Continuity of the First Price Auction Nash Equilibirum Correspondence

by

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Despite the complexity of the first price auction in the general asymmetric case, analytical results have started to emerge in the literature. Authors have also searched to gain insights by computing numerical estimates of the equilibria for some particular probability distributions of the valuations. This paper proves that the Nash equilibrium of the first price auction depends continuously, for the weak topology, on the valuation distributions and thus brings robustness to the numerical results as well as some theoretical results. As an example of application, we disprove a conjecture of comparative statics.

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1.Introduction

One item is being sold at auction to n bidders whose valuations are private and independently distributed. We consider the first price auction, where the highest bidder is awarded the item and pays his bid. This auction procedure is considerably more difficult to study than the English ascending oral bid auction which is equivalent (in this “Independent Private Value" or IPV model) to the second price or Vickrey auction, where the highest bidder is still the winner but pays the second highest bid. For the first price auction, it is only in the symmetric case where the bidders' valuations are identically distributed that there exists a general mathematical formula for the Bayesian equilibrium strategies. Nevertheless, analytical results pertaining to the general (asymmetric) case have started to emerge in the literature (Athey 1997, Griesmer *et al* 1967, Lebrun 1997, 1998, Marshall *et al* 1994, Maskin and Riley 1996 a and b, 1998, Plum 1989, Thomas 1997, Vickrey 1961, Whaerer 1997). Because of its complexity, authors have also searched to gain insights into the asymmetric case by computing numerical estimates of the equilibria for some particular probability distributions of the valuations (Athey 1997, Dalkir *et al* 1998, Li and Riley 1997, Marshal *et al* 1994, Maskin and Riley 1998).

Obviously, the more robust the theoretical results are to deviations from the assumptions on the valuation distributions the more worthwhile they are. It is thus natural to ask whether some results which are known to hold true for particular n-tuples $(F_1,\ldots,F_n)$ of valuation distributions would still hold after slight perturbations of these n-tuples. For example, in Lebrun (1998) it is proved that if there are only two different distributions G_1 and G_2 in the n-tuple of valuation distributions, that is, if there exists m such that $F_i = G_1$, for $1 \leq i \leq m$, and $F_j = G_2$, for $m+1 \leq j \leq n$, then if G_1, for example, is replaced by $\widetilde{G}_1$ such that, by using the same notation for the cumulative distribution functions, d/dv $\widetilde{G}_1(v)/G_1(v) > 0$, for all v, then the new equilibrium bid probability distributions will strictly (first order) stochastically dominate the old ones. As a particular consequence, the auctioneer's expected revenues will increase. An assumption for this result is that $\widetilde{G}_1$ dominates G_1 in a strong sense (it is equivalent to the strict first order stochastic dominance between the conditionals over all intervals of the form $[c,e]$, with $e > c$ and where c is the common minimum of the supports of G_1 and $\widetilde{G}_1$). Will the auctioneer's expected revenues still increase if G_1 is replaced by a distribution H_1 which without satisfying this strong assumption is “close enough", in the sense of the weak topology, to $\widetilde{G}_1$? Since the auctioneer's revenues is a continuous function of the bids (the maximum), the answer to this question is yes if the Nash equilibrium of the first price auction depends continuously, for the weak topology, on the valuation distributions. Here, we show that this is indeed the case.

Clearly, the numerical estimates would be of very little use without this continuity of the Nash equilibrium. In fact, they would be relevant only for the particular and more or less arbitrary choice of valuation distributions and would be of no value even for small perturbations away from these distributions.

It is known that a Nash equilibrium of the standard first price auction does not always exist. Changing the rules regarding the breaking of the ties allows to recover the existence of an equilibrium. Without access to the bidders' private information and thus using only the information they willingly provide, this can be done in several ways that we present in Section 2. They all give equilibria that would also result if the tie breaking rule directly used the bidders' valuations by allocating the item to the bidder involved in the tie with the highest valuation. In Section 3, we investigate the relationships among this “technical" variant (not an auction, strictly speaking) and all the other variants, where the valuations stay private. In particular, one of these other variants is shown to be equivalent to the technical variant. In Section 4, we prove the upper hemicontinuity of the Nash equilibrium correspondence of the technical variant and apply it to prove the existence of an equilibrium for one of the other variants. The upper hemicontinuity immediately implies the continuity when the correspondence is single-valued. In Section 5, we gather assumptions under which the correspondence is single valued and thus continuous. As an example of application, we use this continuity to prove that the property of “monotonicity", alluded to above, of the equilibrium bid distributions with respect to the valuation distributions when there are only two different valuation distributions does not extend to n-tuples of valuation distributions with more than two different distributions even under all the regularity assumptions of Lebrun (1998). Section 6 is the conclusion. Details of te proofs can be found in Appendices 1 to 3.

2. Several First Price Auction Games

We work within the model described in Lebrun (1996). An item is being sold in an auction with n bidders. We denote bidder i's valuation of the item by v_i . The n-tuple of valuations $(v_1, \dots, v_n)$ is chosen randomly according to an n-tuple of independent probability measures $(F_1, \dots, F_n)$. Only bidder i is informed of v_i and submits[1] a bid at least as large as the minimum allowable bid[2] c . The bidders are assumed to be risk-neutral.

As in Lebrun (1996, Assumption A), we assume that the support of F_i is compact and included in [c,L], where L is strictly larger than c. Since at any equilibrium we might define bidders will not bid above L, for the sake of simplicity we introduce the rule that every submitted bid must not be larger than[3] L . We denote by $\mathbb{M}([c,L])$ the set of probability measures over $[c,L]$. We thus have $(F_1,\dots,F_n) \in \mathbb{M}([c,L])^n$.

We consider the first price auction game FPA $= (\mathcal{S},P)$ as defined in Lebrun (1996). It is the standard first price auction game where the highest bidder wins the item and pays his bid, the other bidders do not pay anything, and ties are broken by a fair lottery. A strategy σ_i of bidder i is a probability measure over $[c,L]^2 = [c,L]_1 \times [c,L]_2$ such that its marginal distribution σ_{i1} over the first component space $[c,L]_1$ is equal to F_i. A conditional distribution $\sigma_{i2}(. \mid v_i)$ over $[c,L]_2$ with respect to v_i in $[c,L]_1$ is the probability distribution of the bid b_i when bidder i's valuation is v_i. We denote the set of strategies σ_i by $\mathcal{S}_i$ and the product $\mathcal{S}_1 \times \ldots \times \mathcal{S}_n$ by $\mathcal{S}$. Consistent with the assumption of risk-neutrality, the value $P_i(\sigma_1,\ldots,\sigma_n)$ of the payoff function of bidder i at a n-tuple of strategies $(\sigma_1,\ldots,\sigma_n)$ is the expected value of $p_i(v_i,b_1,\ldots b_n)$ with respect to the measure $\sigma_1 \otimes \ldots \otimes \sigma_n$ over $([c,L]_1 \times [c,L]_2)^n$. The value $p_i(v_i,b_1,\ldots b_n)$ is bidder i's expected payoff when his valuation is v_i and the submitted bids are $(b_1,\ldots b_n)$. It is equal to 0 if $b_i < \max_{1 \leq k \leq n} b_k$ and to $\frac{v_i - b_i}{\#H(b_1,\ldots b_n)}$, where $\#H(b_1,\ldots b_n)$ is the number of highest bidders, if $b_i = \max_{1 \leq k \leq n} b_k$. We denote by P the function $(P_1,\ldots,P_n)$.

