpha_j(\mathrm{b})) \; < \;$ b and thus $\max_{j \neq h} \; \mathrm{b}_{jl}(\alpha_j(\mathrm{b}))$ $< \; \mathrm{b}_{hu}(\alpha_h(\mathrm{b}))$. If $\mathrm{b}_{hu}(\alpha_h(\mathrm{b})) \; > \;$ b, then from Lemma A1-6 we have $\max_{j \neq h} \; \mathrm{b}_{jl}(\alpha_j(\mathrm{b})) \; \leq \;$ b and thus again $\max_{j \neq h} \; \mathrm{b}_{jl}(\alpha_j(\mathrm{b})) \; < \; \mathrm{b}_{hu}(\alpha_h(\mathrm{b}))$. Bidder h's payoff if his valuation is equal to

α_h(b) and he submits $\max_{j \neq h}$ $b_{jl}(\alpha_j(b))$ is strictly larger than his payoff if his valuation is still equal to α_h(b) and if he submits $b_{hu}(\alpha_h(b))$. In fact, his probability of winning is the same but his payment in case of winning is strictly smaller. This contradicts Lemma A1-10 and the definition of an equilibrium and we have proved that there exist at least two indices j and j', with j $\neq$ j', such that b is a point of increase of α_j and $\alpha_{j'}$, for all b in ($\underline{c}$, η). Lemma A1-21 is thus proved. ||

<u>Lemma</u> <u>A1-22</u>: Let $(\beta_1, \ldots, \beta_n)$ *be a Bayesian equilibrium of the first price auction with or without mandatory bidding. Assume that there exist 1 ≤ i ≤ n and an open subinterval* (d', d) *of* ($\underline{c}$, η], *such that* α_i *is constant over* (d', d). *Then* $\alpha_j(b) > \alpha_i(b)$, *for all b in* (d', d) *and all j* $\neq$ *i such that b is a point of increase of* α_j, *that is, such that* α_j *is not constant on any neighborhood of b.*

<u>Proof</u>: Let i and (d', d) be as in the statement of the lemma. We first prove that if b is a point of increase of α_j, then α_j(b) $\geq$ α_i(b). Assume that there exist j $\neq$ i and b in (d', d) such that α_j(b) $<$ α_i(b) and b is a point of increase of α_j, that is, α_j is not constant on any neighborhood of b. Since b > $\underline{c}$, we see from Lemmas A1-3 and A1-13 that α_i(b) > $\underline{c}$. From Lemma A1-17, b is either equal to $b_{jl}(\alpha_j(b))$ or $b_{ju}(\alpha_j(b))$. From Lemmas A1-10 and A1-16, we thus have

$$P(j \mid \alpha_j(b)) \;=\; (\alpha_j(b) - b) \prod_{k \neq j} F_k(\alpha_k(b)) \;=\; (\alpha_j(b) - b)\, F_i(\alpha_i(b)) \prod_{k \neq j,i} F_k(\alpha_k(b)).$$

Let us denote by b' the value of the function b_{il} at α_i(b), that is, b' $=$ b_{il} (α_i(b)). Of course, from the definition of α_i(b) (A1.2), we have b' $\leq$ d' $<$ b. From Lemma A1-15, we have $\alpha_i(b') = \alpha_i(b)$. If b' = $\underline{c}$, from $\alpha_i(b) = \alpha_i(b') > \underline{c}$ and Lemma A1-14 we know that i = j as in Lemma A1-14. Consequently, Lemma A1-16 implies that in all cases (b' = $\underline{c}$ or b' > $\underline{c}$) we have,

$$P(i \mid \alpha_i(b)) \;=\; (\alpha_i(b) - b') \prod_{k \neq i} F_k(\alpha_k(b')) \;=\; (\alpha_i(b) - b')\, F_j(\alpha_j(b')) \prod_{k \neq j,i} F_k(\alpha_k(b')).$$

Since P(i | α_i(b)) $>$ 0, we see that $F_j(\alpha_j(b'))$ $>$ 0. From the definition of an equilibrium, we know that P(j | α_j(b)) $\geq$ P(j | α_j(b), $\widetilde{b}$), for all $\widetilde{b}$ $>$ b', and thus[20] P(j | α_j(b)) $\geq$ $\lim_{\widetilde{b} \underset{>}{\to} b'}$ P(j | α_j(b), $\widetilde{b}$) $=$ $(\alpha_j(b) - b')\ F_i(\alpha_i(b')) \prod_{k \neq j,i} F_k(\alpha_k(b'))$. Substituting its value to P(j | α_j(b)), $(\alpha_j(b) - \alpha_i(b)) + (\alpha_i(b) - b)$ to $(\alpha_j(b) - b)$ and $(\alpha_j(b) - \alpha_i(b)) + (\alpha_i(b) - b')$ to $(\alpha_j(b) - b')$ and simplifying by $F_i(\alpha_i(b'))$ $=$ $F_i(\alpha_i(b))$ (which is strictly positive since α_i(b) > $\underline{c}$) give

(A1.9) $$(\alpha_j(b) - \alpha_i(b)) \prod_{k \neq j,i} F_k(\alpha_k(b)) \;+\; (\alpha_i(b) - b) \prod_{k \neq j,i} F_k(\alpha_k(b))$$

$$\geq\; (\alpha_j(b) - \alpha_i(b)) \prod_{k \neq j,i} F_k(\alpha_k(b')) + (\alpha_i(b) - b') \prod_{k \neq j,i} F_k(\alpha_k(b')).$$

We also know that P(i | α_i(b)) $\geq$ P(i | α_i(b), b) $=$ (α_i(b) $-$ b) $F_j(\alpha_j(b)) \prod_{k \neq j,i} F_k(\alpha_k(b))$. Since b $>$ b' (and α_i(b) $\geq$ b), we have P(i | α_i(b)) $\geq$ (α_i(b) $-$ b) $F_j(\alpha_j(b')) \prod_{k \neq j,i} F_k(\alpha_k(b))$. Substituting its value to P(i | α_i(b)) and simplifying by $F_j(\alpha_j(b')) > 0$, we obtain

$$\text{(A1.10)} \qquad (\alpha_i(b) - b') \prod_{k \neq j,i} F_k(\alpha_k(b')) \geq (\alpha_i(b) - b) \prod_{k \neq j,i} F_k(\alpha_k(b)).$$

Combining the inequalities (A1.9) and (A1.10) and subtracting $(\alpha_i(b) - b) \prod_{k \neq j,i} F_k(\alpha_k(b))$, we obtain the inequality below,

$$(\alpha_j(b) - \alpha_i(b)) \left\{ \prod_{k \neq j,i} F_k(\alpha_k(b)) - \prod_{k \neq j,i} F_k(\alpha_k(b')) \right\} \geq 0.$$

The function α_i is constant over (b', b). Consider b'' in (b', b). From Lemma A1-21, there exists k $\neq$ j such that b'' is a point of increase of α_k. Thus we have $F_k(\alpha_k(b)) > F_k(\alpha_k(b'))$ and $\prod_{k \neq j,i} F_k(\alpha_k(b)) - \prod_{k \neq j,i} F_k(\alpha_k(b')) > 0$. Consequently, we obtain $\alpha_j(b) \geq \alpha_i(b)$.

Assume now that $\alpha_j(b) = \alpha_i(b)$. We know that $b_{jl}(\alpha_j(b)) = b$ or $b_{ju}(\alpha_j(b)) = b$. Assume that $b_{jl}(\alpha_j(b)) = b$ (the proof in the other case is similar). Since b_{jl} is continuous to the left, $b_{jl}(\underline{c}) = \underline{c}$ and b_{jl} has at most a countable number of discontinuities, there exists w $< \alpha_j$(b) such that $b_{jl}(\alpha_j(b)) \geq b_{jl}(w) >$ d' and b_{jl} is continuous at w. Thus $b_{jl}(w)$ is a point of increase of α_j in (d', d) and $\alpha_j(b_{jl}(w)) = w < \alpha_i(b)$. This contradicts the first part of the proof and thus $\alpha_j(b) = \alpha_i(b)$ is also impossible. ||

<u>Lemma</u> <u>A1-23</u>: *Let $(\beta_1, \ldots, \beta_n)$ be a Bayesian equilibrium of the first price auction with or without mandatory bidding. Assume that b_{il} is discontinuous at v_i, that b_{jl} is discontinuous at v_j, with $j \neq i$, and that $(b_{il}(v_i), b_{iu}(v_i)) \cap (b_{jl}(v_j), b_{ju}(v_j)) \neq \emptyset$. Then we have $(b_{il}(v_i), b_{iu}(v_i)) \subseteq (b_{jl}(v_j), b_{ju}(v_j))$ or $(b_{jl}(v_j), b_{ju}(v_j)) \subseteq (b_{il}(v_i), b_{iu}(v_i))$.*

<u>Proof</u>: Assume that we have $b_{il}(v_i) < b_{jl}(v_j) < b_{iu}(v_i) < b_{ju}(v_j)$. Thus $b_{jl}(\underline{c}) > \underline{c}$ and b_{jl} is strictly increasing over a neighborhood of v_j (see Lemma A1-14). Since $b_{jl}(v_j) \in (b_{il}(v_i), b_{iu}(v_i))$, Lemma A1-22 implies $\alpha_j(b_{jl}(v_j)) = v_j \geq \alpha_i(b_{jl}(v_j)) = v_i$. From Lemma A1-14 and $b_{il}(v_i) > \underline{c}$, we see that b_{il} is strictly increasing over a neighborhood of v_i. From Lemma A1-15, we then have $\alpha_i(b_{iu}(v_i)) = v_i$. Consequently, $\alpha_i(b_{iu}(v_i)) = v_i \geq \alpha_j(b_{iu}(v_i)) = v_j$, and $v_i = v_j$.

Since b_{jl} is strictly increasing over a neighborhood of v_j, $b_{jl}(\underline{c}) = \underline{c}$ and b_{jl} is continuous from the left, there exists $w_j < v_j = v_i$ such that $b_{il}(v_i) < b_{jl}(w_j) < b_{jl}(v_j)$ and b_{jl} is strictly increasing over a neighborhood of w_j. This is impossible however, since from Lemma A1-22 we should have $\alpha_j(b_{jl}(w_j)) = w_j \geq \alpha_i(b_{jl}(w_j)) = v_i$ and Lemma A1-23 is proved. ||

Lemma *A1-24*: *Let* $(\beta_1, \ldots, \beta_n)$ *be a Bayesian equilibrium of the first price auction with voluntary bidding. If j and w' are as defined in Lemma A1-14, then* b_{jl} *is continuous at w', that is,* $b_{jl}(w') = b_{ju}(w') = \underline{c}$.

Proof: We already know that b_{jl} is continuous from the left (see Lemma A1-13) and thus $b_{jl}(w') = \underline{c}$. Assume that $b_{ju}(w') > \underline{c}$. Consider $k \neq j$. From Lemma A1-14, b_{kl} is strictly increasing over $[\underline{c}, \overline{c}]$. Since $b_{ku}(\underline{c}) = \underline{c}$ and there is at most a countable number of discontinuities, there exists u in $(\underline{c}, w')$ such that $b_{kl}(u) < b_{ju}(w')$. This contradicts Lemma A1-22 and Lemma A1-24 is proved. ||

Lemma *A1-25*: *If* $(\beta_1, \ldots, \beta_n)$ *is a Bayesian equilibrium of the first price auction with or without mandatory bidding, then the strategies* $\beta_1, \ldots, \beta_n$ *are pure over* $(\underline{c}, \overline{c}]$ *and the functions* $\alpha_1, \ldots, \alpha_n$ *are differentiable and strictly increasing and verify over* $(\underline{c}, \eta]$, *where* η *is defined as in Lemma A1-12, the following system of differential equations considered on the domain* $D = \{ (b, \alpha_1, \ldots, \alpha_n) \in \mathbb{R}^{n+1} \mid \underline{c}, b < \alpha_i, \text{ for all } 1 \leq i \leq n \}$,

$$\frac{d}{db}\mathbb{LNF}(\alpha(b)) = \mathbb{M} \,.\, \mathbb{I}(\alpha(b), b),$$

where $\mathbb{LNF}(\alpha(b))$, $\mathbb{I}(\alpha(b), b)$, $\mathbb{M}$ *and* $\alpha(b)$ *are defined as in the statement of Lemma A1-19 with* $j = 0$.

Proof: Suppose that there exists $1 \leq i \leq n$ and $v_i \in [\underline{c}, \overline{c}]$ such that b_{il} is discontinuous at v_i, that is, such that $b_{il}(v_i) < b_{iu}(v_i)$. Without loss of generality, we can assume that $i = 1$. Either α_k is strictly increasing over all open subsets of $(b_{1l}(v_1), b_{1u}(v_1))$, for all $k \neq 1$, or there exists $k \neq 1$ and $(d_k, d_k') \subseteq (b_{1l}(v_1), b_{1u}(v_1))$ such that α_k is constant over (d_k, d_k') and $d_k < d_k'$. As a consequence, in this latter case b_{kl} is discontinuous at $\alpha_k(d_k) = v_k$ and from Lemma A1-23 we have $(b_{kl}(v_k), b_{ku}(v_k)) \subseteq (b_{1l}(v_1), b_{1u}(v_1))$ or $(b_{1l}(v_1), b_{1u}(v_1)) \subseteq (b_{kl}(v_k), b_{ku}(v_k))$. There is no loss of generality in this case to assume that $k = 2$ and $(b_{kl}(v_k), b_{ku}(v_k)) \subseteq (b_{1l}(v_1), b_{1u}(v_1))$. Also in this case, we repeat this construction and as many times as it is possible. We then see that, without loss of generality, there exist $1 \leq j \leq n$ and $v_1, \ldots, v_j$ in $[\underline{c}, \overline{c}]$ such that $b_{1l}(v_1) \leq \ldots \leq b_{jl}(v_j) < b_{ju}(v_j) \leq \ldots \leq b_{1u}(v_1)$ and such that $\alpha_{j+1}, \ldots, \alpha_n$ are strictly increasing over $(b_{jl}(v_j), b_{ju}(v_j))$. Moreover, $j < n - 1$. Otherwise, no α_k (in the case $j = n$) or only α_n (in the case $j = n - 1$) would be strictly increasing over $(b_{jl}(v_j), b_{ju}(v_j))$ and it would contradict Lemma A1-23.

From Lemma A1-19, we know that the functions $\alpha_{j+1}, \ldots, \alpha_n$ are differentiable over $(b_{jl}(v_j), b_{ju}(v_j))$, differentiable on the right at $b_{jl}(v_j)$, differentiable on the left at $b_{ju}(v_j)$ and form a solution of (A1.5) over this interval. Lemma A1-24 implies that $b_{jl}(v_j) > \underline{c}$. From Lemmas A1-10 and A1-16, we have $P(j \mid v_j) = (v_j - b_{jl}(v_j)) \prod_{k \neq j} F_k(\alpha_k(b_{jl}(v_j))) = (v_j - b_{ju}(v_j)) \prod_{k \neq j} F_k(\alpha_k(b_{ju}(v_j)))$. Moreover, for all b in $[b_{jl}(v_j), b_{ju}(v_j)]$, we have $P(j \mid v_j, b) = (v_j - b) \prod_{k \neq j} F_k(\alpha_k(b))$. However, for all b in $[b_{jl}(v_j), b_{ju}(v_j)]$, the product $\prod_{k=1}^{j-1} F_k(\alpha_k(b))$ is equal to a strictly positive constant K. From the definition of a Bayesian equilibrium, $P(j \mid v_j) \geq P(j \mid v_j, b)$, for all b in $[b_{jl}(v_j), b_{ju}(v_j)]$. Substituting their values to $P(j \mid v_j)$ and $P(j \mid v_j, b)$ and simplifying by K, we obtain

$$(v_j - b_{jl}(v_j)) \prod_{k=j+1}^{n} F_k(\alpha_k(b_{jl}(v_j))) \geq (v_j - b) \prod_{k=j+1}^{n} F_k(\alpha_k(b)),$$

$$(v_j - b_{ju}(v_j)) \prod_{k=j+1}^{n} F_k(\alpha_k(b_{ju}(v_j))) \geq (v_j - b) \prod_{k=j+1}^{n} F_k(\alpha_k(b)),$$

for all b in $[b_{jl}(v_j), b_{ju}(v_j)]$. As a consequence, the function $(v_j - b) \prod_{k=j+1}^{n} F_k(\alpha_k(b))$ and, thus, also its logarithm $\ln(v_j - b) + \sum_{k=j+1}^{n} \ln F_k(\alpha_k(b))$ reaches its maximum over $[b_{jl}(v_j), b_{ju}(v_j)]$ at $b_{jl}(v_j)$ and at $b_{ju}(v_j)$. If they exist, the left-hand derivative of $\ln(v_j - b) + \sum_{k=j+1}^{n} \ln F_k(\alpha_k(b))$ at $b = b_{ju}(v_j)$ must thus be nonnegative and the right-hand derivative of the same function at $b = b_{ju}(v_j)$ must be nonpositive. From Lemma A1-20, we see that the derivative exists and is given by the equation below,

$$\frac{d}{db} \ln(v_j - b) + \sum_{k=j+1}^{n} \frac{d}{db} \ln F_k(\alpha_k(b)) = \frac{-1}{v_j - b} + \frac{1}{(n-j-1)} \sum_{k=j+1}^{n} \frac{1}{\alpha_k(b) - b},$$

for all b in $[b_{jl}(v_j), b_{ju}(v_j)]$. At the lower extremity of this interval, the derivative is a right-hand derivative and at the upper extremity of this interval, the derivative is a left-hand derivative. Consequently, we have $\frac{-1}{v_j - b_{jl}(v_j)} + \frac{1}{(n-j-1)} \sum_{k=j+1}^{n} \frac{1}{\alpha_k(b_{jl}(v_j)) - b_{jl}(v_j)} \leq 0$ and $\frac{-1}{v_j - b_{ju}(v_j)} + \frac{1}{(n-j-1)} \sum_{k=j+1}^{n} \frac{1}{\alpha_k(b_{ju}(v_j)) - b_{ju}(v_j)} \geq 0$. We can rewrite the two inequalities above as follows,

$$\text{(A1.11)} \quad \sum_{k=j+1}^{n} \frac{v_j - b_{jl}(v_j)}{\alpha_k(b_{jl}(v_j)) - b_{jl}(v_j)} \leq (n - j - 1) \leq \sum_{k=j+1}^{n} \frac{v_j - b_{ju}(v_j)}{\alpha_k(b_{ju}(v_j)) - b_{ju}(v_j)}.$$

However, $\sum_{k=j+1}^{n} \frac{v_j - b}{\alpha_k(b) - b}$ is a strictly decreasing function of b. In fact, each term $\frac{v_j - b}{\alpha_k(b) - b}$ is strictly decreasing in b because, for example, its inverse is equal to $1 + \frac{\alpha_k(b) - v_j}{v_j - b}$ and is a strictly increasing function of b since from Lemma A1-22 $\alpha_k(b) > v_j$, for all $k > j$. We know that $b_{jl}(v_j) < b_{ju}(v_j)$ and thus we have $\sum_{k=j+1}^{n} \frac{v_j - b_{jl}(v_j)}{\alpha_k(b_{jl}(v_j)) - b_{jl}(v_j)} > \sum_{k=j+1}^{n} \frac{v_j - b_{ju}(v_j)}{\alpha_k(b_{ju}(v_j)) - b_{ju}(v_j)}$, which contradicts (A1.11). Thus, our assumption made at the beginning of the proof must be wrong. That is, there is no b_{il} which is discontinuous and the functions $b_{1l}, \ldots, b_{nl}$ are continuous and it is enough to apply Lemma A1-19 to end the proof of Lemma A1-25. ||

Proof of the necessity parts in Theorems 1 and 2 (Section 3): Immediate from Lemmas A1-25, A1-3, A1-2, A1-9 and A1-14. ||

Appendix 2.