We also consider several variants of FPA $= (\mathcal{S},P)$ which differ according to the way ties are broken. The game $F\overline{P}A = (\mathcal{M},\overline{P})$ is the first price auction game where bidder i is asked to send a message $m_i \in [c,L]$ together with the bid b_i he submits. In case of a tie between several highest bidders, the highest bidder who has sent the highest message among the highest bidders wins the auction. If there are several such highest bidders, the winner is chosen among these bidders according to a fair lottery. A strategy of bidder i is a probability measure over $[c,L]^3=[c,L]_1 \times [c,L]_2 \times [c,L]_3$ whose marginal distribution over the first component space is equal to F_i. The third component space is the message space. We denote the set of strategies of bidder i by $\mathcal{M}_i$, the payoff function of bidder i by $\overline{P}_i$, the product $\mathcal{M}_1 \times \ldots \times \mathcal{M}_n$ by $\mathcal{M}$, and the function $(\overline{P}_1,\ldots,\overline{P}_n)$ by $\overline{P}$. The complete formal definition of this game can be found in Lebrun (1996).

The game FPA' $= (\mathcal{S}',P')$ is the game where as in $(\mathcal{M},\overline{P})$ each bidder has to send a message along with his bid and the winner of the auction is chosen among the highest bidders who have sent the highest message among the highest bidders. However, in this game the message simply belongs to $\{0,1\}$. In a sense, a message equal to 1 means that the bidder wants to stay in the auction in case of a tie and a message 0 means that the bidder is ready to drop out in case of a tie. A strategy of bidder i is now a probability measure over $[c,L]^2 \times \{0,1\}=[c,L]_1 \times [c,L]_2 \times \{0,1\}$ whose marginal distribution over the first component space is equal to F_i. The third component space is again the message space. A formal definition of FPA' $=$ ($\mathcal{S}$',P') would proceed along the lines of the definition of $\mathrm{F\overline{P}A} = (\mathcal{M},\overline{\mathrm{P}})$.

The game $\mathrm{F\widetilde{P}A} = (\mathcal{S},\widetilde{\mathrm{P}})$ is the variant defined in Lebrun (1996) where bidders only submit bids, as in FPA $= (\mathcal{S}$,P), and where the winner of the auction is chosen among the highest bidders with the highest valuations among the highest bidders. Again, in case of several such bidders a fair lottery determines the winner. Bidder i's payoff function $\widetilde{\mathrm{p}}_i(\mathrm{v}_1,\ldots,\mathrm{v}_n,\mathrm{b}_1,\ldots,\mathrm{b}_n)$ whose expectation is equal to $\widetilde{\mathrm{P}}_i$ is now a function of the whole vectors of valuations and bids. As noticed in Lebrun (1996), strictly speaking $\mathrm{F\widetilde{P}A} = (\mathcal{S},\widetilde{\mathrm{P}})$ cannot be implemented as an auction since determining the winner requires information which is private to the bidders. As in Lebrun (1996), $\mathrm{F\widetilde{P}A} = (\mathcal{S},\widetilde{\mathrm{P}})$ is a technical tool which is useful in the proof and presentation of the results. Moreover, we will see in Theorem 1 in the next section that, as far the as the valuation-bid distributions are concerned, its set of equilibria and the set of equilibria of the first price auction game $\mathrm{F\overline{P}A}$ with the large set of messages coincide, for all n-tuples of valuation distributions.

In order to make explicit the dependency of all these games on the valuation distributions, we sometimes write $(\mathrm{F}_1, \ldots, \mathrm{F}_n)$ as an argument. For example, $\mathrm{FPA}(\mathrm{F}_1, \ldots, \mathrm{F}_n) = (\mathcal{S}(\mathrm{F}_1, \ldots, \mathrm{F}_n),\mathrm{P})$ is the standard first price auction game FPA $= (\mathcal{S}$,P) when the valuation distributions are $(\mathrm{F}_1, \ldots, \mathrm{F}_n)$. Notice that the payoff functions P, $\overline{\mathrm{P}}$, P', and $\widetilde{\mathrm{P}}$ of all games described above take their values in the compact $[\mathrm{c} - \mathrm{L},\mathrm{L} - \mathrm{c}]^n$.

Let $\Gamma = (\Sigma,\Pi)$ be one of the first price auction games defined above, that is, $\Gamma =$ FPA, $\mathrm{F\overline{P}A}$, FPA', or $\mathrm{F\widetilde{P}A}$. The Nash equilibrium correspondence of the game Γ is a correspondence from $\mathbb{M}([\mathrm{c},\mathrm{L}])^n$ to the set of n-tuples of strategies, which is included in $\mathbb{M}([\mathrm{c},\mathrm{L}]^2)^n$ in the cases of FPA and $\mathrm{F\widetilde{P}A}$, $\mathbb{M}([\mathrm{c},\mathrm{L}]^3)^n$ in the case $\mathrm{F\overline{P}A}$, and $\mathbb{M}([\mathrm{c},\mathrm{L}]^2 \times \{0,1\})^n$ in the case FPA'. Its value at $(\mathrm{F}_1, \ldots, \mathrm{F}_n)$ is the (possibly empty) set of the Nash equilibria of $\Gamma(\mathrm{F}_1, \ldots, \mathrm{F}_n)$, that is, the game Γ when the valuation distributions are $(\mathrm{F}_1, \ldots, \mathrm{F}_n)$. All measure spaces are endowed with the weak topology and all products with the product topology. We denote the Nash equilibrium correspondence of FPA by $\mathcal{N}$, of FPA' by $\mathcal{N}$', of $\widetilde{\text{FPA}}$ by $\widetilde{\mathcal{N}}$, and of $\overline{\text{FPA}}$ by $\overline{\mathcal{N}}$ and their graphs by gr$\mathcal{N}$, gr$\mathcal{N}$', gr$\widetilde{\mathcal{N}}$, and $\overline{\mathcal{N}}$, respectively. Moreover, we denote the images of these correspondences by im$\mathcal{N}$, im$\mathcal{N}$', im$\widetilde{\mathcal{N}}$, and im$\overline{\mathcal{N}}$. For example, gr$\mathcal{N}$ is equal to $\{ (\mathrm{F}_1,\ldots,\mathrm{F}_n,\mu) \mid (\mathrm{F}_1,\ldots,\mathrm{F}_n) \in \mathbb{M}([\mathrm{c,L}]^2)^n$ and $\mu \in \mathcal{N}(\mathrm{F}_1,\ldots,\mathrm{F}_n) \}$ and im$\mathcal{N}$ is equal to $\{ \mu \mid$ there exists $(\mathrm{F}_1,\ldots,\mathrm{F}_n) \in \mathbb{M}([\mathrm{c,L}]^2)^n$ such that $\mu \in \mathcal{N}(\mathrm{F}_1,\ldots,\mathrm{F}_n) \}$.

3. Relationships Between the First Price Auction Games.

Lebrun (1996) showed that any equilibrium of the technical variant $\widetilde{\text{FPA}}$ can be extended to an equilibrium of the first price auction $\overline{\text{FPA}}$ with the large set [c,L] of messages, that is, $\widetilde{\mathcal{N}} \subseteq \mathrm{marg} \circ \overline{\mathcal{N}}$ or $\widetilde{\mathcal{N}}(\mathrm{F}_1,\ldots,\mathrm{F}_n) \subseteq \mathrm{marg} \circ \overline{\mathcal{N}}(\mathrm{F}_1,\ldots,\mathrm{F}_n) = \{ \mu \in \mathbb{M}([\mathrm{c,L}]^2)^n \mid$ there exists $\overline{\mu}$ in $\overline{\mathcal{N}}(\mathrm{F}_1,\ldots,\mathrm{F}_n)$ such that $\mu = \mathrm{marg}\overline{\mu} \}$, for all $(\mathrm{F}_1,\ldots,\mathrm{F}_n)$ in $\mathbb{M}([\mathrm{c,L}])^n$, where marg is the function whose value at a measure in $\mathbb{M}([\mathrm{c,L}]^3)^n$ is its marginal distribution over the first two component spaces. We also denote by marg the similar function whose domain is $\mathbb{M}([\mathrm{c,L}]^2 \times \{0,1\})^n$. From statement (i) in Theorem 1 below the reverse inclusion holds true and $\widetilde{\mathcal{N}} = \overline{\mathrm{marg}} \circ \overline{\mathcal{N}}$. Thus, even if the technical variant $\widetilde{\text{FPA}}$ is not strictly speaking an auction all its equilibria and only those can be implemented as the equilibria of the auction $\overline{\text{FPA}}$ whose rules of allocation only makes use of the information supplied by the bidders.

Statement (i) also implies that any equilibrium of any of the first price auction games determines an equilibrium of the technical variant $\widetilde{\text{FPA}}$ and thus of the variant $\overline{\text{FPA}}$ with the large message space. The first inclusion in (i) means that if μ is an equilibrium of the standard first price auction FPA where only the bids are used to determine the winner, then it is also an equilibrium of the first price auction FPA' where in addition to their bids bidders send messages in $\{0,1\}$ indicating their willingness to win a possible tie. Furthermore, statement (ii) implies that the payoffs at an equilibrium of any game agree with the payoffs in the technical variant. Notice that the two inclusions in (i) are in general strict inclusions[4].

The set $\mathcal{U}$ in (iii) is the set of probability distributions over $[c,L]^2$ whose supports lie below the main diagonal, that is, $\mathcal{U} = \{ \mu \in \mathbb{M}([c,L]^2) \mid \mu(\{(v,b) \in [c,L]^2 \mid b \leq v\}) = 1 \}$. Thus the value of the correspondence $\widetilde{\mathcal{N}} \cap \mathcal{U}^n$, for example, at an n-tuple $(F_1,\ldots,F_n)$ is the set $\widetilde{\mathcal{N}}(F_1,\ldots,F_n) \cap \mathcal{U}^n$ of Nash equilibria μ of $F\widetilde{P}A(F_1,\ldots,F_n)$ such that every bidder almost surely submits bids not larger than his valuation. From statement (iii) in Theorem 1 below, if we consider only the equilibria where the bids are smaller than the valuations with probability one then the two variants $F\widetilde{P}A$, $F\overline{P}A$, and the first price auction FPA' with the small set $\{0,1\}$ of messages give the same bid distributions. From statement (iv) in Theorem 1, for all $(F_1,\ldots,F_n)$ such that c belongs to the support of all distributions $F_1,\ldots,F_n$ bidders never bid strictly more than their valuations with a strictly positive probability in any equilibrium of $F\widetilde{P}A(F_1, \ldots, F_n)$, that is, $\widetilde{\mathcal{N}}(F_1, \ldots, F_n) \subseteq \mathcal{U}^n$.

<u>*Theorem*</u> <u>*1*</u>*: According to our definitions, we have*

(i) $\mathcal{N} \subseteq marg \circ \mathcal{N}' \subseteq \widetilde{\mathcal{N}} = marg \circ \overline{\mathcal{N}}$,

(ii) $\widetilde{P} \circ marg\,|_{im\overline{\mathcal{N}}} = \overline{P}\,|_{im\overline{\mathcal{N}}}, \widetilde{P} \circ marg\,|_{im\mathcal{N}'} = P'\,|_{im\mathcal{N}'}, \widetilde{P}\,|_{im\mathcal{N}} = P\,|_{im\mathcal{N}}$

(iii) $\widetilde{\mathcal{N}} \cap \mathcal{U}^n = (marg \circ \mathcal{N}') \cap \mathcal{U}^n$

(iv) $\widetilde{\mathcal{N}} = (marg \circ \mathcal{N}') \subseteq \mathcal{U}^n$, *over the set* $\{(F_1,\ldots,F_n) \in \mathbb{M}([c,L])^n \mid c \in SuppF_i,$ *for all* $1 \leq i \leq n\}$,

<u>Proof</u>: See Appendix 1.

The method of proof of Theorem 1 is similar to the method followed in Lebrun (1996). First, we show (in Lemma 1) that for all equilibrium of any of our first price auction games, any bidder involved in a tie at a bid b' with a strictly positive probability for valuations strictly smaller than b' must have a zero probability of winning the tie. If he did not, he would do better by submitting a smaller bid. Similarly, a bidder involved in a tie at b' with a strictly positive probability for valuations strictly larger than b' must win the tie with probability 1. If he did not, he would do better by submitting a larger bid.