We show the details of the proof of Theorem 3 (Section 4). First we prove Lemma A2-1 below.

Lemma A2-1: Let the assumptions of Section 2 be satisfied. Let $(\alpha_1, \ldots, \alpha_n)$ *be a solution over an interval* $(\gamma, \gamma']$, *with* $\underline{c} \leq \gamma < \gamma' < \overline{c}$ *, of the differential system (2) considered in the domain D. Then the following equations hold true over the interval* $(\gamma, \gamma']$ *and for all* $1 \leq i, j \leq n$,

$$\text{(A2.1)} \qquad \frac{d}{db} \sum_{\substack{k=1 \\ k \neq i}}^{n} \ln F_k(\alpha_k(b)) = \frac{1}{\alpha_i(b) - b},$$

$$\text{(A2.2)} \qquad \frac{d}{db} \ln F_j(\alpha_j(b)) - \frac{d}{db} \ln F_i(\alpha_i(b)) = \frac{1}{\alpha_i(b) - b} - \frac{1}{\alpha_j(b) - b}.$$

Proof: By summing all equations in (6) except the equation corresponding to α_i, we find (A2.1). It suffices to subtract the equation in (6) corresponding to α_i from the equation in (6) corresponding to α_j in order to prove (A2.2). ||

We now prove Lemma A2-2 below which implies that a solution of (2, 18) consists always of strictly increasing functions.

Lemma A2-2: Let the assumptions of Section 2 be satisfied. Let $(\alpha_1, \ldots, \alpha_n)$ *be a solution over an interval* $(\gamma, \gamma']$, *with* $\underline{c} \leq \gamma < \gamma' < \overline{c}$ *, of the differential system (2) considered in the domain D and such that* $\frac{d}{db}\alpha_1(\gamma') > 0, \ldots, \frac{d}{db}\alpha_n(\gamma') > 0$. *Then* $\frac{d}{db}\alpha_1(b) > 0, \ldots, \frac{d}{db}\alpha_n(b) > 0$, *for all b in* $(\gamma, \gamma']$.

Proof: For all $1 \leq i \leq n$, consider b'_i defined as follows, $b'_i = \inf \{b' \in [\gamma, \gamma'] \mid \frac{d}{db}\alpha_i(b) > 0$, for all b in $(b', \gamma]\}$. From equations (2), we see that $\frac{d}{db}\alpha_1, \ldots, \frac{d}{db}\alpha_n$ are continuous over $(\gamma, \gamma']$. Since $\frac{d}{db}\alpha_1(\gamma') > 0, \ldots, \frac{d}{db}\alpha_n(\gamma') > 0$, we have $b'_1 < \gamma', \ldots, b'_n < \gamma'$. We want to prove that $\frac{d}{db}\alpha_1(b) > 0, \ldots, \frac{d}{db}\alpha_n(b) > 0$, for all b in $(\gamma, \gamma']$, that is, that $b'_1 = \gamma, \ldots, b'_n = \gamma$. From their definitions, we know that $b'_1 \geq \gamma, \ldots, b'_n \geq \gamma$. We will have thus proved Lemma A2-2 if we prove that $\max_{1 \leq k \leq n} b'_k \leq \gamma$.

Assume that $\max_{1 \leq k \leq n} b'_k > \gamma$. Let i be such that $b'_i = \max_{1 \leq k \leq n} b'_k$. From the continuity of $\frac{d}{db}\alpha_i$, we have $\frac{d}{db}\alpha_i(b'_i) = 0$. Moreover, since $b'_i \geq b'_k$ we also have $\frac{d}{db}\alpha_k(b'_i) \geq 0$, for all $1 \leq k \leq n$. From the equation in (6) corresponding to α_i, we see that

$$(\alpha_i(\mathrm{b}) - \mathrm{b})\ \frac{\mathrm{d}}{\mathrm{db}}\ln\mathrm{F}_i(\alpha_i(\mathrm{b}))\ =\ \frac{1}{(\mathrm{n}-1)}\ \Big\{\ (-1)(\mathrm{n}-2)\ +\ \sum_{\substack{k=1\\k\neq i}}^{n} \frac{\alpha_i(\mathrm{b})-\mathrm{b}}{\alpha_k(\mathrm{b})-\mathrm{b}}\ \Big\},$$

for all b in $(\gamma, \gamma']$. Taking the derivative of the equation above, we obtain

(A2.3) $\quad \frac{\mathrm{d}}{\mathrm{db}}\{(\alpha_i(\mathrm{b}) - \mathrm{b})\frac{\mathrm{d}}{\mathrm{db}}\ln\mathrm{F}_i(\alpha_i(\mathrm{b}))\} =$

$$=\ \frac{1}{(\mathrm{n}-1)}\Big\{\ \sum_{\substack{k=1\\k\neq i}}^{n} \frac{1}{(\alpha_k(\mathrm{b})-\mathrm{b})^2}\ [\ (\frac{\mathrm{d}}{\mathrm{db}}\alpha_i(\mathrm{b}) - 1)(\alpha_k(\mathrm{b}) - \mathrm{b}) - (\alpha_i(\mathrm{b}) - \mathrm{b})(\frac{\mathrm{d}}{\mathrm{db}}\alpha_k(\mathrm{b}) - 1)\ ]\Big\},$$

for all b in $(\gamma, \gamma']$.

If we substitute b'$_i$ to b in equation (A2.3), we see that the expression between brackets in the sum in the R.H.S. of this equation is equal to $(\alpha_i(\mathrm{b}'_i) - \alpha_k(\mathrm{b}'_i)) - (\alpha_i(\mathrm{b}'_i) - \mathrm{b}'_i)\frac{\mathrm{d}}{\mathrm{db}}\alpha_k(\mathrm{b}'_i)$. Since $\frac{\mathrm{d}}{\mathrm{db}}\alpha_i(\mathrm{b}'_i) = 0$ and $\frac{\mathrm{d}}{\mathrm{db}}\alpha_k(\mathrm{b}'_i) \geq 0$, for all k, we have $\frac{\mathrm{d}}{\mathrm{db}}\ln\mathrm{F}_i(\alpha_i(\mathrm{b}'_i)) = 0$, $\frac{\mathrm{d}}{\mathrm{db}}\ln\mathrm{F}_k(\alpha_k(\mathrm{b}'_i)) \geq 0$, for all k. Equation (A2.2) implies $\alpha_i(\mathrm{b}'_i) \leq \alpha_k(\mathrm{b}'_i)$. Consequently, the term between brackets in the sum in the R.H.S. of (A2.3) is nonpositive. Moreover, there exists k such that the corresponding term is strictly negative. In fact, from equation (A2.1) there exists $k \neq i$ such that $\frac{\mathrm{d}}{\mathrm{db}}\alpha_k(\mathrm{b}'_i) > 0$. Consequently, from (A2.3) we see that the derivative of $(\alpha_i(\mathrm{b}) - \mathrm{b})\ \frac{\mathrm{d}}{\mathrm{db}}\ln\mathrm{F}_i(\alpha_i(\mathrm{b}))$ at $\mathrm{b} = \mathrm{b}'_i$ is strictly negative and this function is thus strictly decreasing in a neighborhood of b'$_i$. However, since $\frac{\mathrm{d}}{\mathrm{db}}\alpha_i(\mathrm{b}'_i) = 0$ and thus $\frac{\mathrm{d}}{\mathrm{db}}\ln\mathrm{F}_i(\alpha_i(\mathrm{b}'_i)) = 0$, the value of this function at $\mathrm{b} = \mathrm{b}'_i$ is equal to zero. Consequently, there exists $\epsilon > 0$ such that $(\alpha_i(\mathrm{b}) - \mathrm{b})\ \frac{\mathrm{d}}{\mathrm{db}}\ln\mathrm{F}_i(\alpha_i(\mathrm{b})) < 0$, for all b in $(\mathrm{b}'_i, \mathrm{b}'_i + \epsilon)$. Since $(\alpha_i(\mathrm{b}) - \mathrm{b}) > 0$, for all b in $(\gamma, \gamma']$, we obtain $\frac{\mathrm{d}}{\mathrm{db}}\ln\mathrm{F}_i(\alpha_i(\mathrm{b})) < 0$, for all b in $(\mathrm{b}'_i, \mathrm{b}'_i + \epsilon)$. This contradicts the definition of b'$_i$ and we have proved Lemma A2-2. ||

As we see in Lemma A2-3 below, it is possible to obtain bounds for the functions $\phi_{ji} = \alpha_j\alpha_i^{-1}$, which “connect" two components of a solution of the problem (2, 19). From the definition of ϕ_{ji}, we see that $\phi_{ji}(\mathrm{v}) = \beta_j^{-1}(\beta_i(\mathrm{v}))$ can be interpreted as the valuation at which bidder j bids the same bid as bidder i at v.

In Lemma A2-3, we use the function ζ_{ji}, with $1 \leq \mathrm{i}, \mathrm{j} \leq \mathrm{n}$, defined as follows,

(A2.4) $\quad \zeta_{ji}(\mathrm{v}) = \mathrm{F}_j^{-1}\Big(\mathrm{F}_i(\mathrm{v}) \min_{v \leq w \leq \overline{c}} \frac{\mathrm{F}_j(\mathrm{w})}{\mathrm{F}_i(\mathrm{w})}\Big),$

for all v in $[\mathrm{w}_{ji}, \overline{\mathrm{c}}\,]$ where w_{ji} belongs to $[\underline{\mathrm{c}}\,, \overline{\mathrm{c}}\,)$ and is such that

(A2.5) $\quad \mathrm{F}_i(\mathrm{w}_{ji}) \min_{w_{ji} \leq w \leq \overline{c}} \frac{\mathrm{F}_j(\mathrm{w})}{\mathrm{F}_i(\mathrm{w})} = \mathrm{F}_j(\underline{\mathrm{c}}\,).$

When $\mathrm{w}_{ji} = \underline{\mathrm{c}}$, $\min_{w_{ji} \leq w \leq \overline{c}} \frac{\mathrm{F}_j(\mathrm{w})}{\mathrm{F}_i(\mathrm{w})}$ in the L.H.S. of the equality above is defined as the continuous extension of $\min_{v \leq w \leq \overline{c}} \frac{\mathrm{F}_j(\mathrm{w})}{\mathrm{F}_i(\mathrm{w})}$ at $\underline{\mathrm{c}}$, that is, its limit for $\mathrm{v} \to \underline{\mathrm{c}}$. The function $\mathrm{F}_i(\mathrm{v})$

$\min_{v \le w \le \overline{c}} \frac{F_j(w)}{F_i(w)}$ is continuous and strictly increasing over $(\underline{c}, \overline{c}]$. It is equal to $1 > F_j(\underline{c})$ at v $= \overline{c}$. Since $\min_{v \le w \le \overline{c}} \frac{F_j(w)}{F_i(w)} \le \frac{F_j(v)}{F_i(v)}$, for all v in $(\underline{c}, \overline{c}]$, we see that $F_i(v) \min_{v \le w \le \overline{c}} \frac{F_j(w)}{F_i(w)} \le F_j(v)$, for all v in $(\underline{c}, \overline{c}]$. In particular, we see that the limit of $F_i(v) \min_{v \le w \le \overline{c}} \frac{F_j(w)}{F_i(w)}$ for v $\to \underline{c}$ is not larger than $F_j(\underline{c})$. As a consequence, there exists one and only one w_{ji} satisfying (A2.5). Moreover, we see that ζ_{ji} is a strictly increasing continuous function, equal to $\overline{c}$ at $v = \overline{c}$ and that $\zeta_{ji}(v) \le v$, for all v in $[w_{ji}, \overline{c}]$. The value $F_j(\zeta_{ji})$ at v is actually the maximum of the values taken by functions not larger than F_j over $[v, \overline{c}]$ and which have the same logarithmic derivative as F_i. If ζ_{ji} is extended continuously at w_{ji}, we have $\zeta_{ji}(w_{ji}) = \underline{c}$. The inverse ζ_{ji}^{-1} of ζ_{ji} is thus defined over $(\underline{c}, \overline{c}]$ and, if extended continuously, is such that $\zeta_{ji}^{-1}(\underline{c}) = w_{ji}$. Notice that if $F_i(\underline{c}) = F_j(\underline{c}) = 0$, then $w_{ji} = w_{ij} = \underline{c}$.

<u>Lemma</u> <u>A2-3</u>: Let the assumptions of Section 2 be satisfied. Let η be such that $\underline{c} < \eta < \overline{c}$ and let $(\alpha_1, \ldots, \alpha_n)$ be a solution over an interval $(\gamma, \eta]$, with $\underline{c} \le \gamma < \eta < \overline{c}$, of the differential system (2) considered in the domain D and such that $\alpha_1(\eta) = \ldots = \alpha_n(\eta) = \overline{c}$. Then the function α_i is strictly increasing over $(\gamma, \eta]$ and the functions α_i^{-1} and $\phi_{ji} = \alpha_j \alpha_i^{-1}$ are differentiable over $(\alpha_i(\gamma), \overline{c}]$, for all $1 \le i, j \le n$. Furthermore, the following inequalities hold true,

$$\zeta_{ji}(w) \le \phi_{ji}(w) \text{ and } \phi_{ji}(v) \le \zeta_{ij}^{-1}(v),$$

for all w in $(max(w_{ji}, \alpha_i(\gamma)), \overline{c}]$, all v in $(\alpha_i(\gamma), \overline{c}]$ and all $1 \le i, j \le n$, where the function ζ_{ji} is defined in (A2.4, A2.5), for all $1 \le i, j \le n$.

<u>Proof</u>: By substituting η and $\overline{c}$ to b and $\alpha_k(b)$ (respectively) in (2), we see that $\frac{d}{db}\alpha_k(\eta) = 1 / (n-1)f_k(\overline{c})(\overline{c} - \eta) > 0$, for all $1 \le k \le n$. From Lemma A2-2, we thus have $\frac{d}{db}\alpha_1(b) > 0, \ldots, \frac{d}{db}\alpha_n(b) > 0$, for all b in $(\gamma, \eta]$. As a consequence, α_i is strictly increasing over $(\gamma, \eta]$ and the functions α_i^{-1} and $\phi_{ji} = \alpha_j \alpha_i^{-1}$ are differentiable over $(\alpha_i(\gamma), \overline{c}]$, for all $1 \le i, j \le n$. We prove the inequality $\zeta_{ji}(v) \le \phi_{ji}(v)$, for all v in $(\alpha_i(\gamma), \overline{c}]$. This inequality is immediate for $v = \overline{c}$. Assume then that $v < \overline{c}$. Let $k > 0$ be such that $k < \min_{v \le w \le \overline{c}} \frac{F_j(w)}{F_i(w)}$. Since the R.H.S. of the last inequality is not larger than 1, we have $k < 1$. Consider the function λ such that $\lambda(u) = F_j^{-1}(kF_i(u))$. For all u in $[v, \overline{c}]$, we have $\lambda(u) < u$. In fact, this inequality is equivalent to $kF_i(u) < F_j(u)$, which is an immediate consequence of the definition of k. From the definition of λ, we have $F_j(\lambda(u)) = (kF_i(u))$, for all u in $[v, \overline{c}]$. Taking the derivative of the logarithm of the last equality gives equation (A2.6) below

(A2.6) $\frac{d}{dv} \ln F_j(\lambda(u)) = \frac{d}{dv} \ln F_i(u)$,

for all u in $[v, \overline{c}]$.