Next we prove (in Lemmas 1 and 2) that for all Nash equilibrium μ of any of the first price auction games above, if there is a strictly positive probability of a tie it must be at the lower extremity of the support of the highest bid, that is, at $\underline{b} = \max_i$ minSuppμ_{i2} and there exists a bidder j which bids $\underline{b}$ with a strictly positive probability for (a set of strictly positive probability of) valuations not smaller than $\underline{b}$ and such that any other bidder i which bids $\underline{b}$ with a strictly positive probability does so for valuations not larger than $\underline{b}$, that is, marg $\circ\, \mu_j([\underline{b}$,L$] \times \{\underline{b}\}) > 0$ and marg $\circ\, \mu_i((\underline{b}$,L$] \times \{\underline{b}\}) = 0$, for all i $\neq$ j such that bidder i submits $\underline{b}$ with a strictly positive probability. The existence of such a bidder j at any tie (occurring with a strictly positive probability) follows easily from the observations in the previous paragraph. The only possible tie must be at b' $= \underline{b}$ otherwise there would be at least one bidder i as above, that is, bidding b' for valuations not larger than b', who would be better off by submitting a smaller bid. Moreover, we prove (in Lemma 3) that if there exists a bidder i which submits $\underline{b}$ with a strictly positive probability for a set of strictly positive probability of valuations strictly smaller than $\underline{b}$, there is a bidder j as above which almost surely does not submit bids srtictly smaller than $\underline{b}$. In fact, if there did not exist such a bidder j there would be a strictly positive probability of a tie involving only bidders with valuations strictly smaller than $\underline{b}$ and it would contradict the results of the previous paragraph.

In order to prove (Lemma 4) that margμ is an equilibrium of F$\widetilde{\text{P}}$A if μ is an equilibrium of any first price auction game $\Gamma = (\Sigma,\Pi)$, we notice that the payoffs Π and $\widetilde{\text{P}}$ are equal at μ. If there is a zero probability of a tie this follows from the equality of π (Π is the expectation of π) and $\widetilde{\text{p}}$ outside ties. If there is a strictly positive probability of a tie, it is of the type described in the previous paragraph. At the tie $\underline{b}$, the payoffs are the same in Γ as in F$\widetilde{\text{P}}$A since if bidder i is involved in the tie with a strictly positive probability for valuations strictly smaller than $\underline{b}$ then with probability one bidder j with a valuation strictly larger than $\underline{b}$ is involved in the tie, resulting in a probability 0 of winning the tie for bidder i in $\widetilde{\text{FPA}}$ and in Γ (see the initial observations above about the probability of winning a tie depending on how the valuation compares with the bid). Similarly, if there is a bidder j which is involved in the tie with a strictly larger valuation than $\underline{b}$, with probability 1 all other bidders involved in the tie have valuations not larger than $\underline{b}$ and bidder j wins the tie with probability 1 in $\widetilde{\text{FPA}}$ and in Γ.

We then observe that if μ was not a Nash equilibrium of $\widetilde{\text{FPA}}$, there would exist a bidder k and a strategy ζ_k in $\widetilde{\mathcal{S}}$ such that $\widetilde{\text{P}}_k(\zeta_k,\mu_{-k}) > \widetilde{\text{P}}_k(\mu) = \Pi_k(\mu)$. However, for all $\epsilon > 0$ we can find a strategy η_k in $\widetilde{\mathcal{S}}$ which gives bidder i against μ_{-k} a payoff not smaller than $\widetilde{\text{P}}_k(\zeta_k,\mu_{-k})$ by more than ϵ, that is, such that $\widetilde{\text{P}}_k(\eta_k,\mu_{-k}) \geq \widetilde{\text{P}}_k(\zeta_k,\mu_{-k}) - \epsilon$, and such that (η_k,μ_{-k}) involves almost surely no tie. For all bid b' where there is a strictly positive probability of a tie, it suffices to alter ζ_k slightly by submitting a smaller bid when bidder k's valuation is smaller than b' and by submitting a larger bid when bidder k's valuation is larger. Since $\widetilde{\text{p}}$ and π agree outside ties, we have $\widetilde{\text{P}}_k(\eta_k,\mu_{-k}) = \Pi_k(\eta_k,\mu_{-k})$ and if we choose $\epsilon > 0$ small enough we would have $\Pi_k(\eta_k,\mu_{-k}) > \Pi_k(\mu)$, which is impossible since μ is a equilibrium of Γ. Consequenlty, marg $\circ\, \overline{\mathcal{N}} \subseteq \widetilde{\mathcal{N}}$ and thus (the reverse inclusion was proved in Lebrun 1996) $\widetilde{\mathcal{N}} =$ marg $\circ\, \overline{\mathcal{N}}$. We also have proved the inclusions $\mathcal{N} \subseteq \widetilde{\mathcal{N}}$ and marg $\circ\, \mathcal{N}' \subseteq \widetilde{\mathcal{N}}$ as well as the equalities in (ii) stating that the equilibrium payoffs are the same in all games as in $\widetilde{\text{FPA}}$.

In order to prove (iii), it now suffices to prove (Lemma 6) the inclusion $\widetilde{\mathcal{N}} \cap \mathcal{U}^n \subseteq$ marg $\circ\, \mathcal{N}'$. Let μ be an equilibrium of $\widetilde{\text{FPA}}$ such that almost surely bidders submit bids not larger than their valuations. If there is a strictly positive probability of a tie then any bidder i $\neq$ j, as in the previous proof, which is involved in the tie with a strictly positive probability can bid $\underline{b}$ with a strictly positive probability only when his valuation is equal to $\underline{b}$, that is, $\mu_i(([c,L]\backslash\{\underline{b}\}) \times \{\underline{b}\}) = 0$, for all i $\neq$ j. In this case if μ' is the n-tuple of strategies in $\mathcal{S}'$ such that marg$\mu' = \mu$, μ'_j always sends the message 1, and μ'_k always sends the message 0, for all k $\neq$ j, we can show by using the same arguments as above that μ' is a Nash equilibrium of FPA'.

Because no bidder involved in the tie at $\underline{b}$ has strictly smaller valuations than $\underline{b}$ and thus since all bidders involved in the tie have their valuations equal to $\underline{b}$ except possibly one, bidder j, whose valuation can be strictly larger than $\underline{b}$, the messages 0 and 1 suffice. The same bidder, bidder j, is the bidder who has to win the tie no matter who the other bidders involved in the ties are. It was because this was not the case in the three bidder example of footnote 4 that we needed a larger message space[5]. If μ implies a probability zero of a tie, any n-tuple μ' of strategies in $\mathcal{S}'$ such that marg$\mu' = \mu$ is a Nash equilibrium of FPA'[6]. We can similaryly prove (Lemma 5) the inclusion $\mathcal{N} \subseteq$ marg $\circ\, \mathcal{N}'$ in (i).