From equation (A2.2) with the change of variable $b = \alpha_i^{-1}(u)$, we have the following equation, (A2.7) $\frac{d}{dv}\ \ln F_j(\phi_{ji}(u)) = \frac{d}{dv}\ \ln F_i(u) + \frac{1}{[\frac{d}{db}\alpha_i(b)]_{b=\alpha_i^{-1}(u)}} \left\{ \frac{1}{u-\alpha_i^{-1}(u)} - \frac{1}{\phi_{ji}(u)-\alpha_i^{-1}(u)} \right\}$,

for all u in $(\alpha_i(\gamma), \overline{c}\,]$. Comparing this equation with equation (A2.6), we see that if $\lambda(u) = \phi_{ji}(u)$ then $\frac{d}{dv}\ \ln F_j(\phi_{ji}(u)) < \frac{d}{dv}\ \ln F_j(\lambda(u))$. In fact, since $\lambda(u) < u$ we have $\phi_{ji}(u) < u$ and the expression between braces in (A2.7) is strictly negative. Obviously, $\lambda(\overline{c}\,) < F_j^{-1}(F_i(\overline{c}\,)) = \overline{c} = \phi_{ji}(\overline{c}\,)$. We then apply Lemma A5-1, a variant of Lemma 2 in Milgrom and Weber (1982), to $[a, b] = [v, \overline{c}\,]$, $l = \ln F_j(\phi_{ji}(u))$ and $h = \ln F_j(\lambda)$ and we see that $\ln F_j(\lambda(u)) \leq \ln F_j(\phi_{ji}(u))$ and thus $\lambda(u) \leq \phi_{ji}(u)$, for all u in $[v, \overline{c}\,]$. In particular, we have $\lambda(v) = F_j^{-1}(kF_i(v)) \leq \phi_{ji}(v)$. The result then follows by making k tend towards $\min_{v \leq w \leq \overline{c}} \frac{F_j(w)}{F_i(w)}$. The inequality $\phi_{ji} \leq \zeta_{ij}^{-1}$ is obtain by inverting the inequality $\zeta_{ij} \leq \phi_{ij}$ and using $\phi_{ji} = \phi_{ij}^{-1}$. ||

From the theory of ordinary differential equations, we know that under the assumptions of Section 2, there exists one and only one maximal solution $(\psi_1, \ldots, \psi_n)$ over the interval $(\underline{c}\,, \eta]$ of the differential system (18) considered in the domain $\mathcal{D}$ which satisfies the initial condition (19), for all $\underline{c} < \eta < \overline{c}$. From the equivalence of the systems (2) and (18), $(\alpha_1, \ldots, \alpha_n) = (F_1^{-1}(\psi_1), \ldots, F_n^{-1}(\psi_n))$ is the only maximal solution over $(\underline{c}\,, \eta]$ of (2) considered in D which satisfies (19). Let $(\underline{\gamma}\,, \eta]$ be the definition sub-interval of the maximal solution $(\alpha_1, \ldots, \alpha_n)$, or, according to the terminology from Section 4, the maximal interval. When the solution can be extended to the whole interval $(\underline{c}\,, \eta]$, that is, when $\underline{\gamma} = \underline{c}$, we say that the solution is of type I (see Figure 3, Section 4). We denote by Λ_I the set of parameters $\underline{c} < \eta < \overline{c}$ corresponding to such maximal solutions. Again from the theory of ordinary differential equations, we know that, when the solution cannot be extended to the whole interval $(\underline{c}\,, \eta]$, the $(n+1)$-tuple $(b, \psi_1(b), \ldots, \psi_n(b))$ has an accumulation point in the boundary of $\mathcal{D}$, or, equivalently, $(b, \alpha_1(b), \ldots, \alpha_n(b))$ has an accumulation point in the boundary of $D = \{ (b, \alpha_1, \ldots, \alpha_n) \in \mathbb{R}^{n+1} \mid \underline{c}\,, b < \alpha_i \leq \overline{c}$, for all $1 \leq i \leq n \}$ when $b \underset{>}{\to} \underline{\gamma}$. From Lemma A2-2 and the initial condition (19), we can rule out $\alpha_i(b) \to \overline{c}$. The case $\alpha_i(\underline{\gamma}\,) = \underline{c}$, for some i, is impossible since, when $\underline{\gamma} > \underline{c}$, we have $\alpha_i(\underline{\gamma}\,) \geq \underline{\gamma} > \underline{c}$. In the only remaining possible case when $\underline{\gamma} > \underline{c}$, there exists $1 \leq i \leq n$ such that $\alpha_i(\underline{\gamma}\,) = \underline{\gamma}$ (see Figure 4, Section 4). We then say that the solution is of type II and the set of parameters η in $(\underline{c}\,, \overline{c}\,)$ corresponding to such solutions is denoted by Λ_{II}. We obtain next a result concerning the type I solutions.

<u>Lemma</u> <u>A2-4</u>: Let the assumptions of Section 2 be satisfied. Suppose that $F_1(\underline{c}\,) = \ldots F_n(\underline{c}\,) = 0$. *Let* η *be such that* $\underline{c} < \eta < \overline{c}$ *and let* $(\alpha_1, \ldots, \alpha_n)$ *be a solution over the interval* $(\underline{c}\,, \eta]$ *of the problem (2, 19) where the differential system (2) is considered in D. Then, either* $\alpha_1(\underline{c}\,) = \ldots = \alpha_n(\underline{c}\,) = \underline{c}$ *or* $\alpha_1(\underline{c}\,), \ldots, \alpha_n(\underline{c}\,) > \underline{c}$.

<u>Proof</u>: Because $(b, \alpha_1, \ldots, \alpha_n)$ lies in the domain D over the interval $(\underline{c}\,, \eta]$, we have $\alpha_1(\underline{c}\,), \ldots, \alpha_n(\underline{c}\,) \geq \underline{c}$. From Lemma A2-3 and from the definition of ϕ_{ji}, we know that $\zeta_{ji}(\alpha_i(b)) \leq \alpha_j(b) \leq \zeta_{ij}^{-1}(\alpha_i(b))$, for all b in $(\underline{c}\,, \eta]$ and for all $1 \leq i, j \leq n$. Assume that there exists i such that $\alpha_i(\underline{c}\,) = \underline{c}$. It suffices to make b tend towards $\underline{c}$ in the previous

inequalities and to use the equalities $\zeta_{ji}(\underline{c}) = \underline{c}$ and $\zeta_{ij}^{-1}(\underline{c}) = \underline{c}$, to find the result $\alpha_j(\underline{c}) = \underline{c}$, for all $1 \leq$ i, j $\leq$ n. ||

Before obtaining results pertaining to type II solutions, we prove Lemmas A2-5 and A2-6 below.

Lemma A2-5: Let the assumptions of Section 2 be satisfied. Let $(\alpha_1, \ldots, \alpha_n)$ be a solution over an interval $(\gamma, \gamma']$, with $\underline{c} \leq \gamma < \gamma' < \overline{c}$, of the differential system (2) considered in the domain D. Suppose that there exists $1 \leq i \leq n$ such that $\frac{d}{db}\alpha_i(b) > 0$, for all b in $(\gamma, \gamma']$. Then the functions $\phi_{ji} = \alpha_j\alpha_i^{-1}$, $1 \leq j \leq n$ and $j \neq i$, and $\beta_i = \alpha_i^{-1}$ are differentiable and solutions over the interval $(\alpha_i(\gamma), \alpha_i(\gamma')]$ of the following system of differential equations considered in the domain $D_i = \Big\{(v, (\phi_{ji})_{j\neq i}, \beta_i) \mid \underline{c} < v \leq \overline{c}, \underline{c} < \phi_{ji} \leq \overline{c}, \beta_i < \phi_{ji},$ $\beta_i < v$, for all $1 \leq j \leq n$ such that $j \neq i$, and $\frac{(-1)(n-2)}{v-\beta_i(v)} + \sum_{\substack{l=1\\l\neq i}}^{n} \frac{1}{\phi_{li}(v)-\beta_i(v)} > 0 \Big\}$,

$$(A2.8) \quad \frac{d}{dv}\phi_{ji}(v) = \frac{f_i(v)}{F_i(v)} \frac{F_j(\phi_{ji}(v))}{f_j(\phi_{ji}(v))} \frac{\frac{(-1)(n-2)}{\phi_{ji}(v)-\beta_i(v)} + \sum_{\substack{l=1\\l\neq j}}^{n} \frac{1}{\phi_{li}(v)-\beta_i(v)}}{\frac{(-1)(n-2)}{v-\beta_i(v)} + \sum_{\substack{l=1\\l\neq i}}^{n} \frac{1}{\phi_{li}(v)-\beta_i(v)}}, \quad 1 \leq j \leq n \text{ and } j \neq i,$$

$$(A2.9) \quad \frac{d}{dv}\beta_i(v) = \frac{f_i(v)}{F_i(v)} \frac{(n-1)}{\frac{(-1)(n-2)}{v-\beta_i(v)} + \sum_{\substack{l=1\\l\neq i}}^{n} \frac{1}{\phi_{li}(v)-\beta_i(v)}}.$$

Inversely, if $((\phi_{ji})_{j\neq i}, \beta_i)$ is a solution over an interval $(w, w']$, with $\underline{c} \leq w < w' \leq \overline{c}$, of the system (A2.8, A2.9) considered on the domain D_i, then $\alpha_j = \phi_{ji}\beta_i^{-1}$, for $j \neq i$, and $\alpha_i = \beta_i^{-1}$, are differentiable and form a solution over the interval $(\beta_i(w), \beta_i(w')]$ of the system (2) considered in the domain D.

Proof: Let $(\alpha_1, \ldots, \alpha_n)$ be a solution over an interval $(\gamma, \gamma']$, with $\underline{c} \leq \gamma < \gamma' \leq \overline{c}$, of the differential system (2) considered in the domain D and let i be between 1 and n such that $\frac{d}{db}\alpha_i(b) > 0$, for all b in $(\gamma, \gamma']$. The function α_i is thus strictly increasing with a derivative strictly positive over the interval $(\gamma, \gamma']$. As a consequence, the functions $\beta_i = \alpha_i^{-1}$ and $\phi_{ji} = \alpha_j\alpha_i^{-1}$, $1 \leq j \leq n$ and $j \neq i$, are differentiable over the interval $(\alpha_i(\gamma), \alpha_i(\gamma')]$ and from the equation in (2) corresponding to α_i, we see that $((\phi_{ji})_{j\neq i}, \beta_i)$ lies in D_i. Moreover, $\frac{d}{dv}\phi_{ji}(v) = [\frac{d}{dv}\alpha_j(b)]_{b=\alpha_i{-1}(v)} \frac{1}{[\frac{d}{dv}\alpha_i(b)]_{b=\alpha_i{-1}(v)}}$, for all v in $(\alpha_i(\gamma), \alpha_i(\gamma')]$. It then suffices to substitute to $\frac{d}{dv}\alpha_j(b)$ and $\frac{d}{dv}\alpha_i(b)$ the expressions given in equations (2) in order to find equations (A2.8). The equation (A2.9) can be proved similarly.

Let $((\phi_{ji})_{j\neq i}, \beta_i)$ be a solution over an interval (w, w'], with $\underline{c} \leq w < w' \leq \overline{c}$, of the system (A2.8)-(A2.9) considered on the domain D_i. From equation (A2.9) and the definition of D_i, we see that $\frac{d}{dv}\beta_i(v) > 0$, for all v in (w, w']. Consequently, the functions $\alpha_j = \phi_{ji}\beta_i^{-1}$, for $j \neq i$, and $\alpha_i = \beta_i^{-1}$ are differentiable over the interval $(\beta_i(w), \beta_i(w')]$. By applying the formula $\frac{d}{dv}\alpha_j(b) = [\frac{d}{dv}\phi_{ji}(v)]_{v=\beta_i^{-1}(v)} \frac{1}{[\frac{d}{dv}\beta_i(v)]_{v=\beta_i^{-1}(v)}}$ and the equations (A2.8, A2.9), we find that $\alpha_j = \phi_{ji}\beta_i^{-1}$, $j \neq i$, and $\alpha_i = \beta_i^{-1}$ form a solution over $(\beta_i(w), \beta_i(w')]$ of (2) in D. ‖

Through the change of variables $(p, (\chi_{ji})_{j\neq i}, \rho_i) = (F_i(b), (F_j(\phi_{ji}(F_i^{-1})))_{j\neq i}, \beta_i(F_i^{-1}))$ and its inverse $(v, (\phi_{ji})_{j\neq i}, \beta_i) = (F_i^{-1}(p), (F_j^{-1}(\chi_{ji}(F_i)))_{j\neq i}, \rho_i(F_i))$, the system (A2.8, A2.9) in the domain D_i is equivalent to the system (A2.10, A2.11) in the domain $\mathcal{D}_i = \Big\{(p, (\chi_{ji})_{j\neq i}, \rho_i) \mid 0 < p \leq 1,\ 0 < \chi_{ji} \leq 1,\ \rho_i < \chi_{ji},\ \rho_i < F_i^{-1}(p)$, for all $1 \leq j \leq n$ such that $j \neq i$, and $\frac{(-1)(n-2)}{F_i^{-1}(p)-\rho_i(p)} + \sum_{\substack{l=1\\ l\neq i}}^{n} \frac{1}{\chi_{li}(p)-\rho_i(v)} > 0 \Big\}$,

$$\text{(A2.10)} \quad \frac{d}{dp}\chi_{ji}(p) = \frac{\chi_{ji}(p)}{p} \frac{\frac{(-1)(n-2)}{F_j^{-1}(\chi_{ji}(p))-\rho_i(p)} + \sum_{\substack{l=1\\ l\neq j}}^{n} \frac{1}{F_l^{-1}(\chi_{li}(p))-\rho_i(p)}}{\frac{(-1)(n-2)}{F_i^{-1}(p)-\rho_i(p)} + \sum_{\substack{l=1\\ l\neq i}}^{n} \frac{1}{F_l^{-1}(\chi_{li}(p))-\rho_i(p)}}, \quad 1 \leq j \leq n \text{ and } j \neq i,$$

$$\text{(A2.11)} \quad \frac{d}{dp}\rho_i(p) = \frac{1}{p} \frac{(n-1)}{\frac{(-1)(n-2)}{F_i^{-1}(p)-\rho_i(p)} + \sum_{\substack{l=1\\ l\neq i}}^{n} \frac{1}{F_l^{-1}(\chi_{li}(p))-\rho_i(p)}}.$$

The system (A2.10, A2.11) in its domain satisfies the standard requirements from the theory of ordinary differential equations and we are thus able to apply the results of this theory through (A2.10, A2.11) to (A2.8, A2.9).

__Lemma__ __A2-6__: Let the assumptions of Section 2 be satisfied. Let $(\alpha_1, \ldots, \alpha_n)$ be a solution over an interval $(\gamma, \gamma']$, with $\underline{c} \leq \gamma < \gamma' < \overline{c}$, of the differential system (2) considered in the domain D. Suppose that there exists $1 \leq i \leq n$ such that $\frac{d}{db}\alpha_i(b) > 0$, for all b in $(\gamma, \gamma']$. Then the functions $\beta_i = \alpha_i^{-1}$ and $\phi_{ji} = \alpha_j\alpha_i^{-1}$, $j \neq i$, are differentiable over the interval $(\alpha_i(\gamma), \alpha_i(\gamma')]$ and we have

$$\text{(A2.12)} \quad \frac{d}{dv}\Big\{ (v - \beta_i(v)) \prod_{\substack{k=1\\ k\neq i}}^{n} F_k(\phi_{ki}(v)) \Big\} = \prod_{\substack{k=1\\ k\neq i}}^{n} F_k(\phi_{ki}(v)),$$

for all v in $(\alpha_i(\gamma), \alpha_i(\gamma')]$.

Proof: Since $\frac{d}{db}\alpha_i(b) > 0$, for all b in $(\gamma, \gamma']$, the functions $\beta_i = \alpha_i^{-1}$ and $\phi_{ji} = \alpha_j\alpha_i^{-1}$, $j \neq i$, are differentiable over the interval $(\alpha_i(\gamma), \alpha_i(\gamma')]$. The function $\sum_{\substack{k=1\\k\neq i}}^{n} \ln F_k(\phi_{ki}(v))$ is differentiable over the same interval and its derivative is given by $[\frac{d}{db} \sum_{\substack{k=1\\k\neq i}}^{n} \ln F_k(\alpha_k(b))]_{b=\beta_i(v)} \frac{d}{dv}\beta_i(v)$. Using equation (A2.1) with $b = \beta_i(v)$, we obtain

$$\frac{d}{dv} \ln \prod_{\substack{k=1\\k\neq i}}^{n} F_k(\phi_{ki}(v)) = \frac{d}{dv}\beta_i(v) \frac{1}{v-\beta_i(v)},$$

all v in $(\alpha_i(\gamma), \alpha_i(\gamma')]$. Substituting $\{ \frac{d}{dv} \prod_{\substack{k=1\\k\neq i}}^{n} F_k(\phi_{ki}(v))\} / \prod_{\substack{k=1\\k\neq i}}^{n} F_k(\phi_{ki}(v))$ to $\frac{d}{dv} \ln \prod_{\substack{k=1\\k\neq i}}^{n} F_k(\phi_{ki}(v))$ and rearranging, we find

$$\frac{d}{dv} \{ (v - \beta_i(v)) \prod_{\substack{k=1\\k\neq i}}^{n} F_k(\phi_{ki}(v)) \} = \prod_{\substack{k=1\\k\neq i}}^{n} F_k(\phi_{ki}(v)),$$

for all v in $(\alpha_i(\gamma), \alpha_i(\gamma')]$, and Lemma A2-6 is proved. ||

We now state and prove Lemma A2-7 from which properties of type II solutions can be derived.

Lemma A2-7: Let the assumptions of Section 2 be satisfied. Let η and $\underline{\gamma}$ be such that $\underline{c} \leq \underline{\gamma} < \eta < \overline{c}$ and let $(\alpha_1, \ldots, \alpha_n)$ be a solution over the interval $(\underline{\gamma}, \eta]$ of the differential system (2) considered in the domain D with initial condition (19). If there exists j such that $\alpha_j(\underline{\gamma}) > \underline{\gamma}$, then $F_k(\alpha_k(\underline{\gamma})) > 0$, for all $k \neq i$. If there exists i such that $\alpha_i(\underline{\gamma}) = \underline{\gamma}$, then $\alpha_k(\underline{\gamma}) = \underline{\gamma}$, for all but at most one k between 1 and n.

Proof: Assume that there exists j such that $\alpha_j(\underline{\gamma}) > \underline{\gamma}$. From equation (A2.1), we see that the limit of $\frac{d}{db} \sum_{\substack{k=1\\k\neq j}}^{n} \ln F_k(\alpha_k(b))$ for $b \underset{>}{\rightarrow} \gamma$ exists and is equal to $\frac{1}{\alpha_j(\underline{\gamma})-\underline{\gamma}}$. Since $\frac{d}{db} \ln F_k(\alpha_k(b))$ is strictly positive over $(\underline{\gamma}, \eta]$, we see that every term in the sum $\sum_{\substack{k=1\\k\neq j}}^{n} \frac{d}{db} \ln F_k(\alpha_k(b))$ is bounded for $b \underset{>}{\rightarrow} \underline{\gamma}$. If there existed $k \neq j$ such that $F_k(\alpha_k(\underline{\gamma})) = 0$, we would have $\ln F_k(\alpha_k(b)) \rightarrow -\infty$, for $b \underset{>}{\rightarrow} \underline{\gamma}$, and $\frac{d}{db} \ln F_k(\alpha_k(b))$ could not be bounded, for $b \underset{>}{\rightarrow} \underline{\gamma}$. The first part of Lemma A2-7 is thus proved.

Assume next that there exist i and j such that $\alpha_i(\underline{\gamma}) = \underline{\gamma}$, $\alpha_j(\underline{\gamma}) \neq \underline{\gamma}$ (and thus $i \neq j$). Because $(b, \alpha_1, \ldots, \alpha_n)$ lies in the domain D over the interval $(\underline{\gamma}, \eta]$, we have $\alpha_j(\underline{\gamma}) > \underline{\gamma}$.