Proving (iv) in Theorem 1 is now equivalent to proving $\widetilde{\mathcal{N}} \subseteq \mathcal{U}^n$ over the set $\{(F_1,\ldots,F_n) \in \mathbb{M}([c,L])^n \mid c \in \text{Supp}F_i, \text{ for all } 1 \leq i \leq n\}$. First we notice (Lemma 7) that if μ is a Nash equilibrium of F$\widetilde{\text{P}}$A for an n-tuple of valuation distributions in this set then $c \in \text{Supp}\mu_{i2}$, for all $1 \leq i \leq n$, that is, the support of the bid distribution of every bidder includes c. Otherwise, there would exist some bidders who would bid almost surely above $c + \epsilon$, with $\epsilon > 0$, and at least one bidder would experience strictly negative payoffs for valuations in $[c, c + \epsilon)$ while he can always obtain at least zero (by submitting his valuation). This property implies (Lemma 8) that any bid strictly larger than c from any bidder has a strictly positive probability of winning. Consequently, no bidder will bid strictly higher than his valuation since it would result in a strictly negative payoff.

4.Upper Hemicontinuity

In Lebrun (1996, Lemma 1 p. 430), we showed that under our assumptions there always exists an equilibrium of F$\widetilde{\text{P}}$A and thus of F$\overline{\text{P}}$A. Equivalently, the Nash equilibrium correspondence $\widetilde{\mathcal{N}} = \overline{\text{marg}} \circ \overline{\mathcal{N}}$ has non-empty values. Theorem 2 below states that the graph of this correspondence is closed.

<u>Theorem</u> <u>2</u>: The Nash equilibrium correspondence $\widetilde{\mathcal{N}} = m\bar{a}rg \circ \overline{\mathcal{N}}$ *has non-empty values and its graph is closed. Moreover, the payoff function* $\widetilde{\mathrm{P}}$ *is continuous on the image im*$\widetilde{\mathcal{N}}$ *of* $\widetilde{\mathcal{N}}$ *.*

<u>Proof</u>: See Appendix 2.

Since $\mathbb{M}([\mathrm{c,L}]^2)^n$ is compact, Corollary 1 below follows immediately from Theorem 2 (see, for example, Duffie 1988, exercise 19.2 (B) p.199).

<u>Corollary</u> <u>1</u>: The Nash equilibrium correspondence $\widetilde{\mathcal{N}}$ *is upper hemicontinuous.*

The statement in Corollary 1 means that $\{\ (\mathrm{F}_1,\ldots,\mathrm{F}_n) \in \mathbb{M}([\mathrm{c,L}]^2)^n \mid \widetilde{\mathcal{N}}(\mathrm{F}_1,\ldots,\mathrm{F}_n) \subset \mathrm{O}\ \}$ is open, for all open set $\mathrm{O} \subset \mathbb{M}([\mathrm{c,L}]^2)^n$. The statements in Theorem 2 mean that for all sequence $(\mathrm{F}_1^l,\ldots,\mathrm{F}_n^l)_{l\geq 1}$ in $\mathbb{M}([\mathrm{c,L}]^2)^n$ which converges weakly towards $(\mathrm{F}_1,\ldots,\mathrm{F}_n)$, if $(\mu^l)_{l\geq 1}$ is a sequence in $\mathbb{M}([\mathrm{c,L}]^2)^n$ such that μ^l is a Nash equilibrium of $\mathrm{F\widetilde{P}A}(\mathrm{F}_1^l,\ldots,\mathrm{F}_n^l)$, that is, $\mu^l \in \widetilde{\mathcal{N}}(\mathrm{F}_1^l,\ldots,\mathrm{F}_n^l)$, for all $l \geq 1$, and which converges weakly towards μ then μ is a Nash equilibrium of $\mathrm{F\widetilde{P}A}(\mathrm{F}_1,\ldots,\mathrm{F}_n)$, that is, $\mu \in \widetilde{\mathcal{N}}(\mathrm{F}_1,\ldots,\mathrm{F}_n)$ and moreover $\widetilde{\mathrm{P}}(\mu^l)$ tends towards $\widetilde{\mathrm{P}}(\mu)$.

Although the limit of Nash equilibria of the standard first price auction FPA may not be an equilibrium of the same auction game, from Theorem 1 (i) and the upper hemicontinuity of $\widetilde{\mathcal{N}}$ stated in Corollary 1 the limit is a Nash equilibrium of the technical variant $\mathrm{F\widetilde{P}A}$ and it can be extended to an equilibrium of the first price auction $\mathrm{F\overline{P}A}$ with the large set $[\mathrm{c,L}]$ of messages used in breaking the ties. From Theorem 1 (ii) and the continuity of $\widetilde{\mathrm{P}}$ over im$\widetilde{\mathcal{N}}$ stated in Theorem 2, the payoffs in the standard auction will tend towards the payoffs in $\mathrm{F\widetilde{P}A}$ and $\mathrm{F\overline{P}A}$. From Theorem 1, Theorem 2, and Corollary 1 the same statements hold true with, instead of the standard auction FPA, the auction FPA' with the small set $\{0,1\}$ of messages.

The proof of Theorem 2 proceeds as follows. We consider any subsequence such that the n-tuple of payoffs $(\widetilde{\mathrm{P}}_1,\ldots,\widetilde{\mathrm{P}}_n)(\mu^l)$ is convergent. We first prove that in the game $\widetilde{\mathrm{FPA}}(\mathrm{F}_1,\ldots,\mathrm{F}_n)$ bidder i cannot obtain a higher payoff than the limit $\lim_{l\to+\infty}\widetilde{\mathrm{P}}_i(\mu^l)$ of his payoffs in the converging games. In fact, if it was the case there would exist a strategy θ_i which would give bidder i against μ_{-i} in the game $\widetilde{\mathrm{FPA}}(\mathrm{F}_1,\ldots,\mathrm{F}_n)$ a payoff larger by a certain strictly positive number $\delta > 0$ than his equilibrium payoffs and thus any payoffs he can obtain against μ^l_{-i} in the games $\widetilde{\mathrm{FPA}}(\mathrm{F}^l_1,\ldots,\mathrm{F}^l_n)$, for all l large enough. That is, there would exist θ_i such that