From the previous paragraph, $\frac{d}{db}$ ln $F_k(\alpha_k(b))$ is bounded, for b $\underset{>}{\rightarrow}$ $\underline{\gamma}$ and all k $\neq$ j. From equation (A2.2) and because $\frac{d}{db}$ ln $F_k(\alpha_k(b))$ and $\frac{d}{db}$ ln $F_i(\alpha_i(b))$ are bounded for b $\underset{>}{\rightarrow}$ $\underline{\gamma}$ and for k $\neq$ i, j, we see that $\frac{1}{\alpha_i(b)-b} - \frac{1}{\alpha_k(b)-b}$ is also bounded for b $\underset{>}{\rightarrow}$ $\underline{\gamma}$. Consequently, $\alpha_k(b) - b$ must tend towards zero as b $\underset{>}{\rightarrow}$ $\underline{\gamma}$,that is, $\alpha_k(\underline{\gamma}) = \underline{\gamma}$, for all k $\neq$ j. We have thus proved that $\alpha_k(\underline{\gamma}) = \underline{\gamma}$, for all but at most one k between 1 and n, and the proof of Lemma A2-7 is complete. ||

We now prove the monotonicity of the solution of (2, 19) with respect to η.

<u>Lemma</u> <u>A2-8</u>: Let the assumptions of Section 2 be satisfied. Let $(\alpha_1, \ldots, \alpha_n)$ be the solution of the problem (2, 19) for a parameter $\underline{c} < \eta < \overline{c}$ and let $(\alpha'_1, \ldots, \alpha'_n)$ be the solution of the problem (2, 19) for a parameter $\underline{c} < \eta' < \overline{c}$ with $\eta' < \eta$. Assume that $(\alpha_1, \ldots, \alpha_n)$ and $(\alpha'_1, \ldots, \alpha'_n)$ are defined over the interval $(\gamma, \eta']$, with $\gamma < \eta'$. Then $\alpha'_i(b) > \alpha_i(b)$, for all b in $(\gamma, \eta']$ and all $1 \leq i \leq n$.

<u>Proof</u>: As in the proof of Lemma A2-3, we see that the functions $\alpha_1, \ldots, \alpha_n$ are strictly increasing. Consequently, $\alpha_i(\eta') < \alpha'_i(\eta') = \overline{c}$, for all $1 \leq i \leq n$. Consider d in $(\gamma, \eta']$, defined as follows, d $=$ inf $\{ b \in [\gamma, \eta'] \mid \alpha'_i(b) > \alpha_i(b)$, for all $1 \leq i \leq n \}$. We have to prove that d $= \gamma$. We already know that d $< \eta'$. Assume that d $> \gamma$. Let $1 \leq i \leq n$ an index such that $\alpha'_i(d) = \alpha_i(d)$. By continuity, there is at least one such index. From the definition of d, we also have $\alpha'_j(d) \geq \alpha_j(d)$, for all $1 \leq j \leq n$. Moreover, there exists j $\neq$ i such that $\alpha'_j(d) > \alpha_j(d)$. In fact, if it was not the case the solutions $(\alpha_1, \ldots, \alpha_n)$ and $(\alpha'_1, \ldots, \alpha'_n)$ of the differential system (2) would be equal at d and would thus be equal over their common definition domain (here we use the uniqueness of the solutions of (18) and, thus, (2) with initial condition), which is impossible since $\alpha_k(\eta') < \alpha'_k(\eta') = \overline{c}$, for all $1 \leq k \leq n$. From equation (2), we see that $\frac{d}{db}\alpha_i(d)$ is a strictly decreasing function of $\alpha_j(d)$, for all j $\neq$ i. Consequently, $\frac{d}{db}\alpha_i(d) > \frac{d}{db}\alpha'_i(d)$. There thus exists $\delta > 0$, such that $\alpha_i(b) > \alpha'_i(b)$, for all b $\in$ $(d, d + \delta)$. However, this contradicts the definition of d and Lemma A2-8 is proved. ||

<u>Lemma</u> <u>A2-9</u>: Let the assumptions of Section 2 be satisfied. The lower extremity $\underline{\gamma}$ of the definition interval of the maximal solution $(\alpha_1, \ldots, \alpha_n)$ of the differential system (2) over $(\underline{c}, \eta]$ considered in the domain D with initial condition (19) is a nondecreasing function of $\eta \in (\underline{c}, \overline{c})$.

<u>Proof:</u> Consider η and η' such that $\underline{c} \leq \eta' < \eta < \overline{c}$. Let $\underline{\gamma}$ be the lower extremity of the definition interval of the maximal solution $(\alpha_1, \ldots, \alpha_n)$ of (2, 19) for the parameter η. Let $(\alpha'_1, \ldots, \alpha'_n)$ be the solution of (2, 19) for the parameter η' and let $\underline{\gamma}'$ be the lower extremity of its maximal definition interval. Suppose that $\underline{\gamma}' > \underline{\gamma}$. In this case, we have $\underline{\gamma}' > \underline{c}$ and from Lemma A2-7 there exists $1 \leq i \leq n$ such that $\alpha'_i(\underline{\gamma}') = \underline{\gamma}'$. From Lemma A2-8, we know that $\alpha'_i > \alpha_i$ over the intersection of their definition intervals. We would thus have $\alpha'_i(b) > \alpha_i(b)$, for all b in $(\underline{\gamma}', \eta]$. By making b tend towards $\underline{\gamma}'$, we obtain $\alpha'_i(\underline{\gamma}') = \underline{\gamma}' \geq \alpha_i(\underline{\gamma}')$, which is impossible since $\underline{\gamma}'$ belongs to the definition interval of the maximal

definition interval of $(\alpha_1, \ldots, \alpha_n)$ and $\alpha_i(\text{b}) > \text{b}$ everywhere over this interval. We have thus proved that $\underline{\gamma}' \leq \underline{\gamma}$ and that the lower extremity of the maximal definition interval is a nondecreasing function of η. ||

As defined before Lemma A2-4, Λ_{I} is the set of parameters η corresponding to type I solutions.

Lemma A2-10: Let the assumptions of Section 2 be satisfied. Assume further that $F_1(\underline{c}) = \ldots = F_n(\underline{c}) = 0$. Let Λ'_{I} be the set of all parameters $\underline{c} < \eta < \overline{c}$ in Λ_{I} such that not all values $\alpha_1(\underline{c}), \ldots, \alpha_n(\underline{c})$ of the maximal solution $(\alpha_1, \ldots, \alpha_n)$ of the differential system (2) over $(\underline{c}, \eta]$ considered in the domain D with initial condition (19) are equal to $\underline{c}$. Then Λ'_{I} is an open set.

Proof: Let η be an element of Λ'_{I}. The definition interval of the maximal solution $(\alpha_1, \ldots, \alpha_n)$ of the differential system (2) considered in the domain D with initial condition (19) is thus equal to $(\underline{c}, \eta]$ and we have (see Lemma A2-4) $\alpha_1(\underline{c}), \ldots, \alpha_n(\underline{c}) > \underline{c}$. Take i such that $\min_{1 \leq l \leq n} \alpha_l(\underline{c}) = \alpha_i(\underline{c})$. From Lemma A2-5, we know that $\beta_i = \alpha_i^{-1}$ and $\phi_{ji} = \alpha_j \alpha_i^{-1}$, $1 \leq j \leq n$ and $j \neq i$, form a solution over $(\alpha_i(\underline{c}), \overline{c}\,]$ of the differential system (A2.8, A2.9) considered in D_i such that $\beta_i(\overline{c}) = \eta$ and $\phi_{ji}(\overline{c}) = \overline{c}$, $j \neq i$. We see that $(\text{v}, (\phi_{ji}(\text{v}))_{j \neq i}, \beta_i(\text{v}))$, for $\text{v} = \alpha_i(\underline{c})$, belongs to the domain D_i. In fact, $\beta_i(\alpha_i(\underline{c})) = \underline{c}$, $\phi_{ji}(\alpha_i(\underline{c})) = \alpha_j(\underline{c})$, $j \neq i$, and the last denominators in the R.H.S.'s of (A2.8) and of (A2.9) are equal to $\frac{(-1)(\text{n}-2)}{\alpha_i(\underline{c})-\underline{c}} + \sum_{\substack{l=1 \\ l \neq i}}^{n} \frac{1}{\alpha_l(\underline{c})-\underline{c}} = \frac{1}{\alpha_j(\underline{c})-\underline{c}} + \sum_{\substack{l=1 \\ l \neq i,j}}^{n} \left\{ \frac{1}{\alpha_l(\underline{c})-\underline{c}} - \frac{1}{\alpha_i(\underline{c})-\underline{c}} \right\} > 0$, since $\alpha_i(\underline{c}) \leq \alpha_l(\underline{c})$, for all $l \neq i$. From the theory of ordinary differential equations applied to the equivalent system (A2.10, A2.11), we thus see that the solution $((\phi_{ji})_{j \neq i}, \beta_i)$ can be continued beyond $\alpha_i(\underline{c})$ over an interval $(\underline{\text{v}}, \overline{c}\,]$, with $\underline{c} < \underline{\text{v}} < \alpha_i(\underline{c})$.

Let ϵ be an arbitrary strictly positive number. By decreasing ϵ if necessary, we can assume that $\epsilon < \alpha_i(\underline{c}) - \underline{\text{v}}$. Consider $\alpha_i(\underline{c}) - \epsilon > \underline{\text{v}}$. Since $\beta_i(\alpha_i(\underline{c})) = \underline{c}$ and β_i is strictly increasing, we have $\beta_i(\alpha_i(\underline{c}) - \epsilon) < \underline{c}$. From the theory of ordinary differential equations applied to the equivalent system (A2.10, A2.11), there exists $\delta > 0$ such that if $\eta < \eta' < \eta + \delta$, then the solution $((\phi'_{ji})_{j \neq i}, \beta'_i)$ corresponding to η' is defined at $\alpha_i(\underline{c}) - \epsilon$ and $\beta'_i(\alpha_i(\underline{c}) - \epsilon) < \underline{c}$. From Lemma A2-5, $\alpha'_j = \phi'_{ji} \beta'^{-1}_i$, for $j \neq i$, and $\alpha'_i = \beta'^{-1}_i$ form a solution of (2, 19) for the parameter η' and is defined at $\beta'_i(\alpha_i(\underline{c}) - \epsilon) < \underline{c}$. Consequently, $(\alpha'_1, \ldots, \alpha'_n)$ is not of type II, is of type I and $\alpha'_1(\underline{c}), \ldots, \alpha'_n(\underline{c}) \geq \underline{c}$. Since α'_i is strictly increasing, we have $\alpha'_i(\beta'_i(\alpha_i(\underline{c}) - \epsilon)) = \alpha_i(\underline{c}) - \epsilon < \alpha'_i(\underline{c})$. Since $\alpha_i(\underline{c}) - \epsilon > \underline{\text{v}} > \underline{c}$, we have proved that if $\eta < \eta' < \eta + \delta$ then $\alpha'_i(\underline{c}) > \underline{c}$ and $\eta' \in \Lambda'_{\text{I}}$.

If $\eta' < \eta$, Lemma A2-9 implies that $\underline{\gamma}' = \underline{c}$ and $(\alpha_1', \ldots, \alpha_n')$ is of type I. Moreover, from Lemma A2-8 we have $\alpha_k(\underline{c}) \geq \alpha_k(\underline{c}) > \underline{c}$, for all k, and η' belongs to Λ'_{I}. The openness of Λ'_{I} follows[21]. ||

In the two next lemmas, we prove the continuity of the lower extremity of the maximal definition interval of the solution of (2, 19) with respect to η.

<u>Lemma</u> <u>A2-11</u>: Let the assumptions of Section 2 be satisfied. The lower extremity $\underline{\gamma}$ of the definition interval of the maximal solution $(\alpha_1, \ldots, \alpha_n)$ of the differential system (2) over $(\underline{c}$, $\eta]$ considered in the domain D with initial condition (19) is a function of $\eta \in (\underline{c}, \overline{c})$ which is continuous from the right at every η for which there exists $1 \leq i \leq n$ such that $\alpha_i(\underline{\gamma}) = \underline{\gamma}$. In particular, $\underline{\gamma}$ is continuous to the right at η, for all η in Λ_{II}.

<u>Proof:</u> Let η be an element of $(\underline{c}, \overline{c})$, $\underline{\gamma}$ the lower extremity of the maximal definition interval of the solution of (2, 19) corresponding to η and i an index such that $\alpha_i(\underline{\gamma}) = \underline{\gamma}$. Let ϵ be an arbitrary strictly positive number. Without loss of generality, we can assume that $\epsilon < \overline{c} - \underline{\gamma}$. From Lemma A2-5, the n-tuple $((\phi_{ji})_{j\neq i}, \beta_i)$, where $\beta_i = \alpha_i^{-1}$ and $\phi_{ji} = \alpha_j\alpha_i^{-1}$, j $\neq$ i, is a solution of (A2.8, A2.9) over $(\underline{\gamma}, \overline{c}]$ such that $\phi_{ji}(\overline{c}) = \overline{c}$, j $\neq$ i, $\beta_i(\overline{c}) = \eta$. Obviously, $\underline{\gamma} + \epsilon$ belongs to the definition interval $(\underline{\gamma}, \overline{c}]$. From the theory of ordinary differential equations applied to the equivalent system (A210, A211), we know that there exists $\delta > 0$ such that if $\eta < \eta' < \eta + \delta$ then the solution $((\phi'_{ji})_{j\neq i}, \beta'_i)$ of (A2.8, A2.9) with initial condition $\phi'_{ji}(\overline{c}) = \overline{c}$, j $\neq$ i, $\beta'_i(\overline{c}) = \eta'$, is also defined at $\underline{\gamma} + \epsilon$. From Lemma A2-5 again, $\alpha'_j = \phi'_{ji}\beta'^{-1}_i$, for j $\neq$ i, and $\alpha'_i = \beta'^{-1}_i$ form the solution of (A2.8, A2.9) corresponding to η' and this solution is defined at $\beta_i'(\underline{\gamma} + \epsilon)$. Since $\beta_i'(\underline{\gamma} + \epsilon) \leq \underline{\gamma} + \epsilon$, we see that $\underline{\gamma} + \epsilon$ also belongs to the definition interval of $(\alpha_1', \ldots, \alpha_n')$. Consequently, $\underline{\gamma}' \leq \underline{\gamma} + \epsilon$. From Lemma A2-9, we know that $\underline{\gamma} \leq \underline{\gamma}'$. Consequently, $|\underline{\gamma}' - \underline{\gamma}| \leq \epsilon$ and $\underline{\gamma}$ is continuous from the right with respect to η. ||

<u>Lemma</u> <u>A2-12</u>: Let the assumptions of Section 2 be satisfied. Assume further that $F_1(\underline{c}) = \ldots = F_n(\underline{c}) = 0$. The lower extremity $\underline{\gamma}$ of the definition interval of the maximal solution $(\alpha_1, \ldots, \alpha_n)$ of the differential system (2) over $(\underline{c}$, $\eta]$ considered in the domain D with initial condition (19) is a nondecreasing continuous function of $\eta \in (\underline{c}, \overline{c})$. The set Λ_{II} of parameters in $(\underline{c}, \overline{c})$ corresponding to type II solutions is open.

<u>Proof</u>: Let η be an element of $(\underline{c}, \overline{c})$ and let $\underline{\gamma}$ be the lower extremity of the definition interval of the maximal solution $(\alpha_1, \ldots, \alpha_n)$ of the differential system (2) considered in D with initial condition (19). We know from Lemma A2-9 that $\underline{\gamma}$ is a nondecreasing function of η. We prove that $\underline{\gamma}$ considered as a function of η is continuous from the left. Suppose first that $\underline{\gamma} = \underline{c}$, that is, that the solution $(\alpha_1, \ldots, \alpha_n)$ is of type I or, equivalently, that $\eta \in \Lambda_{\mathrm{I}}$. From Lemma A2-9, we see that the lower extremity $\underline{\gamma}'$ of the maximal definition interval of the solution $(\alpha'_1, \ldots, \alpha'_n)$ corresponding to $\eta' < \eta$ is not larger than $\underline{c}$ and thus is equal to $\underline{c}$. The lower extremity of the maximal definition interval is equal to $\underline{c}$ for all $\eta' < \eta$ and is thus continuous from the left at η.

Suppose next that $\underline{\gamma} > \underline{c}$, that is, that $(\alpha_1, \ldots, \alpha_n)$ is of type II. Let ϵ be an arbitrary strictly positive number. By decreasing ϵ if necessary, we can assume that $\epsilon < \underline{\gamma} - \underline{c}$. From Lemma A2-7, we have $\alpha_i(\underline{\gamma}) = \underline{\gamma}$, for all but possibly one α_i. Let $1 \leq$ i $\leq$ n be such that $\alpha_i(\underline{\gamma}) = \underline{\gamma}$. Let δ' be defined as follows,

$$\delta' = \int_{\underline{\gamma}-\epsilon}^{\underline{\gamma}} \prod_{\substack{j=1 \\ j\neq i}}^{n} \mathrm{F}_j(\zeta_{ji}(\mathrm{v}))\, \mathrm{dv}.$$

Since ζ_{ji} is strictly larger than $\underline{\mathrm{c}}$ over $(\underline{\mathrm{c}}, \overline{\mathrm{c}})$, we see that $\delta' > 0$. Since $\beta_i = \alpha_i^{-1}$, we see that $\beta_i(\underline{\gamma}) = \underline{\gamma}$. Let w be an element of $(\underline{\gamma}, \overline{\mathrm{c}})$ such that

(A2.13) $\;| \beta_i(\mathrm{w}) - \mathrm{w} | < \delta'/2.$

From Lemma A2-5, the n-tuple $((\phi_{ji})_{j\neq i}, \beta_i)$, where $\phi_{ji} = \alpha_j\alpha_i^{-1}$, j ≠ i, is a solution of (A2.8, A2.9) over $(\underline{\gamma}, \overline{\mathrm{c}}\,]$ such that $\phi_{ji}(\overline{\mathrm{c}}) = \overline{\mathrm{c}}$, j ≠ i, $\beta_i(\overline{\mathrm{c}}) = \eta$. From the continuity of the solution of a differential equation with respect to the initial condition applied to the system (A2.10, A2.11), we see that there exists $\delta > 0$ such that if $\eta - \delta < \eta' < \eta$, then the solution $((\phi'_{ji})_{j\neq i}, \beta'_i)$ of (A2.8, A2.9) such that $\phi'_{ji}(\overline{\mathrm{c}}) = \overline{\mathrm{c}}$, j ≠ i, $\beta'_i(\overline{\mathrm{c}}) = \eta'$, is defined at w and thus w $\geq \alpha'_i(\underline{\gamma}')$ and

(A2.14) $\;| \beta_i(\mathrm{w}) - \beta'_i(\mathrm{w}) | < \delta'/2.$

Consider η' such that $\eta - \delta < \eta' < \eta$ and $\underline{\gamma}'$ the lower extremity of the maximal interval of definition of the solution $(\alpha'_1, \ldots, \alpha'_n)$ corresponding to η'. Because $\underline{\gamma}$ is a nondecreasing function of η, we have $\underline{\gamma}' \leq \underline{\gamma}$. From Lemma A2-5, the n-tuple $((\phi'_{ji})_{j\neq i}$, $\beta'_i)$, where $\beta'_i = \alpha'^{-1}_i$ and $\phi'_{ji} = \alpha'_j\alpha'^{-1}_i$, j ≠ i, is a solution of (A2.8, A2.9) over $(\alpha'_i(\underline{\gamma}'), \overline{\mathrm{c}}\,]$ such that $\phi'_{ji}(\overline{\mathrm{c}}) = \overline{\mathrm{c}}$, j ≠ i, $\beta'_i(\overline{\mathrm{c}}) = \eta'$. From Lemma A2-6 we have

$$\frac{\mathrm{d}}{\mathrm{dv}}\left\{ (\mathrm{v} - \beta'_i(\mathrm{v})) \prod_{\substack{j=1 \\ j\neq i}}^{n} \mathrm{F}_j(\phi'_{ji}(\mathrm{v})) \right\} = \prod_{\substack{j=1 \\ j\neq i}}^{n} \mathrm{F}_j(\phi'_{ji}(\mathrm{v})),$$

for all v in $(\alpha'_i(\underline{\gamma}'), \overline{\mathrm{c}}\,]$. As a consequence, we have

$$\int_{\alpha_i(\underline{\gamma}')}^{w} \prod_{\substack{j=1 \\ j\neq i}}^{n} \mathrm{F}_j(\phi'_{ji}(\mathrm{v}))\, \mathrm{dv} = (\mathrm{w} - \beta'_i(\mathrm{w})) \prod_{\substack{j=1 \\ j\neq i}}^{n} \mathrm{F}_j(\phi'_{ji}(\mathrm{w})) - (\underline{\gamma}' - \beta'_i(\underline{\gamma}')) \prod_{\substack{j=1 \\ j\neq i}}^{n} \mathrm{F}_j(\phi'_{ji}(\underline{\gamma}')),$$

and

$$\int_{\alpha'_i(\underline{\gamma}')}^{w} \prod_{\substack{j=1 \\ j\neq i}}^{n} \mathrm{F}_j(\phi'_{ji}(\mathrm{v}))\, \mathrm{dv} \leq (\mathrm{w} - \beta'_i(\mathrm{w})).$$

From (A2.13) and (A2.14) we have $(\mathrm{w} - \beta'_i(\mathrm{w})) < \delta'$ and we obtain $\int_{\alpha'_i(\underline{\gamma}')}^{\underline{\gamma}} \prod_{\substack{j=1 \\ j\neq i}}^{n} \mathrm{F}_j(\phi'_{ji}(\mathrm{v}))$ dv $< \delta'$ and, since $\phi'_{ji} \geq \zeta_{ji}$,

$$\int_{\alpha'_i(\underline{\gamma}')}^{\underline{\gamma}} \prod_{\substack{j=1 \\ j \neq i}}^{n} \mathrm{F}_j(\zeta_{ji}(\mathrm{v}))\, \mathrm{dv} \; < \; \delta'.$$

From the definition of δ', we have

$$\int_{\alpha'_i(\underline{\gamma}')}^{\underline{\gamma}} \prod_{\substack{j=1 \\ j \neq i}}^{n} \mathrm{F}_j(\zeta_{ji}(\mathrm{v}))\, \mathrm{dv} \; < \; \int_{\underline{\gamma}-\epsilon}^{\underline{\gamma}} \prod_{\substack{j=1 \\ j \neq i}}^{n} \mathrm{F}_j(\zeta_{ji}(\mathrm{v}))\, \mathrm{dv}.$$

This last inequality is possible only if $\alpha'_i(\underline{\gamma}') > \underline{\gamma} - \epsilon$ and we have proved that $\lim_{\eta' \underset{<}{\to} \eta} \alpha'_i(\underline{\gamma}') = \underline{\gamma}$ and thus (as we show in the Addendum) the continuity from the left of the lower extremity of the interval of the maximal solution of (2, 19) with respect to η.

From Lemma A2-11, the lower extremity is continuous from the right at all η in Λ_{II}. It is thus continuous at all η in Λ_{II}. The openness of Λ_{II} then follows immediately from its definition.

Let η^{**} be the infimum of Λ_{II}. From Lemma A2-9 and from the openness of Λ_{II}, we have $\Lambda_{\mathrm{II}} = (\eta^{**}, \overline{c})$. If $\eta^{**} = \underline{c}$, Lemma A2-12 is proved. If $\eta^{**} = \overline{c}$, the lower extremity is always equal to $\underline{c}$, is continuous and Lemma A2-12 is proved. Assume $\underline{c} < \eta^{**} < \overline{c}$. We have $\Lambda_{\mathrm{I}} = (\underline{c}, \eta^{**}]$. From Lemma A2-10, $\Lambda'_{\mathrm{I}} \subseteq \Lambda_{\mathrm{I}}$ is an open set. Consequently $\eta^{**} \notin \Lambda'_{\mathrm{I}}$ and $\alpha_1(\underline{c}) = \ldots = \alpha_n(\underline{c}) = \underline{c}$, where $(\alpha_1, \ldots, \alpha_n)$ is the solution of (2, 19) corresponding to η^{**}. From Lemma A2-11, the lower extremity is continuous from the right at η^{**}. We have proved that $\underline{\gamma}$ is continuous from the left and from the right and is thus continuous at all η in $[\eta^{**}, \overline{c})$. The lower extremity $\underline{\gamma}$ is constant over $(\underline{c}, \eta^{**})$ (it is equal to $\underline{c}$). It is thus continuous over $(\underline{c}, \overline{c})$ and Lemma A2-12 is proved. ||

Lemma _A2-13_: *Let the assumptions of Section 2 be satisfied. If* $\eta \in \left(\underline{c}, \overline{c} - \max_{1 \le i \le n} \int_{\underline{c}}^{\overline{c}} \prod_{\substack{j=1 \\ j \neq i}}^{n} F_j(\zeta_{ij}^{-1}(v))\, dv\right)$, *then the solution of the problem (2, 19) is of type I, that is,*

$\underline{\gamma} = \underline{c}$. *If* $\eta \in \left(\overline{c} - \min_{1 \le i \le n} \int_{\underline{c}}^{\overline{c}} \prod_{\substack{j=1 \\ j \neq i}}^{n} F_j(\underline{\zeta}_{ji}(v))\, dv, \overline{c}\right)$, *where* $\underline{\zeta}_{ji} = \zeta_{ji}$ *over* $[w_{ji}, \overline{c}]$

and

$\underline{\zeta}_{ji} = \underline{c}$ *over* $[\underline{c}, w_{ji}]$, *for all* $1 \le j, i \le n$, *then the solution of the problem (2, 19) is of type II, that is,* $\underline{\gamma} > \underline{c}$. *Moreover, if* η *tends towards* $\overline{c}$ *then* $\underline{\gamma}$ *tends towards* $\overline{c}$.

Proof: Take η in $\left(\underline{c}, \overline{c} - \max_{1 \le i \le n} \int_{\underline{c}}^{\overline{c}} \prod_{\substack{j=1 \\ j \neq i}}^{n} \mathrm{F}_j(\zeta_{ij}^{-1}(\mathrm{v}))\, \mathrm{dv}\right)$ and assume that the corresponding solution is of type II, that is, $\underline{\gamma} > \underline{c}$. For all i's but possibly one, we have $\alpha_i(\underline{\gamma}) = \underline{\gamma}$. Let i be such an index. From Lemma A2-6, we have $\int_{\underline{\gamma}}^{\overline{c}} \prod_{\substack{j=1\\j\neq i}}^{n} F_j(\phi_{ji}(v))\, dv = [\, (v - \beta_i(v)) \prod_{\substack{j=1\\j\neq i}}^{n} F_j(\phi_{ji}(v))\,]_{v=\underline{\gamma}}^{v=\overline{c}}$ and thus

(A2.15) $$\eta = \overline{c} - \int_{\underline{\gamma}}^{\overline{c}} \prod_{\substack{j=1\\j\neq i}}^{n} F_j(\phi_{ji}(v))\, dv.$$

From Lemma A2-3, we know that $\phi_{ji} \leq \zeta_{ij}^{-1}$, for all j, i. Consequently, we obtain

$$\eta \geq \overline{c} - \int_{\underline{\gamma}}^{\overline{c}} \prod_{\substack{j=1\\j\neq i}}^{n} F_j(\zeta_{ji}^{-1}(v))\, dv,$$

which contradicts the choice of η. We thus have $\underline{\gamma} = \underline{c}$ and the first part of Lemma A2-11 is proved.

Take η in $(\, \overline{c} - \min_{1 \leq i \leq n} \int_{\underline{c}}^{\overline{c}} \prod_{\substack{j=1\\j\neq i}}^{n} F_j(\underline{\zeta}_{ji}(v))\, dv, \overline{c}\,)$ and assume that the corresponding solution is of type I, that is, $\underline{\gamma} = \underline{c}$. For all i's, we have $\alpha_i(\underline{c}) \geq \underline{c}$. Let i be an arbitrary index. From Lemma A2-6 again, we have $\int_{\alpha_i(\underline{c})}^{\overline{c}} \prod_{\substack{j=1\\j\neq i}}^{n} F_j(\phi_{ji}(v))\, dv = [\, (v - \beta_i(v)) \prod_{\substack{j=1\\j\neq i}}^{n} F_j(\phi_{ji}(v))\,]_{v=\alpha_i(\underline{c})}^{v=\overline{c}}$ and thus

$$(\overline{c} - \eta) - (\alpha_i(\underline{c}) - \underline{c}) \prod_{\substack{j=1\\j\neq i}}^{n} F_j(\phi_{ji}(\alpha_i(\underline{c}))) = \int_{\alpha_i(\underline{c})}^{\overline{c}} \prod_{\substack{j=1\\j\neq i}}^{n} F_j(\phi_{ji}(v))\, dv.$$

From Lemma A2-3, we know that $\phi_{ji} \geq \underline{\zeta}_{ji}$, for all j, i. Consequently, we obtain $(\overline{c} - \eta) \geq (\alpha_i(\underline{c}) - \underline{c}) \prod_{\substack{j=1\\j\neq i}}^{n} F_j(\underline{\zeta}_{ji}(\alpha_i(\underline{c}))) + \int_{\alpha_i(\underline{c})}^{\overline{c}} \prod_{\substack{j=1\\j\neq i}}^{n} F_j(\underline{\zeta}_{ji}(v))\, dv$. Since $\underline{\zeta}_{ji}$ is nondecreasing over

$[\underline{c}, \overline{c}]$, the last inequality implies $(\overline{c} - \eta) \geq \int_{\underline{c}}^{\overline{c}} \prod_{\substack{j=1\\j\neq i}}^{n} F_j(\underline{\zeta}_{ji}(v))\, dv$, which contradicts the choice

of η. We thus have $\underline{\gamma} > \underline{c}$ and the second part of Lemma A2-13 is proved.

Finally, from equation (A2.15) we see that if η tends towards $\overline{c}$, the integral $\int_{\underline{\gamma}}^{\overline{c}} \prod_{\substack{j=1 \\ j \neq i}}^{n}$ $F_j(\zeta_{ji}^{-1}(v))$ dv tends towards zero and thus $\underline{\gamma}$ tends towards $\overline{c}$. ||

Proof of Theorem 3 (Section 4): Assume that $F_1(\underline{c}) = \ldots = F_n(\underline{c}) = 0$. According to Theorem 1 (Section 2), there exists a Bayesian equilibrium of the first price auction with mandatory bidding if and only if there exists $\underline{c} < \eta < \overline{c}$, such that the maximal solution $(\alpha_1, \ldots, \alpha_n)$ of the problem (2, 19) is defined over $(\underline{c}, \eta]$ and such that $\alpha_1(\underline{c}) = \ldots = \alpha_n(\underline{c}) = \underline{c}$. Let η^* be the supremum of Λ'_{I} and η^{**} be, as in the proof of Lemma A2-12, the infimum of Λ_{II}. From Lemmas A2-9 and A2-13, we have $\underline{c} < \eta^* \leq \eta^{**} < \overline{c}$. From Lemmas A2-10 and A2-12, we also have[21] $\Lambda'_{\mathrm{I}} = (\underline{c}, \eta^*)$ and $\Lambda_{\mathrm{II}} = (\eta^{**}, \overline{c})$. Let the closed interval $[\eta^*, \eta^{**}]$ be denoted by Λ^*.

Take η in Λ^* and consider the solution $(\alpha_1, \ldots, \alpha_n)$ corresponding to η. Since $\eta \notin \Lambda_{\mathrm{II}}$, $(\alpha_1, \ldots, \alpha_n)$ is not of type II and is thus of type I, that is, $\underline{\gamma} = \underline{c}$. From Lemma A2-4, we have $\alpha_1(\underline{c}), \ldots, \alpha_n(\underline{c}) \geq \underline{c}$. Moreover, $\eta \notin \Lambda'_{\mathrm{I}}$ and the inequalities $\alpha_1(\underline{c}), \ldots, \alpha_n(\underline{c}) > \underline{c}$ are thus impossible. Consequently, $\alpha_1(\underline{c}) = \ldots = \alpha_n(\underline{c}) = \underline{c}$, $(\alpha_1, \ldots, \alpha_n)$ satisfies the conditions of Theorem 1 (Section 2) and the first part of Theorem 3 is proved.

Assume next that the right-hand derivatives of $F_1, \ldots, F_n$ exist at $\underline{c}$, $\frac{d}{dv}F_1 = f_1, \ldots, \frac{d}{dv}F_n = f_n$ are bounded away from zero over $[\underline{c}, \overline{c}]$, and $F_1(\underline{c}), \ldots, F_n(\underline{c}) > 0$. Extend the density functions $f_1, \ldots, f_n$, for example in a piecewise linear way, to an interval $[\underline{c}_0, \overline{c}]$, with $0 \leq \underline{c}_0 < \underline{c}$, such that they be locally bounded away from zero over $(\underline{c}_0, \overline{c}]$ and $\int_{\underline{c}_0}^{\underline{c}} f_1(u)\, du = F_1(\underline{c}), \ldots, \int_{\underline{c}_0}^{\underline{c}} f_n(u)\, du = F_n(\underline{c})$. The new functions are density functions and they define probability distributions $H_1, \ldots, H_n$ over $[\underline{c}_0, \overline{c}]$.

From Lemma A2-12, the lower extremity $\underline{\gamma}$ of the maximal definition interval of the solution of (2, 19), where $H_1, \ldots, H_n$ have been substituted to $F_1, \ldots, F_n$, is a continuous function of η in $(\underline{c}, \overline{c})$. From Lemma A2-13, we know that $\underline{\gamma}$ tends towards $\overline{c}$ if η tends towards $\overline{c}$ and that $\underline{\gamma}$ is equal to $\underline{c}_0$ if η is close enough to $\underline{c}_0$. From the intermediate value theorem, there exists $\eta^* < \overline{c}$ such that the lower extremity $\underline{\gamma}^*$ of the maximal definition interval of the corresponding solution $(\alpha_1^*, \ldots, \alpha_n^*)$ is equal to $\underline{c} > \underline{c}_0$. The solution $(\alpha_1^*, \ldots, \alpha_n^*)$ is of type II for the new distributions $H_1, \ldots, H_n$. However, the system (2) for the new extended distributions $H_1, \ldots, H_n$ coincide over $(\underline{c}, \overline{c}]$ with the system for the initial distributions $H_1, \ldots, H_n$. From Lemma A2-7, we see that the conditions (4) and (5) are fulfilled. The second part of Theorem 3 then follows from Theorem 2 (Section 2). ||

Appendix 3

Proof of Corollay 2: From Theorem 3, there exists an equilibrium $(\beta_1, \ldots, \beta_n)$ of the first price auction with voluntary bidding. From Theorem 2 (Section 2), there exists η such that $\alpha_1 = \beta_1^{-1}, \ldots, \alpha_n = \beta_n^{-1}$ form a solution of (2, 4, 5). Suppose that there exists another equilibrium $(\widetilde{\beta}_1, \ldots, \widetilde{\beta}_n)$ which differs from $(\beta_1, \ldots, \beta_n)$ over $(\underline{c}, \overline{c}]$. Similarly, $\widetilde{\alpha}_1 = \widetilde{\beta}_1^{-1}, \ldots, \widetilde{\alpha}_n = \widetilde{\beta}_n^{-1}$ form a solution of (2, 4, 5) for a value $\widetilde{\eta}$ of the parameter. From the uniqueness (under our assumptions) of the solutions of the differential system (18) and thus (2) with initial conditions, we have $\eta \neq \widetilde{\eta}$. Without loss of generality, we can assume that $\widetilde{\eta} < \eta$. If there exists j such that $\alpha_j(\underline{c}) > \underline{c}$, the monotonicity of the solution of (2, 19) with respect to η (Lemma A2-8) implies that $\widetilde{\alpha}_j(\underline{c}) > \underline{c}$, and thus $\widetilde{\alpha}_i(\underline{c}) = \alpha_i(\underline{c}) = \underline{c}$, for all $i \neq j$. Consequently, there always exists j such that $\sum_{k\neq j} \ln F_k(\alpha_k(\underline{c})) = \sum_{k\neq j} \ln F_k(\widetilde{\alpha}_k(\underline{c})) = \sum_{k\neq j} \ln F_k(\underline{c})$. However, from (A2.1) we see that $\frac{d}{db}\{\sum_{k\neq j} \ln F_k(\widetilde{\alpha}_k(b)) - \sum_{k\neq j} \ln F_k(\alpha_k(b))\} = \frac{1}{\widetilde{\alpha}_j(b)-b} - \frac{1}{\alpha_j(b)-b}$, over $(\underline{c}, \widetilde{\eta}]$. From the property of monotonicity, $\widetilde{\alpha}_j(b) > \alpha_j(b)$ over $(\underline{c}, \widetilde{\eta}]$ and thus the derivative is striclty negative over this interval and the function $\sum_{k\neq j} \ln F_k(\widetilde{\alpha}_k(b)) - \sum_{k\neq j} \ln F_k(\alpha_k(b))$ is striclty decreasing over $[\underline{c}, \widetilde{\eta}]$. Consequently, $0 = \sum_{k\neq j} \ln F_k(\widetilde{\alpha}_k(\underline{c})) - \sum_{k\neq j} \ln F_k(\alpha_k(\underline{c})) > \sum_{k\neq j} \ln F_k(\widetilde{\alpha}_k(\widetilde{\eta})) - \sum_{k\neq j} \ln F_k(\alpha_k(\widetilde{\eta})) = 1 - \sum_{k\neq j} \ln F_k(\alpha_k(\widetilde{\eta}))$ and $\sum_{k\neq j} \ln F_k(\alpha_k(\widetilde{\eta})) > 1$, which is impossible. We have proved that there cannot be two equilibria different over $(\underline{c}, \overline{c}]$ and the equilibrium is thus essentially unique. ||

<u>Proof</u> <u>of</u> <u>Corollary</u> <u>4:</u> (i). From Theorem 1 and 2 (Section 2), $(\alpha_1 = \beta_1^{-1}, \ldots, \alpha_n = \beta_n^{-1})$ is a solution over $(\underline{c}, \overline{c}]$ of (2, 3) or (2, 4, 5), for the parameter $\eta = \beta_1(\overline{c}) = \ldots = \beta_n(\overline{c})$. From Lemma A2-3, we have $\phi_{ji}(v) \leq \zeta_{ij}^{-1}(v)$, for all v in $(\underline{c}, \overline{c}]$, where $\phi_{ji} = \alpha_j\beta_i$ and $\zeta_{ij}(v) = F_i^{-1}\Big(F_j(v) \min_{v \leq w \leq \overline{c}} \frac{F_i(w)}{F_j(w)}\Big)$. Since $F_j(w) \leq F_i(w)$ and thus $F_i(w)/F_j(w) \geq 1$, for all w in $(\underline{c}, \overline{c}]$, we have $\min_{v \leq w \leq \overline{c}} \frac{F_i(w)}{F_j(w)} = 1$, for all v in $(\underline{c}, \overline{c}]$. We obtain then the inequality $\phi_{ji}(v) = \alpha_j\beta_i(v) \leq \zeta_{ij}^{-1}(v) = F_j^{-1}(F_i(v))$, for all v in $(\underline{c}, \overline{c}]$. It suffices then to substitute $\alpha_i(b)$ to v in the last inequality.