(1) $\widetilde{\mathrm{P}}_i(\theta_i,\mu_{-i}) \;>\; \widetilde{\mathrm{P}}_i(\nu^l_i,\mu^l_{-i}) \;+\; \delta,$

for all l large enough and for all strategy ν^l_i of bidder i in $\widetilde{\mathrm{FPA}}(\mathrm{F}^l_1,\ldots,\mathrm{F}^l_n)$. However, as we show in Lemmas 11 and 12 (Appendix 2), for all $\epsilon > 0$ there exists a function ζ_i of v_i such that the strategy of bidder i consisting in bidding according to this function is an ϵ-best response to μ_{-i} in the game $\widetilde{\mathrm{FPA}}(\mathrm{F}_1,\ldots,\mathrm{F}_n)$ and gives expected payoffs against μ^l_{-i} in the games $\widetilde{\mathrm{FPA}}(\mathrm{F}^l_1,\ldots,\mathrm{F}^l_n)$ which tend towards the expected payoff against μ_{-i} in the game $\widetilde{\mathrm{FPA}}(\mathrm{F}_1,\ldots,\mathrm{F}_n)$. The existence of such a function for $\epsilon \;<\; \delta$ rules out the existence of θ_i since (1) would imply $\widetilde{\mathrm{P}}_i(\theta_i,\mu_{-i}) \;>\; \widetilde{\mathrm{P}}_i(\zeta_i,\mu^l_{-i}) \;+\; \delta$, while the limit of $\widetilde{\mathrm{P}}_i(\zeta_i,\mu^l_{-i})$ is not smaller than $\widetilde{\mathrm{P}}_i(\theta_i,\mu_{-i}) - \epsilon$. The function ζ_i is a step function constructed by taking an increasing finite sequence $\mathrm{c} = \mathrm{w}_0,\ \mathrm{w}_1,\ \ldots,\ \mathrm{w}_m = \mathrm{L}$ of valuations in $[\mathrm{c,L}]$ which are not mass points of $\mu_{i1} = \mathrm{F}_i$ and such that the distances between two consecutive elements are small enough and by defining the constant value of ζ_i over[7] $[\mathrm{w}_{k-1},\mathrm{w}_k)$ as a bid which is not a mass point of the highest bid from the other bidders using μ_{-i} and which is a ϵ-best response to μ_{-i} when bidder i's valuation is equal to w_{k-1}. Such a bid exists for all $\mathrm{k} \geq 1$ since if it happens that an ϵ-best response bid is equal to a mass point of the highest bid of μ_{-i}, that is, there is a strictly positive probability of a tie if this bid is submitted, it suffices to change it slightly to find a suitable bid. Following this procedure, we can construct a “bid function" ζ_i such that the set of discontinuities of bidder i's payoff $\widetilde{\mathrm{p}}_i\,(\mathrm{v}_i,\zeta_i(\mathrm{v}_i),\mathrm{v}_{-i},\mathrm{b}_{-i})$

when he follows ζ_i has a $\mu_{i1} \otimes \mu_{-i} = \mathrm{F}_i \otimes \mu_{-i}$ measure equal to 0 and such that the strategy it determines in $\mathrm{F\widetilde{P}A}(\mathrm{F}_1,\ldots,\mathrm{F}_n)$ is an ϵ-best response to μ_{-i}. Such a function ζ_i fulfills our requirements.

The second part of the proof consists in showing that bidder i can obtain at least the limit of his equilibrium payoffs by playing the limit μ_i of his equilibrium strategies, that is, that $\widetilde{\mathrm{P}}_i(\mu) \geq \lim_{l \to +\infty} \widetilde{\mathrm{P}}_i(\mu^l)$. This inequality is an easy consequence of the first part of the proof and the property of upper semicontinuity of the sum of the payoffs $\sum_{i=1}^{n} \widetilde{\mathrm{p}}_i$ which implies the upper semicontinuity with respect to ν in $\mathbb{M}([\mathrm{c,L}]^2)$ of $\sum_{i=1}^{n} \widetilde{\mathrm{P}}_i(\nu)$. In fact, if this inequality did not hold true, as μ^l tends towards μ the sequence $\widetilde{\mathrm{P}}_i(\mu^l)$ would exhibit a “jump down" to $\widetilde{\mathrm{P}}_i(\mu)$. Since $\sum_{i=1}^{n} \widetilde{\mathrm{P}}_i$ can never exhibit a “jump down", there would exist at least one j $\neq$ i such that $\widetilde{\mathrm{P}}_j(\mu^l)$ would jump up to $\widetilde{\mathrm{P}}_j(\mu)$. This is impossible from the first part of the proof and this completes the proof of the closedness of the graph. Finally, in the course of this proof we showed that the limit of any convergent subsequence of $(\widetilde{\mathrm{P}}(\mu^l))_{l \geq 1}$ is equal to $\widetilde{\mathrm{P}}(\mu)$. The statement about $\widetilde{\mathrm{P}}$ in Theorem 2 then follows.

A first consequence of the upper hemicontinuity of $\widetilde{\mathcal{N}}$ is Corollary 2 below stating, for all $(\mathrm{F}_1,\ldots,\mathrm{F}_n)$ in $\mathbb{M}([\mathrm{c,L}])^n$, the existence of an equilibrium of $\mathrm{F\widetilde{P}A}(\mathrm{F}_1,\ldots,\mathrm{F}_n)$, or equivalently (see (i) in Theorem1) of $\mathrm{F\overline{P}A}(\mathrm{F}_1,\ldots,\mathrm{F}_n)$, where bidders do not bid higher than their valuations with a strictly positive probability. From (iii) in Theorem 1, it is equivalent to stating that there exists such an equilibrium of the first price auction $\mathrm{FPA'}(\mathrm{F}_1,\ldots,\mathrm{F}_n)$ with the set $\{0,1\}$ of messages or that the correspondence $\widetilde{\mathcal{N}} \cap \mathcal{U}^n = (\mathrm{marg} \circ \mathcal{N}') \cap \mathcal{U}^n$ has non-empty values. Notice that it implies that $\mathcal{N}'$ has non-empty values and thus that there always exists at least a equilibrium of FPA'.

<u>*Corollary*</u> *<u>2</u>: The correspondence* $\widetilde{\mathcal{N}} \cap \mathcal{U}^n = (\mathit{marg} \circ \mathcal{N}') \cap \mathcal{U}^n$ *has non-empty values.*

Proof: See Appendix 2.

To prove Corollary 2 it suffices to approximate any $(F_1,\ldots,F_n)$ by a sequence $(F_1^l,\ldots,F_n^l)_{l\geq 1}$ such that c belongs to the supports of F_1^l, …, F_n^l, for all l, and to apply Theorem 1 (iv) and the upper hemicontinuity of $\widetilde{\mathcal{N}}$ (Corollary 1).