(ii). From Theorems 1 and 2 (Section 2), $(\alpha_1 = \beta_1^{-1}, \ldots, \alpha_n = \beta_n^{-1})$ is a solution over $(\underline{c}, \overline{c}]$ of (2, 3) or (2, 4, 5), for the parameter $\eta = \beta_1(\overline{c}) = \ldots = \beta_n(\overline{c})$. From Lemma A3-3, we have $\phi_{ji}(v) \geq \zeta_{ji}(v)$, for all v in $(w_{ji}, \overline{c}]$, where $\phi_{ji} = \alpha_j\beta_i$, $\zeta_{ji}(v) = F_j^{-1}\Big(F_i(v) \min_{v \leq w \leq \overline{c}} \frac{F_j(w)}{F_i(w)}\Big)$ and w_{ji} is such that $F_i(w_{ji}) \min_{w_{ji} \leq w \leq \overline{c}} \frac{F_j(w)}{F_i(w)} = F_j(\underline{c})$. Since F_i/F_j is nonincreasing over $(\underline{c}, \overline{c}]$, F_j/F_i is nondecreasing over $(\underline{c}, \overline{c}]$ and thus $\min_{v \leq w \leq \overline{c}} \frac{F_j(w)}{F_i(w)} = \frac{F_j(v)}{F_i(v)}$, for all v in $[\underline{c}, \overline{c}]$. Consequently, $w_{ji} = \underline{c}$ and $\zeta_{ji}(v) = v$, for all v in $[\underline{c}, \overline{c}]$.

(iii). If we compute the derivative in $\frac{d}{dv} F_i/F_j(v) < 0$, we obtain the inequality $f_i(v)/F_i(v) < f_j(v)/F_j(v)$, for all v in $(\underline{c}, \overline{c}]$. From equation (A2.8), we see that, for all v in $(\underline{c}, \overline{c}]$ such that $\phi_{ji}(v) = v$, we have $\frac{d}{dv}\phi_{ji}(v) = (f_i(v)/F_i(v))\ (F_j(v)/f_j(v))$. Consequently, for such v, $\frac{d}{dv}\phi_{ji}(v) < 1$. Furthermore, we know that $\phi_{ji}(\overline{c}) = \overline{c}$. We can thus apply Lemma A5-1 to $a = \underline{c}$, $b = \overline{c}$, $l(v) = \phi_{ji}(v)$ and $h(v) = v$, for all v in $(\underline{c}, \overline{c}]$ and we obtain $\phi_{ji}(v) > v$, for all v in $(\underline{c}, \overline{c})$. Consequently, $\beta_i(v) > \beta_j(v)$, for all v in $(\underline{c}, \overline{c})$, and (iii) is proved.

(iv). It is an immediate consequence from (i) or (ii).

(v). It follows[22] from Riley and Samuelson (1981) and the observation that from (iv) we have $\beta_i = \beta_j$, over $(\underline{c}, \overline{c}]$, for all $1 \leq$ i, j $\leq$ n. ‖

Proof of Corollary 5: When m = n, Theorem 4 is an immediate consequence of Corollary 4 (v) (Section 4). We can thus assume that m < n. Let $(\alpha_1, \ldots, \alpha_n)$ be a maximal solution of (2, 19) and $(\underline{\gamma}, \eta]$ its definition interval, with $\underline{c} \leq \underline{\gamma} < \eta < \overline{c}$. Lemma A2-5 implies that the functions $\beta_1 = \alpha_1^{-1}$ and $\phi_{j1} = \beta_j^{-1}\beta_1 = \alpha_j\alpha_1^{-1}$, $1 \leq j \leq n$ and $j \neq 1$, form a solution over the interval $(\alpha_1(\underline{c}) = \alpha'_1(\underline{c}), \overline{c}]$ of the system (A2.8, A2.9) of differential equations considered on the domain $D_1 = \Big\{(v, (\phi_{j1})_{j\neq 1}, \beta_1) \mid \underline{c} < v \leq \overline{c}, \underline{c} < \phi_{j1} \leq \overline{c}, \beta_1 < \phi_{j1}$, for all $1 \leq j \leq n$ such that $j \neq 1$, and $\frac{(-1)(n-2)}{v-\beta_1(v)} + \sum_{\substack{l=1 \\ l\neq 1}}^{n} \frac{1}{\phi_{l1}(v)-\beta_1(v)} > 0 \Big\}$, with initial conditions $\beta_1(\overline{c}) = \eta$ and $\phi_{j1}(\overline{c}) = \overline{c}$, for all $j \neq 1$. Here, it is possible to simplify somewhat the differential system (A2.8, A2.9). From Lemma A2.3, there exist α_1' and α_2' such that $\alpha_i = \alpha'_1$, for all $1 \leq i \leq m$, and $\alpha_i = \alpha'_2$, for all $m < i \leq n$. By substituting $\beta'_1 = \alpha'^{-1}_1$ to β_1, G_1 to F_1, g_1 to f_1, G_2 to F_j, g_2 to f_j and $\phi'_{21} = \alpha'^{-1}_2\alpha'_1$ to ϕ_{j1}, for j such that $m < j \leq n$, and by rearranging and simplifying we obtain

(A3.1) $$\frac{d}{dv}\phi'_{21}(v) = \frac{g_1(v)}{G_1(v)} \frac{G_2(\phi'_{21}(v))}{g_2(\phi'_{21}(v))} \frac{m\phi'_{21}(v) - (m-1)v - \beta'_1(v)}{(n-m)v - (n-m-1)\phi'_{21}(v) - \beta'_1(v)},$$

(A3.2) $$\frac{d}{dv}\beta'_1(v) = \frac{g_1(v)}{G_1(v)} \frac{(n-1)(v-\beta'_1(v))(\phi'_{21}(v)-\beta'_1(v))}{(n-m)v - (n-m-1)\phi'_{21}(v) - \beta'_1(v)}.$$

Consequently, from Lemma A2-5 β'_1 and ϕ'_{21} form a solution over $(\alpha_1(\underline{\gamma}) = \alpha'_1(\underline{\gamma}), \overline{c}]$ of (A3.1, A3.2) considered in the domain $D'_1 = \Big\{(v, \phi'_{21}, \beta'_1) \mid \underline{c} < v \leq \overline{c}, \underline{c} < \phi'_{21} \leq \overline{c}$, $\beta'_1 < \phi'_{21}$ and $\frac{(-1)(n-m-1)}{v-\beta'_1(v)} + \frac{n-m}{\phi'_{21}(v)-\beta'_1(v)} > 0 \Big\}$ with initial conditions (A3.3) below

(A3.3) $\beta'_1(\overline{c}) = \eta$ and $\phi'_{21}(\overline{c}) = \overline{c}$.

We see that we can consider the system (A3.1, A3.2) over the domain $D''_1 = \{ (v, \phi'_{21}, \beta'_1) \mid \underline{c} < v, \phi'_{21}(v) \leq \overline{c}$ and $(n-m)v > (n-m-1)\phi'_{21}(v) + \beta'_1(v) \}$. Through the change of variables $(p, \chi'_{21}, \rho'_1) = (G_1(v), G_2(\phi'_{21}(G_1^{-1})), \beta'_1(G_1^{-1}))$, the system (A3.1, A3.2) in D''_1 is equivalent to the system (A3.4, A3.5) below in the domain $\mathcal{D}''_1 = \{ (p, \chi'_{21}, \rho'_1) \mid 0 < p, \chi'_{21}(p) \leq 1$ and $(n-m)G_1^{-1}(p) > (n-m-1)G_2^{-1}(\chi'_{21}(p)) + \rho'_1(p) \}$,

(A3.4) $$\frac{d}{dp}\chi'_{21}(p) = \frac{\chi'_{21}(p)}{p} \frac{mG_2^{-1}(\chi'_{21}(p)) - (m-1)G_1^{-1}(p) - \rho'_1(p)}{(n-m)G_1^{-1}(p) - (n-m-1)G_2^{-1}(\chi'_{21}(p)) - \rho'_1(p)},$$

(A3.5) $$\frac{d}{dp}\rho'_1(p) = \frac{1}{p} \frac{(n-1)(G_1^{-1}(p)-\rho'_1(p))(G_2^{-1}(\chi'_{21}(p))-\rho'_1(p))}{(n-m)G_1^{-1}(p) - (n-m-1)G_2^{-1}(\chi'_{21}(p)) - \rho'_1(p)}.$$

Under our assumptions, the system (A3.1, A3.2) satisfies the standard requirements from the theory of ordinary differential equations.

Let $(\beta_1, \ldots, \beta_n)$ be a Bayesian equilibrium. From Corollary 3 (Section 4), we can assume that $(\beta_1, \ldots, \beta_n)$ is a Bayesian equilibrium of the first price auction with mandatory bidding and thus (see Theorem 1, Section 2) that $\beta_1(\underline{c}) = \ldots = \beta_n(\underline{c}) = \underline{c}$. From Corollary 4 (iv) (Section 4, or Lemma A2-3), there exist β'_1 and β'_2 such that $\beta_i = \beta'_1$, for all $1 \leq i \leq m$, and $\beta_i = \beta'_2$, for all $m < i \leq n$, over $(\underline{c}, \overline{c}]$. Moreover, from the previous paragraph there exists $\underline{c} < \eta < \overline{c}$ such that β'_1 and $\phi'_{21} = \beta'^{-1}_2\beta'_1$ form a solution over $(\alpha'_1(\underline{c}), \eta] = (\underline{c}, \eta]$ of the differential system (A3.1, A3.2) considered in the domain D"$_1$ with initial conditions (A3.3). From Corollary 4 (iii) (Section 4), we see that $\beta'_1(v) > \beta'_2(v)$ and thus $\phi'_{21}(v) > v$, for all v in $(\underline{c}, \overline{c})$.

Suppose that there exists another equilibrium $(\widetilde{\beta}_1, \ldots, \widetilde{\beta}_n)$. Proceeding as above, we see that there exist $\widetilde{\beta}'_1$ and $\widetilde{\beta}'_2$ such that $\widetilde{\beta}_1 = \ldots = \widetilde{\beta}_m = \widetilde{\beta}'_1, \widetilde{\beta}_{m+1} = \ldots = \widetilde{\beta}_n = \widetilde{\beta}'_2$ and there exists $\widetilde{\eta}$ such that $\widetilde{\beta}'_1$ and $\widetilde{\phi}'_{21} = \widetilde{\beta}'^{-1}_2\widetilde{\beta}'_1$ form a solution of the differential system (A3.1, A3.2) considered in the domain D"$_1$ with initial conditions (A3.3) for the parameter $\widetilde{\eta}$. Moreover, we can assume that $\widetilde{\beta}'_1(\underline{c}) = \widetilde{\beta}'_2(\underline{c}) = \underline{c}$ and thus $\widetilde{\phi}'_{21}(\underline{c}) = \underline{c}$. From the uniqueness of the solution of the differential system (A3.4, A3.5) and thus of the system (A3.1, A3.2) with initial conditions, we have $\widetilde{\eta} \neq \eta$. Without loss of generality, assume that $\widetilde{\eta} < \eta$. From Lemma A2-8, $\widetilde{\beta}'_1(v) < \beta'_1(v)$, for all v in $(\underline{c}, \overline{c}]$.

We can rewrite equation (A3.4) as $\frac{d}{d\ln p}\ln\chi'_{21}(p) = \frac{mG_2^{-1}(\chi'_{21}(p)) - (m-1)G_1^{-1}(p) - \rho'_1(p)}{(n-m)G_1^{-1}(p) - (n-m-1)G_2^{-1}(\chi'_{21}(p)) - \rho'_1(p)}$, for all p in $(0, 1]$. From this equation, we see that $\frac{d}{d\ln p}\ln\chi'_{21}(p)$ is differentiable at 1 and thus that $\frac{d^2}{d(\ln p)^2}\ln\chi'_{21}(1)$ exists. By taking the derivative of this equation and substituting its value $(n-1)(\overline{c} - \eta)$, from (A3.5), to $\frac{d}{dp}\rho'_1(1)$, 1 to $\frac{d}{dp}\chi'_{21}(1)$, and η to $\rho'_1(1)$, we find

(A3.6) $\frac{d^2}{d(\ln p)^2}\ln\chi'_{21}(1) = \frac{(n-1)\,(g_1(\overline{c}) - g_2(\overline{c}))}{(\overline{c} - \eta)\, g_2(\overline{c})}$.

From (21) at $v = \overline{c}$, we know that $g_1(\overline{c}) - g_2(\overline{c}) < 0$. Equation (A3.6) then implies that $\frac{d^2}{d(\ln p)^2}\ln\chi'_{21}(1)$ is a strictly decreasing of η. Since $\eta > \widetilde{\eta}$, we have $\frac{d^2}{d(\ln p)^2}\ln\chi'_{21}(1) < \frac{d^2}{d(\ln p)^2}\ln\widetilde{\chi}'_{21}(1)$, where $\widetilde{\chi}'_{21} = G_2(\widetilde{\phi}'_{21}(G_1^{-1}))$. Since $\widetilde{\phi}'_{21}(\overline{c}) = \phi'_{21}(\overline{c}) = \overline{c}$ and, from equation (A3.4), $\frac{d}{d\ln p}\ln\widetilde{\chi}'_{21}(1) = \frac{d}{d\ln p}\ln\chi'_{21}(1) = 1$, this inequality implies the existence of $\epsilon > 0$ such that $\widetilde{\phi}'_{21}(v) > \phi'_{21}(v)$, for all v in $(\overline{c} - \epsilon, \overline{c})$.

Equation (A3.1) can also be rewritten as

$$\frac{d}{dv}\phi'_{21}(v) = \frac{g_1(v)}{G_1(v)}\ \frac{G_2(\phi'_{21}(v))}{g_2(\phi'_{21}(v))}\left\{ 1 + \frac{(n-m)\,(\phi'_{21}(v) - v)}{(n-m)v - (n-m-1)\phi'_{21}(v) - \beta'_1(v)} \right\},$$

for all v in $(\underline{c}, \overline{c}]$. We then see that if $\phi'_{21}(v) > v$, the derivative $\frac{d}{dv}\phi'_{21}(v)$ is a strictly increasing function of $\beta'_1(v)$. Because $\widetilde{\beta}'_1(v) < \beta'_1(v)$ over $(\underline{c}, \overline{c}]$, we have $\frac{d}{dv}\phi'_{21}(v) > \frac{d}{dv}\widetilde{\phi}'_{21}(v)$, for all v in $(\underline{c}, \overline{c}]$ such that $\widetilde{\phi}_{21}(v) = \phi'_{21}(v)$. The assumptions of Lemma A5-1 are satisfied for h $= \phi'_{21}$, l $= \widetilde{\phi}'_{21}$, a $= \underline{c}$ and b $= \overline{c} - \epsilon/2$. From this lemma, we obtain $\widetilde{\phi}'_{21}(v) > \phi'_{21}(v)$, for all v in $(\underline{c}, \overline{c} - \epsilon/2]$ and thus for all v in $(\underline{c}, \overline{c})$.