5.Continuity

The upper hemicontinuity of $\widetilde{\mathcal{N}} = \overline{\mathrm{marg}} \circ \overline{\mathcal{N}}$ immediately implies its continuity over any set of n-tuples $(F_1,\ldots,F_n)$ where $\widetilde{\mathcal{N}}$ is single-valued. Examples of assumptions under which $\widetilde{\mathcal{N}}$ is single valued can be found in Lebrun (1997 a, b, and 1999). These papers consider the case where the continuous from the right variants of the valuation cumulative distribution functions, which we also denote by $F_1,\ldots,F_n$, satisfy Assumption A below:

A. $F_1,\ldots,F_n$ have their supports equal to the same interval $[c,d]$, with $c < d$, are differentiable[8] over $(c,d]$, and are such that their derivatives – the density functions $f_1,\ldots,f_n$, – are locally bounded away from zero over $(c,d]$.

The results in Lebrun (1997 a, b, and 1999) about the characterization, existence, and uniqueness of the Bayesian-Nash equilibrium under mandatory and voluntary bidding can easily be shown to imply that the Nash equilibrium (as we have defined it in the present paper) of $\widetilde{FPA}(F_1,\ldots,F_n)$ is unique, that is, $\widetilde{\mathcal{N}}$ and thus (from Theorem 1 (iv) and (iii)) marg' $\circ\,\mathcal{N}$' are single-valued at $(F_1,\ldots,F_n)$ under any of the following assumptions :

B. In addition to Assumption A, $F_1(c),\ldots,F_n(c) > 0$, $F_1,\ldots,F_n$ are right-differentiable at c, and the derivatives $f_1,\ldots,f_n$ are bounded away from zero over $[c,d]$.

C. In addition to Assumption A, $F_1(c),\ldots,F_n(c) = 0$ and there exists $\delta > 0$ such that $F_1,\ldots,F_n$ are strictly log-concave over $(c,c+\delta)$.

D. In addition to Assumption A, there exists $1 \leq m \leq n$ and two distributions G_1 and G_2 absolutely continuous over $[c,d]$ such that $F_i = G_1$, for $1 \leq i \leq n$, $F_j = G_2$, for $m < j \leq n$, and there exists $\delta > 0$ such that $\frac{d}{dv}\frac{G_1}{G_2}(v) < 0$, for all v in $(c,c+\delta]$.

E. In addition to Assumption A, $F_1 = \ldots = F_n$.

In Assumption B, c is a mass point of all distributions $F_1,\ldots, F_n$. In Assumption C, c is not a mass point af any of the distributions $F_1,\ldots,F_n$ and these distributions are strictly log-concave in a (right) neighborhood of c. In Assumption D, there are up to two different valuation distributions and there exists a relation of stochastic dominance[9] between them in a neighborhood of the lower extremity c. Assumption E describes the standard symmetric case. The uniqueness under Assumption B follows from Corollary 1 in Lebrun (1997a) and under Assumption E from Corollary 3 (v) in the same paper (or from Corollaries 2 and 4 (v) in Lebrun 1997b). The uniqueness under Assumption C or D follows from Lebrun (1999). We thus have Corollary 2 below.

<u>Corollary</u> <u>2</u>: Let $\mathbb{S}$ be a subset of $\mathbb{M}([c,L])^n$ over which $\widetilde{\mathcal{N}}$ is single valued. Then the correspondence $\widetilde{\mathcal{N}} = m\bar{a}rg \circ \overline{\mathcal{N}}$ is continuous over $\mathbb{S}$. In particular, the correspondence $\widetilde{\mathcal{N}} = m\bar{a}rg \circ \overline{\mathcal{N}}$ is single valued and continuous over the set of n-tuples $(F_1,\ldots,F_n)$ which satisfy one or more of the assumptions B, C, D, E. Moreover, over this latter set marg' $\circ\, \mathcal{N}'$ coincides with $\widetilde{\mathcal{N}} = m\bar{a}rg \circ \overline{\mathcal{N}}$ and is thus single valued and continuous.

Consequently, any numerical estimate (see references in the introduction) or any property of the Nash equilibrium (see references in the introduction) for particular n-tuples $(F_1,\ldots,F_n)$ is somewhat robust to the choice of valuation distributions in the set of Corollary 2 and no “razor-edge" effect can appear. For example, when $n = 2$ Lebrun (1996) analytically proves that the seller's expected revenues are strictly larger at the unique equilibrium of the first price auction than at the unique equilibrium in weakly dominant strategies of the second price auction for the couples (F_1,F_2) of distributions over $[0,1]$ such that $F_1(v) = v^\gamma$ and $F_2(v) = v^\delta$, for all v in $[0,1]$, with $\gamma\delta \geq 1/2$, and thus for any couples of distributions of maxima of uniformly distributed independent random variables. Since the seller's expected revenues in both cases are the expectations of continuous functions of the bidders' strategies (as we have defined them here), this ranking still holds true in the intersection of an open neighborhood[10]of any such couple (F_1,F_2) in $\mathbb{M}([0,1])^2$ with the set of couples of distributions satisfying one or more of the Assumptions[11] B, C, D, or E.

The upper hemicontinuity of $\widetilde{\mathcal{N}}$ cannot only be used to extend properties, it can also be used to rule out some extensions. For example, Lebrun (1998) proves the result of comparative statics stating that if the n-tuple of distributions is composed of up to two different distributions and if one of these two distributions is replaced by a new distribution which dominates it stochastically according to the relation of stochastic dominance introduced around c in Assumption C and if the equilibria of the first price auction before and after the change are unique, then the new bid probability distributions first-order stochastically dominate the old distributions and, in particular, the bid function of the bidders whose valuation distribution has not changed will increase. We first remark (see Theorem 1 in Lebrun 1997a and Theorems 1 and 2 in Lebrun 1997b) that under Assumption A, at an equilibrium of any first price auction game (among $\mathrm{F\widetilde{P}A}$, $\mathrm{F\overline{P}A}$, and FPA') bidder i's bid probability distribution conditional on the valuation v_i is concentrated at one point denoted by $\beta_i(v_i)$, for all $1 \leq i \leq n$ and F_i-almost all valuation v_i. Any equilibrium, as defined in this present paper, is thus determined by an n-tuple of bid functions unique $F_1 \otimes \ldots \otimes F_n$- almost everywhere. The variant we take for such an n-tuple is the n-tuple of bid functions defining a Bayesian equilibrium, that is, such that $\beta_i(v_i)$ maximizes bidder i's expected payoff[12] when his valuation is equal to v_i . We now state the result from Lebrun (1998, Theorem A1) more precisely: for any two n-tuples of distributions $(F_1,\ldots,F_n)$ and $(F_1^*,\ldots,F_n^*)$ satisfying Assumption A such that there exist $1 \leq m \leq n$ and G_1, G_2, and G_2^* with