From Lemma A2-6, we have $\frac{d}{dv}\{(v - \beta'_2(v))\ G_1^m(\phi'_{12}(v))\ G_2^{(n-m-1)}(v)\} = G_1^m(\phi'_{12}(v))\ G_2^{(n-m-1)}(v)$ over $(\underline{c}, \overline{c}]$. Since $\beta'_2(\underline{c}) = \underline{c}$, $\beta'_2(\overline{c}) = \eta$ and $\phi'_{12}(\overline{c}) = \overline{c}$, we obtain

$$\int_{\underline{c}}^{\overline{c}} G_1^m(\phi'_{12}(v))\ G_2^{(n-m-1)}(v)\ dv = \overline{c} - \eta.$$

Similarly, we have $\int_{\underline{c}}^{\overline{c}} G_1^m(\widetilde{\phi}'_{12}(v))\ G_2^{(n-m-1)}(v)\ dv = \overline{c} - \widetilde{\eta}$. From the previous paragraph, we know that $\phi'_{21}(v) < \widetilde{\phi}'_{21}(v)$, for all v in $(\underline{c}, \overline{c})$, and thus $\phi'_{12}(v) = \phi'^{-1}_{21}(v) > \widetilde{\phi}'_{12}(v) = \widetilde{\phi}'^{-1}_{21}(v)$, for all v in $(\underline{c}, \overline{c})$, and

$$\int_{\underline{c}}^{\overline{c}} G_1^m(\phi'_{12}(v))\ G_2^{(n-m-1)}(v)\ dv \geq \int_{\underline{c}}^{\overline{c}} G_1^m(\widetilde{\phi}'_{12}(v))\ G_2^{(n-m-1)}(v)\ dv.$$

This inequality and the two previous equalities imply $\overline{c} - \eta \geq \overline{c} - \widetilde{\eta}$. This contradicts our initial assumption $\widetilde{\eta} < \eta$ and Corollary 5 is proved. ||

Proof of Corollary 6: Without loss of generality, we can assume that m = 1. Assume first that contrary to the hypothesis, $G_1(\underline{c}) = G_2(\underline{c}) = 0$. We will later relax this assumption. Let $\underline{c} < \eta < \overline{c}$ be such that the corresponding solution $(\alpha_1, \ldots, \alpha_n)$ of (2, 19) is of type II, that is, such that $\underline{\gamma} > \underline{c}$. We prove that $\alpha_1(\underline{\gamma}) = \ldots = \alpha_n(\underline{\gamma}) = \underline{\gamma}$. From Lemma A2-3, $\alpha_i = \alpha_2$, for all m $<$ i $\leq$ n. Suppose that there exists i such that $\alpha_i(\underline{\gamma}) \neq \underline{\gamma}$. From Lemma A2-7, there cannot be more than one such i. We can thus assume that i = 1. Consequently, $\alpha_1(\underline{\gamma}) > \underline{\gamma}$ and $\alpha_2(\underline{\gamma}) = \underline{\gamma}$. From the first part of the proof of Theorem 4, $\rho_1 = \beta_1(G_1^{-1})$ and $\chi_{21} = G_2(\phi_{21}(G_1^{-1})) = G_2(\alpha_2\beta_1(G_1^{-1}))$ form a solution over $(G_1(\alpha_1(\underline{\gamma})), 1]$ of (A3.4, A3.5) considered in the domain $\mathcal{D}''_1 = \{ (p, \chi'_{21}, \rho'_1) \mid G_1(\underline{c}) < p,\ \chi'_{21}(p) \leq 1$ and $(n-1)G_1^{-1}(p) > (n-2)G_2^{-1}(\chi'_{21}(p)) + \rho'_1(p) \}$ (we substituted its value to m) with initial conditions (A3.3). Since $(n-1)\ \alpha_1(\underline{\gamma}) > (n-2)\ \phi_{21}(\alpha_1(\underline{\gamma})) + \beta_1(\alpha_1(\underline{\gamma})) = (n-1)\ \underline{\gamma}$, we see that $(G_1(\alpha_1(\underline{\gamma})), G_2(\phi_{21}(\alpha_1(\underline{\gamma}))), \beta_1(\alpha_1(\underline{\gamma}))) = (G_1(\alpha_1(\underline{\gamma})), G_2((\underline{\gamma})), \underline{\gamma})$ lies in this domain $\mathcal{D}''_1$.

Since m = 1, the system (A3.4, A3.5) reduces to

$$\frac{d}{dp}\chi'_{21}(p) = \frac{\chi'_{21}(p)}{p}\ \frac{G_2^{-1}(\chi'_{21}(p)) - \rho'_1(p)}{(n-1)G_1^{-1}(p) - (n-2)G_2^{-1}(\chi'_{21}(p)) - \rho'_1(p)},$$

$$\frac{d}{dp}\rho'_1(p) = \frac{1}{p}\ \frac{(n-1)(G_1^{-1}(p) - \rho'_1(p))(G_2^{-1}(\chi'_{21}(p)) - \rho'_1(p))}{(n-1)G_1^{-1}(p) - (n-2)G_2^{-1}(\chi'_{21}(p)) - \rho'_1(p)}.$$

Notice that $(\widetilde{\chi}_{21}, \widetilde{\rho}_1)$, where $\widetilde{\chi}_{21}$ is equal to the constant function $G_2(\underline{\gamma})$ and $\widetilde{\rho}_1$ is equal to the constant function $\underline{\gamma}$, is a solution over $(G_1(\alpha_1(\underline{\gamma})), 1]$ of this system considered in the domain $\mathcal{D}''_1$. The solutions (χ_{21}, ρ_1) and $(\widetilde{\chi}_{21}, \widetilde{\rho}_1)$ coincide at $G_1(\alpha_1(\underline{\gamma}))$ and from the uniqueness (under our assumptions) of the solution of this differential system with initial conditions, they must coincide everywhere. However, this is impossible since $\chi_{21}(1) =$

$G_2(\phi_{21}(\overline{c}\,)) = G_2(\overline{c}\,) = 1$ and $\widetilde{\chi}_{21}(1) = \; = G_2(\underline{\gamma}\,) < 1$. We have thus proved that for every type II solution $(\alpha_1, \ldots, \alpha_n)$ of (2, 19), we have $\alpha_1(\underline{\gamma}\,) = \ldots = \alpha_n(\underline{\gamma}\,) = \underline{\gamma}$.

Suppose now that the right-hand derivatives of G_1 and G_2 exist at $\underline{c}$, $\frac{d}{dv}G_1 = g_1$ and $\frac{d}{dv}G_2 = g_2$ are bounded away from zero over $[\underline{c}\,, \overline{c}\,]$, and $G_1(\underline{c}\,), G_2(\underline{c}\,) > 0$. The existence of an equilibrium of the first price auction with mandatory bidding is then proved by extending the density functions g_1, g_2 as we did in the proof of Theorem 3 (statement in Section 4, proof in Appendix 2) to an interval $[\underline{c}_{\,0}, \overline{c}\,]$, with $0 \leq \underline{c}_{\,0} < \underline{c}$, such that the distributions they determine are atomless, applying the continuity of $\underline{\gamma}$ with respect to η and the property of type II solutions we proved in the previous paragraph. ||

Appendix 4

As explained in Section 2, a strategy of bidder i specifies his bidding plans for every possible valuation. As suggested in footnote 7, OUT is supposed to be a real number stricly smaller than $\underline{c}$. We formally define a strategy β_i of bidder i as a function from the Cartesian product of the set of possible valuations $[\underline{c}\,, \overline{c}\,]$ with the family $\mathcal{B}(\mathbf{A})$ of the Borel subsets of the set of admissible actions $\mathbf{A} = \{\mathrm{OUT}\} \cup [\underline{c}\,, +\infty)$ or $[\underline{c}\,, +\infty)$ to the interval $[0, 1]$, that is,

$$\begin{array}{llcl} \beta_i: & [\underline{c}\,, \overline{c}\,] \times \mathcal{B}(\mathbf{A}) & \longrightarrow & [0, 1] \\ & (v, B) & \longrightarrow & \beta_i(v, B), \end{array}$$

such that $\beta_i(v, .)$ is a probability measure over $\mathbf{A}$, for all v in $[\underline{c}\,, \overline{c}\,]$, and $\beta_i(., B)$ is a measurable function (for the σ-algebras of the Borel subsets), for all B in $\mathcal{B}(\mathbf{A})$. The topology over $\{\mathrm{OUT}\} \cup [\underline{c}\,, +\infty)$ and $[\underline{c}\,, +\infty)$ is the topology of the Euclidien distance. For v in $[\underline{c}\,, \overline{c}\,]$, the probability measure $\beta_i(v, .)$ should be interpreted as the bid probability distribution bidder i uses if his valuation is equal to v and if he follows the strategy β_i.

A strategy β_i of bidder i and the valuation probability distribution F_i determine a probability measure $\beta_i{*}F_i$ over the product $[\underline{c}\,, \overline{c}\,] \times \mathbf{A}$ of the set of possible valuations $[\underline{c}\,, \overline{c}\,]$ with the set of allowable actions $\mathbf{A}$. The probability measure $\beta_i{*}F_i$ is defined as follows,

$$\beta_i{*}F_i\,(V \times B) = \int_V \beta_i(v, B)\, dF_i(v),$$

for all Borel subset V of $[\underline{c}\,, \overline{c}\,]$ and all Borel subset B of $\mathbf{A}$.

Appendix 5.

<u>Lemma</u> <u>A5-1</u>: Let h and l be two functions continuous over $[a, b]$ and differentiable over $(a, b]$ with $a < b$. If $l(b) \geq h(b)$ and $\frac{d}{dx}h(x) > \frac{d}{dx}l(x)$ for all those x in $(a, b]$ such that $l(x) = h(x)$, then

$$l(x) > h(x),$$

for all x in $(a, b]$ and $l(a) \geq h(a)$.

<u>Proof</u>: Without loss of generality, we can assume that $\mathrm{h(x)} = 0$, for all x in [a, b]. We can also assume that $\mathrm{l(b)} > 0$. Otherwise we would have $\mathrm{l(b)} = 0$ and $\frac{\mathrm{d}}{\mathrm{dx}}\mathrm{l(b)} < 0$ and thus $\mathrm{l(x)} > 0$ in a neighborhood of b and we would have to consider a smaller interval with a different upper extremity. Suppose that the set $\{\mathrm{x} \in (\mathrm{a, b}] \mid \mathrm{l(x)} \leq 0\}$ is not empty and let y be defined as the supremum of this set, that is, $\mathrm{y} = \sup \{\mathrm{x} \in (\mathrm{a, b}] \mid \mathrm{l(x)} \leq 0\}$. From the continuity of l, we have $\mathrm{l(y)} = 0$, $\mathrm{y} = \max \{\mathrm{x} \in (\mathrm{a, b}] \mid \mathrm{l(x)} \leq 0\}$ and also $\mathrm{y} = \min \{\mathrm{x} \in (\mathrm{a, b}] \mid \mathrm{l(z)} > 0$, for all z in $(\mathrm{x, b}]\}$. By assumption, we have $\frac{\mathrm{d}}{\mathrm{dx}}\mathrm{l(y)} < 0$. For $\mathrm{z} > \mathrm{y}$, we have $\mathrm{l(z)} = \frac{\mathrm{d}}{\mathrm{dx}}\mathrm{l(y)}\ (\mathrm{z} - \mathrm{y}) + \mathrm{o}(\mid \mathrm{z} - \mathrm{y} \mid)\ (\mathrm{z} - \mathrm{y})$ and thus l(z) is strictly negative for z close enough to y. This conclusion contradicts the definition of y and Lemma A5-1 is proved. ||

Footnotes.

1. I thank Mamoru Kaneko for discussions on an earlier draft. Comments by Ming Huang and by referees are gratefully acknowledged. A two part draft of this paper circulated under the titles: “First Price Auction: the Asymmetric Case with N Bidders" and “First Price Auction: Properties of the Equilibria in the Asymmetric N Bidder Case."

2. Throughout our paper, an absolutely continuous measure means a measure absolutely continuous with respect to the Lebesgue measure.

3. In the frameworks of Lebrun (1996) and this present paper, it is easily seen that an equilibrium of the first price auction with mandatory bidding is an equilibrium of the auction with voluntary bidding. However, the reverse is not generally true. For a counterexample, see the introduction of Lebrun (1996).

4. For example, when the measures are atomless the authors use but do not prove the alleged differentiability of the bid functions at the lower extremity of the support. When a measure has a mass point at this lower extremity, the existence result in Maskin and Riley (December 1994) does not apply. The endpoint conditions are not precisely specified nor fully proved.

5. Corollary 4 (v) (Section 4) gives other existence results (see footnote 17).

6. The support of a probability measure μ is the largest closed set of μ-measure one.

7. This last assumption is satisfied if, for example, $\mathrm{f}_1, \mathrm{f}_2, \ldots, \mathrm{f}_n$ are stricltý positive and continuous over $(\underline{\mathrm{c}}, \overline{\mathrm{c}}]$.

8. Although this assumption is convenient, it is necessary.

9. For example, OUT can be any real number stricly smaller than $\underline{\mathrm{c}}$.

10. Remark that, from our definition of a strategy (see Appendix 4), if a strategy is pure then the bid function is measurable. In fact, if B is a Borel subset of $[\underline{\mathrm{c}}, +\infty)$ or $\{\mathrm{OUT}\} \cup [\underline{\mathrm{c}}, +\infty)$ and if β is pure, $\beta^{-1}(\mathrm{B}) = (\beta(., \mathrm{B}))^{-1}(\{1\})$ and is thus a Borel subset of $[\underline{\mathrm{c}}, \overline{\mathrm{c}}]$.

11. From Corollary 4 (i) in Section 4, we can show that if $F_n(x) \leq F_i(x)$, for all i and x, only bidder n can have such a bid function and only when $F_i(\underline{c}) > 0$, for all $i \neq n$, and, form Corollary 4 (iv) (Section 4) and Corollary 6 (Section 5), no other bidder has the same valuation distribution as bidder n, that is, $F_n \neq F_i$, for all $i \neq n$.

12. When bidding is voluntary, $b = \underline{c}$ and $i = j$ as in (4), we use the fact that, if a function f is continuous over an interval $[a, b]$ and is differentiable over the interval (a, b) and if the limit of the derivative of f at x for $x \underset{<}{\rightarrow} b$ exists, then the function f is differentiable on the left at b and the left-hand derivative at b is equal to the limit of the derivatives.

13. Bidder i does not necessarily bid everywhere in this interval. That is, the support of $\beta_i(v, .)$ may be a proper subset of the interval $[b_{il}(v), b_{iu}(v)]$ when his valuation is equal to v.

14. The graphs of the functions b_{il} may cross each other. In our diagrams we represented simple cases where they do not.

15. In the case of the closed interval $[\underline{c}, \overline{c}]$, being locally bounded away from zero is equivalent to being “uniformly" bounded away from zero.

16. Remark that from Lemma A2-11 we are able to obtain the following bounds of η^* and η^{**}, $\overline{c} - \max_{1 \leq i \leq n} \int_{\underline{c}}^{\overline{c}} \prod_{\substack{j=1 \\ j \neq i}}^{n} F_j(\underline{\zeta}_{ij}^{-1}(v))\, dv \leq \eta^* \leq \eta^{**} \leq \overline{c} - \min_{1 \leq i \leq n} \int_{\underline{c}}^{\overline{c}} \prod_{\substack{j=1 \\ j \neq i}}^{n} F_j(\underline{\zeta}_{ji}(v))\, dv$,

where $\underline{\zeta}_{ji}$ is defined in Lemma A2-13 from ζ_{ji} which in turn is defined in (A2.4) and (A2.5) in Appendix 2. Notice that when $F_1(\underline{c}) = \ldots = F_n(\underline{c}) = 0$, we have $\underline{\zeta}_{ji} = \zeta_{ji}$, for all $1 \leq i, j \leq n$.

17. Remark that Corollary 4 (v) extends our existence results to the symmetric case with mandatory bidding where there is a mass point at $\underline{c}$. Actually the existence of an equilibrium in this case follows from more general results. In order to obtain the existence of an equilibrium of the first price auction with mandatory bidding, it suffices to add to the assumptions of Theorem 3 concerning the case with simultaneous mass points at $\underline{c}$ the requirement that there be two identical distribution functions which stochastically dominate the others. We can instead require that for every bidder there exists another one with the same valuation probability distribution. The proofs are simple and rely on the property of the type II solutions that at most one function α_i can be such that $\alpha_i(\underline{c}) > \underline{c}$ (Lemma A2-7). See also Corollary 6. Remark also that in Corollary 4, we do not require that the density function be locally bounded away from zero at $\underline{c}$.

18. As it is the case of the system (2), this system is equivalent to a system which satisfies the standard requirements of the theory of ordinary differential equations, for intial conditions in the domain.

19. Such ϵ and δ exist since the L.H.S. of $\eta \frac{[\beta_j * F_j]_2(\{b\})}{2[\beta_j * F_j]_2([\underline{c}, b])} - \epsilon \frac{[\beta_j * F_j]_2([\underline{c}, b)) + [\beta_j * F_j]_2(\{b\})/2}{[\beta_j * F_j]_2([\underline{c}, b])} > \delta$ tends towards $\eta \frac{[\beta_j * F_j]_2(\{b\})}{2[\beta_j * F_j]_2([\underline{c}, b])}$ as ϵ tends towards zero.

20. Actually P(j | α_j(b), b') $=$ (α_j(b) $-$ b') $F_i(\alpha_i$(b')) $\prod_{k \neq j,i} F_k(\alpha_k$(b')) when b' $>$ $\underline{c}$. It is the case b' $=$ $\underline{c}$ and i $=$ j as in Lemma A1-14 which requires us to consider the limit $\widetilde{b} \underset{>}{\to}$ b'.

21. We have actually proved also the continuity of $\alpha_i(\underline{c}\,)$ with respect to η.

22. Λ_I $=$ $(\underline{c}\,, \eta^{**}]$.

23. Here, one way to prove (v) directly is as follows. We know that ϕ_{ji}(v) $=$ $\beta_j^{-1}\beta_i$(v) $=$ v, for all v in $(\underline{c}\,, \overline{c}\,]$ and all 1 $\leq$ i, j $\leq$ n. Let β denote the continuous function over $[\underline{c}\,, \overline{c}\,]$ such that β $=$ β_i, for all 1 $\leq$ i $\leq$ n. From (3) in Theorem 1 (Section 2) or (4) in Theorem 2 (Section 2), we have $\beta(\underline{c}\,) = \underline{c}$. From Lemma A2-6, we obtain $\frac{d}{dv}$ { (v $-$ β(v)) F^{n-1}(v) } $=$ F^{n-1}(v), for all v in $(\underline{c}\,, \overline{c}\,]$. By integrating the equation above from $\underline{c}$ to v, for v in $[\underline{c}\,, \overline{c}\,]$, by using the equality $\underline{c}$ $=$ $\beta(\underline{c}\,)$ and by solving for β(v), we obtain the expression in (v). The way to see that the formula and the conditions in (v) give Bayesian equilibria is by noticing that the functions ϕ_{ki}, k $\neq$ i, with ϕ_{ki}(v) $=$ v, for all v in $[\underline{c}\,, \overline{c}\,]$, and $\beta_i = \beta$ form a solution over $(\underline{c}\,, \overline{c}\,]$ of (A3.8, A3.9) in D_i. Moreover, the value of the continuous extension of β at $\underline{c}$ is $\underline{c}$ since $\int_{\underline{c}}^{v}$ F^{n-1}(w) dw/F^{n-1}(v) $\leq$ (v $-$ $\underline{c}$). It suffices then to apply Lemma A2-5 and Theorems 1 and 2.

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Figures

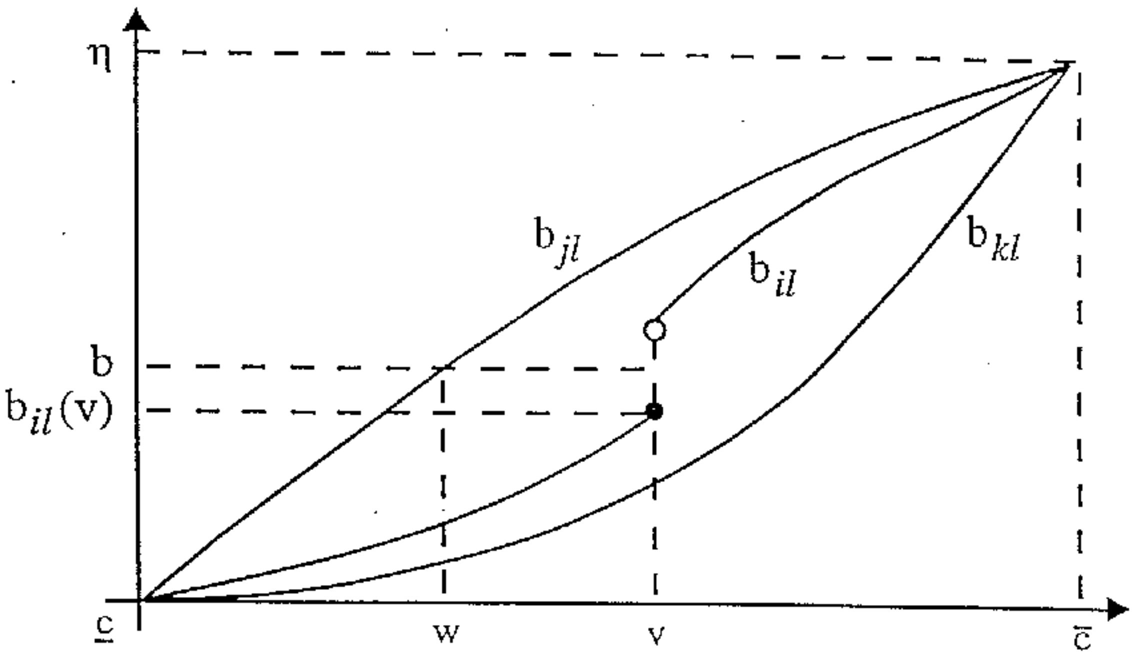

Figure 1

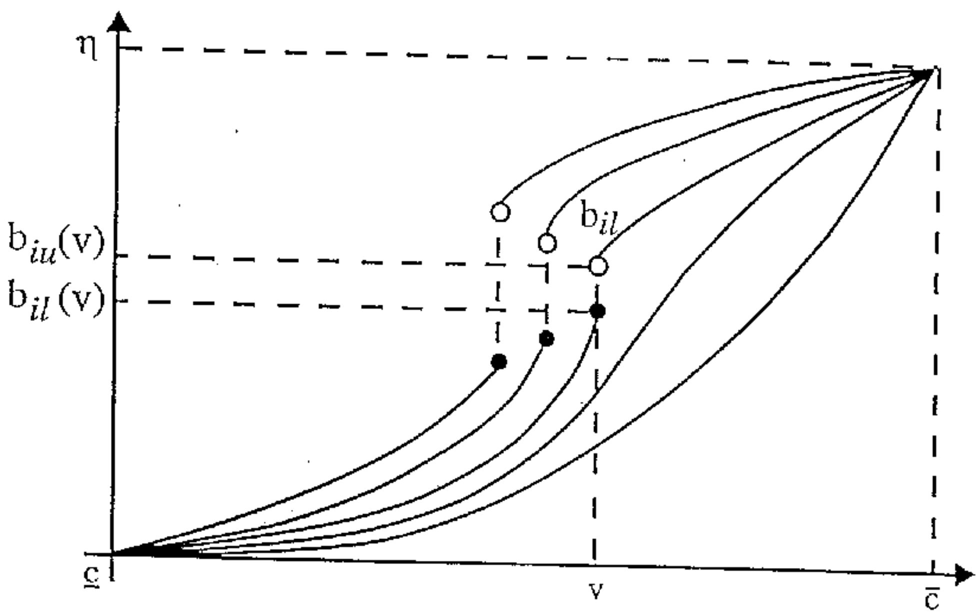

Figure 2

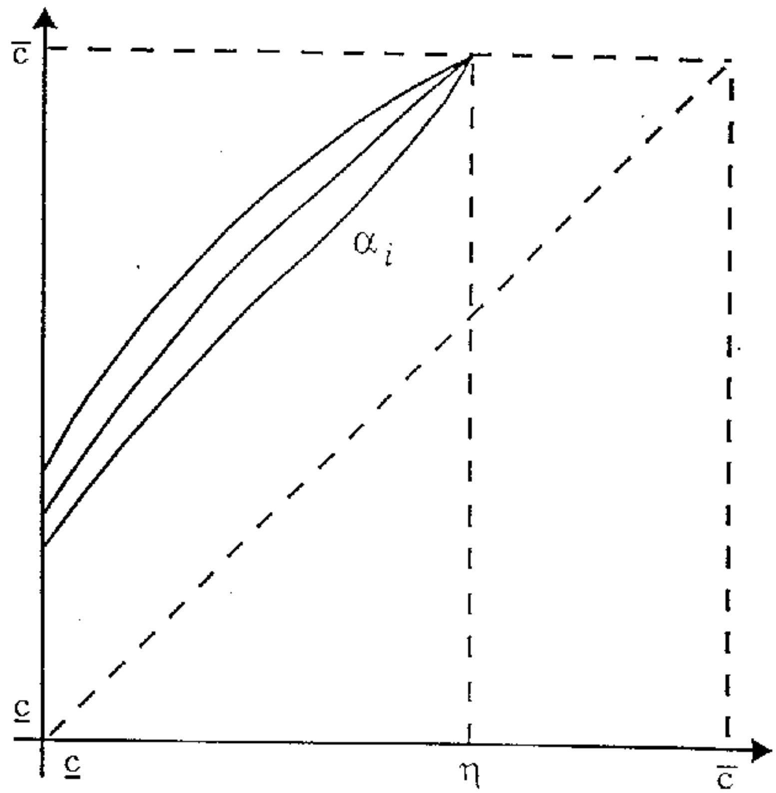

Figure 3

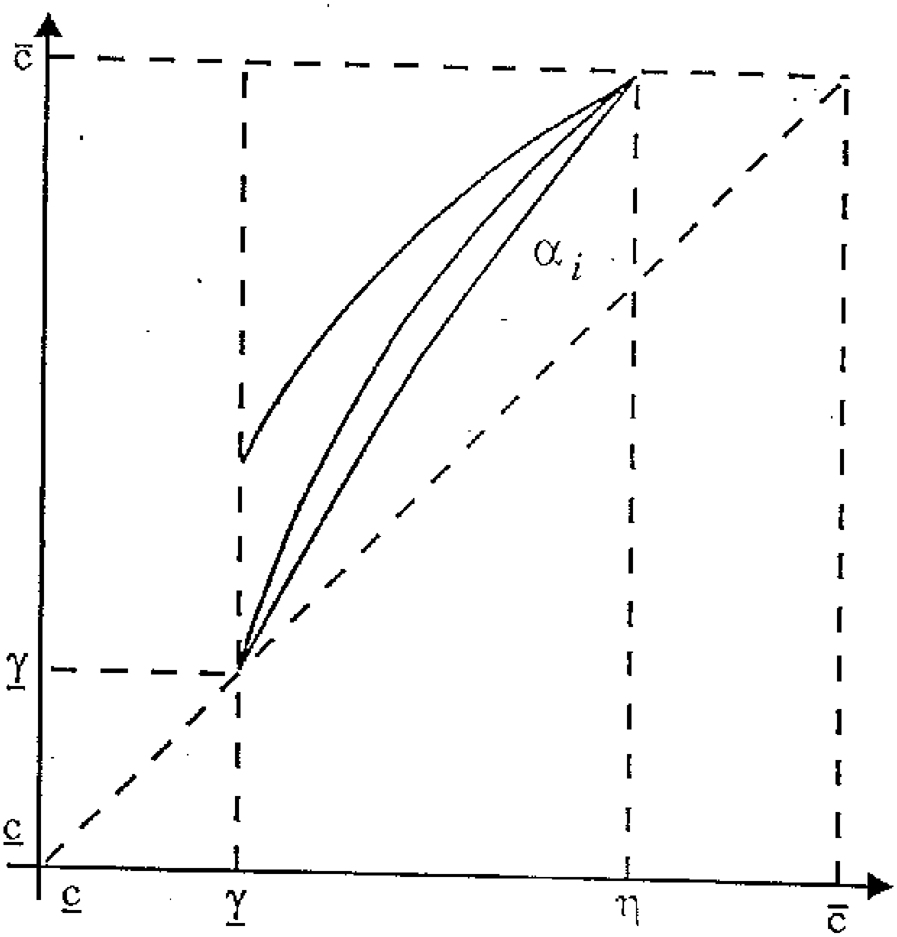

Figure 4

ADDENDUM

In the proof of Lemma A2-12, we showed that $\lim\limits_{\eta' \underset{<}{\rightarrow} \eta} \alpha_i'(\underline{\gamma}') = \underline{\gamma}$, for all i such that $\alpha_i(\underline{\gamma}) = \underline{\gamma}$, and we stated that this property implies the continuity from the left of the lower extremity of the interval of the maximal solution with respect to η. Here, we prove this statement.

We first show that there exists $\eta' < \eta$ such that the corresponding maximal solution is of type II, that is, is such that $\underline{\gamma}' > \underline{c}$. Suppose, on the contrary, that $\underline{\gamma}' = \underline{c}$, for all $\eta' < \eta$. Let b be an arbitrary bid in $(\underline{\gamma}, \eta)$. By monotonicity, we have

$$\alpha_i'(\underline{c}) = \alpha_i'(\underline{\gamma}') \leq \alpha_i'(b') \leq \alpha_i'(b)$$

for all b' in $[\underline{c}, \underline{\gamma}]$. Relying, as previously in the proof, on the continuity of the solution of the differential system with respect to the initial conditions, we have $\lim\limits_{\eta' \underset{<}{\rightarrow} \eta} \alpha_i'(b) = \alpha_i(b)$ and consequently we find

$$\underline{\gamma} = \lim_{\eta' \underset{<}{\rightarrow} \eta} \alpha_i'\left(\underline{\gamma}'\right) \leq \liminf_{\eta' \underset{<}{\rightarrow} \eta} \alpha_i'(b') \leq \limsup_{\eta' \underset{<}{\rightarrow} \eta} \alpha_i'(b') \leq \lim_{\eta' \underset{<}{\rightarrow} \eta} \alpha_i'(b) = \alpha_i(b)$$

By making b tend towards $\underline{\gamma}$ we then find that $\lim\limits_{\eta' \underset{<}{\rightarrow} \eta} \alpha_i'(b')$ exists and is equal to $\underline{\gamma}$, for all b' in $[\underline{c}, \underline{\gamma}]$.

Let $\underline{b}$ be an arbitrary element of $\left(\underline{c}, \underline{\gamma}\right)$. Let ϵ be a strictly positive number strictly smaller than $\left(\underline{\gamma} - \underline{b}\right)/2$ and let $\delta > 0$ be such that $\alpha_i'\left(\underline{\gamma} - 2\epsilon\right) \geq \underline{\gamma} - \epsilon$, for all η' such that $\eta - \delta \leq \eta' \leq \eta$. From equation (A2.1), we have $\frac{d}{db}\ln\left\{\prod\limits_{k \neq i} F_k(\alpha_k'(b))\left(\underline{\gamma} - b\right)\right\} = \frac{1}{\alpha_i'(b) - b} - \frac{1}{\underline{\gamma} - b}$, for all b in $\left(\underline{c}, \underline{\gamma}\right)$, and thus, by integrating from $\underline{b}$ to $\underline{\gamma} - 2\epsilon$, we have

$$\prod_{k \neq i} F_k\left(\alpha_k'\left(\underline{\gamma} - 2\epsilon\right)\right)2\epsilon - \prod_{k \neq i} F_k(\alpha_k'(\underline{b}))\left(\underline{\gamma} - \underline{b}\right) = \int_{\underline{b}}^{\underline{\gamma} - 2\epsilon} \left\{\frac{1}{\alpha_i'(b) - b} - \frac{1}{\underline{\gamma} - b}\right\} db.$$

Over the integration interval we have $\left|\frac{1}{\alpha_i'(b) - b} - \frac{1}{\underline{\gamma} - b}\right| \leq \frac{1}{\epsilon} + \frac{1}{2\epsilon}$ and we can thus apply the Lebesgue dominated convergence theorem. We find $\lim\limits_{\eta' \underset{<}{\rightarrow} \eta}\left\{\prod\limits_{k \neq i} F_k\left(\alpha_k'\left(\underline{\gamma} - 2\epsilon\right)\right)2\epsilon - \prod\limits_{k \neq i} F_k(\alpha_k'(\underline{b}))\left(\underline{\gamma} - \underline{b}\right)\right\} = 0$. Since $0 \leq \prod\limits_{k \neq i} F_k(\alpha_k'(\underline{b}))\left(\underline{\gamma} - \underline{b}\right) \leq 2\epsilon + \left\{\prod\limits_{k \neq i} F_k(\alpha_k'(\underline{b}))\left(\underline{\gamma} - \underline{b}\right) - \prod\limits_{k \neq i} F_k\left(\alpha_k'\left(\underline{\gamma} - 2\epsilon\right)\right)2\epsilon\right\}$, we obtain

$$0 \leq \liminf_{\eta' \underset{<}{\to} \eta} \prod_{k \neq i} F_k(\alpha'_k(\underline{b}))\left(\underline{\gamma} - \underline{b}\right) \leq \limsup_{\eta' \underset{<}{\to} \eta} \prod_{k \neq i} F_k(\alpha'_k(\underline{b}))\left(\underline{\gamma} - \underline{b}\right) \leq 2\epsilon.$$

Since these inequalities hold true for all $\epsilon > 0$, we find $\lim\limits_{\eta' \underset{<}{\to} \eta} \prod\limits_{k \neq i} F_k(\alpha'_k(\underline{b}))\left(\underline{\gamma} - \underline{b}\right) = 0$. However, this is impossible since $\prod\limits_{k \neq i} F_k(\alpha'_k(\underline{b}))\left(\underline{\gamma} - \underline{b}\right) \geq \prod\limits_{k \neq i} F_k(\underline{b})\left(\underline{\gamma} - \underline{b}\right)$, for all $\eta' < \eta$, and $\prod\limits_{k \neq i} F_k(\underline{b})\left(\underline{\gamma} - \underline{b}\right)$ is strictly positive. We have thus proved that there exists $\widetilde{\eta}' < \eta$ such that $\widetilde{\underline{\gamma}}' > \underline{c}$. By monotonicity of the solution of the differential system with initial condition with respect to η, this is the case for all values of the parameter in $\left(\widetilde{\eta}', \eta\right)$.

We show in the proof of Corollary 6 in Appendix 3 (see the paragraph preceding the statement of Corollary 6 in Section 5) that when all distributions except at most one are identical then $\alpha_1\left(\underline{\gamma}\right) = ... = \alpha_n\left(\underline{\gamma}\right) = \underline{\gamma}$, for all type II solutions. Thus, when $n = 2$ we have $\alpha_i\left(\underline{\gamma}\right) = \underline{\gamma}$, for all i, and $\alpha'_i\left(\underline{\gamma}'\right) = \underline{\gamma}'$, for all i and all η' in $\left(\widetilde{\eta}', \eta\right)$, since then $\underline{\gamma}' > \underline{c}$. Take any $i = 1, 2$. Then $\lim\limits_{\eta' \underset{<}{\to} \eta} \alpha'_i\left(\underline{\gamma}'\right) = \underline{\gamma}$ immediately implies $\lim\limits_{\eta' \underset{<}{\to} \eta} \underline{\gamma}' = \underline{\gamma}$. Assume $n > 2$. From Lemma A2-7, $\alpha_i\left(\underline{\gamma}\right) = \underline{\gamma}$ for at least $(n-1)$ values of the index i. From the same lemma, for all η' in $\left(\widetilde{\eta}', \eta\right)$ there exist at least $(n-1)$ values of i such that $\alpha'_i\left(\underline{\gamma}'\right) = \underline{\gamma}'$. Since $2(n-1) > n$, for all η' in $\left(\widetilde{\eta}', \eta\right)$ there exists at least one such common value of i, that is, there exists i such that $\alpha_i\left(\underline{\gamma}\right) = \underline{\gamma}$ and $\alpha'_i\left(\underline{\gamma}'\right) = \underline{\gamma}'$. Take a sequence $(\eta'_k)_{k \geq 1}$ in $\left(\widetilde{\eta}', \eta\right)$ such that $\eta'_k \to \eta$ as $k \to +\infty$ and let $(i_k)_{k \geq 1}$ be the sequence of corresponding index values. Since there is only a finite number (n) of possible values for i, there exists at least one value i that is repeated infinitely in the sequence $(i_k)_{k \geq 1}$. By considering the corresponding subsequence of $(\eta'_k)_{k \geq 1}$, we can thus assume that ${\alpha'}_i^k\left({\underline{\gamma}'}^k\right) = {\underline{\gamma}'}^k$, where ${\alpha'}_i^k$ and ${\underline{\gamma}'}^k$ are the i-th component and the extremity of the maximal definition interval of the solution of the differential system with initial condition for the value η'_k of the parameter, for all $k \geq 1$. From $\lim\limits_{\eta' \underset{<}{\to} \eta} \alpha'_i\left(\underline{\gamma}'\right) = \underline{\gamma}$, we then have $\lim\limits_{k \to +\infty} {\underline{\gamma}'}^k = \underline{\gamma}$. The monotonicity of $\underline{\gamma}$ with respect to η (Lemma A2-9) then implies $\lim\limits_{\eta' \underset{<}{\to} \eta} \underline{\gamma}' = \underline{\gamma}$.