(2) $\frac{d}{dv}\frac{G_2^*}{G_2}(v) < 0$, for all v in $(c,d]$,

$F_1 = \ldots = F_m = F_1^* = \ldots = F_m^* = G_1, \quad F_{m+1} = \ldots = F_n = G_2, \quad F_{m+1}^* = \ldots = F_n^* = G_2^*,$ and such that the equilibria of $\widetilde{FPA}(F_1,\ldots,F_n)$ and $\widetilde{FPA}(F_1^*,\ldots,F_n^*)$ are unique, we have

$\gamma_1^*(v) > \gamma_1(v)$,

for all v in $(c,d]$, and

$G_2^*(\delta_2^{*-1}(b)) \;<\; G_2(\delta_2^{-1}(b))$,

for all b in $(c, \gamma_1(d) = \delta_2(d)]$, when $(\beta_1,\ldots,\beta_n)$ and $(\beta_1^*,\ldots,\beta_n^*)$ are the “bid functions" defining the equilibria of $\widetilde{FPA}(F_1,\ldots,F_n)$ and $\widetilde{FPA}(F_1^*,\ldots,F_n^*)$, respectively, with[13] $\beta_1 = \ldots = \beta_m = \gamma_1$, $\beta_1^* = \ldots = \beta_m^* = \gamma_1^*$, $\beta_{m+1} = \ldots = \beta_n = \delta_2$, and $\beta_{m+1}^* = \ldots = \beta_n^* = \delta_2^*$.

We now show that this result of comparative statics does not generalize to situations where the n-tuple of valuation distributions counts strictly more than two distributions. Actually, we show that if (F_{-i},F_i) and (F_{-i},F_i^*) are two n-tuples of valuation distributions satisfying Assumption A with the same distributions F_j, $j \neq i$, such that $\widetilde{FPA}(F_{-i},F_i)$ and $\widetilde{FPA}(F_{-i},F_i^*)$ have unique equilibria (β_{-i},β_i) and (β_{-i}^*,β_i^*) (respectively), then the stochastic dominance of F_i by F_i^* in the (strong) sense (2) does *not* imply in general that the probability distribution of the bid from bidder j, $j \neq i$, in the equilibrium (β^*_{-i},β^*_i) first-order dominates this distribution in the equilibrium (β_{-i},β_i), nor does it imply that the probability distribution of the highest bid from the bidders j, $j \neq i$, in the equilibrium (β^*_{-i},β^*_i) dominates this distribution in the equilibrium (β_{-i},β_i). In general, we have

(3) $\frac{\mathrm{d}}{\mathrm{dv}}\frac{\mathrm{F}^*_i}{\mathrm{F}_i}(\mathrm{v}) < 0$, for all v in $(\mathrm{c},\mathrm{d}]$,

does *not* imply $\beta^*_j(\mathrm{v}) > \beta_j(\mathrm{v})$, for all v in $(\mathrm{c},\mathrm{d}]$ and $j \neq i$,

does *not* imply $\prod_{j\neq i}\mathrm{F}_j(\beta^{*-1}_j(\mathrm{b})) < \prod_{j\neq i}\mathrm{F}_j(\beta^{-1}_j(\mathrm{b}))$, for all b in $(\mathrm{c},\beta_1(\mathrm{d}) = \ldots = \beta_n(\mathrm{d})]$.

To establish this result, we first exhibit in the following proposition an example which does not satisfy Assumption A but where the equilibria can easily be obtained and where a particular increase of a bidder's valuation distribution does not result in increases of the probability distributions of the bids from the other bidders. Following standard lines, it is not too difficult to prove Proposition 1 below (simultaneously with Proposition 2 in Appendix 3).

Proposition 1: Consider the example with 3 bidders where bidder 1's valuation is equal to 0 with probability p and 1 with probability $1 - p$, bidder 2's valuation is equal to 0 with probability q and 1 with probability $1 - q$, and bidder 3's valuation is equal to x with probability 1 with $0 < p$, $q < 1$ and $0 < x < 1$. Without loss of generality, we assume that $q \geq p$. Then there exists one and only equilibrium (μ_1,μ_2,μ_3) of $F\widetilde{P}A(F_1,F_2,F_3)$[14]*. We denote by I, J, and H the cumulative distribution functions of the equilibrium bid distributions, that is, the marginals over the bid space of μ_1, μ_2, and μ_3, respectively. If $x > \frac{q}{p+q}$, the supports of the equilibrium bid distributions are as in Figure 1, and*

$J(b) = I(b) = H(b) = 1$, *for all* $b \geq \eta$

$J(b) = I(b) = \frac{2(pqx(1-x))^{1/2}}{1-b}$ *and* $H(b) = 1$*, for* b *in Range A, that is, for* b *such that* $\widetilde{b} \leq b \leq \eta$*,*

$J(b) = I(b) = \left(\frac{pqx}{x-b}\right)^{1/2}$ *and* $H(b) = 2(1-x)^{1/2}\ \frac{(x-b)^{1/2}}{1-b}$*, for* b *in Range B, that is, for* b *such that* $\underset{\sim}{b} \leq b \leq \widetilde{b}$ *,*

$J(b) = q$*,* $I(b) = \frac{px}{x-b}$*, and* $H(b) = \frac{2(px(1-x))^{1/2}}{q^{1/2}(1-b)}$*, for* b *in Range C, that is,* $0 \leq b \leq \underset{\sim}{b}$ *,*

where

$\eta\ =\ 1 - 2(pqx(1-x))^{1/2}$*,* $\widetilde{b}\ = 2x - 1$*, and* $\underset{\sim}{b}\ = \frac{q-p}{q}x$*.*

[FIGURE 1]

In Figure 1, the vertical lines represent the supports of the bid distributions and the full dots represent possible mass points. In the example of Proposition 1, assume that $x > \frac{q}{p+q}$ and consider F_2^* corresponding to q^* such that $q^* < q$ and thus $x > \frac{q^*}{p+q^*}$. Let (I,J,H) and (I^*,J^*,H^*) be the equilibrium bid cumulative distribution functions when the valuation distributions are (F_1,F_2,F_3) and (F_1,F_2^*,F_3), respectively. Then, from the proposition above we see that $H^*(b) = 2(px(1-x))^{1/2}/(q^{*1/2}(1-b)) > H(b) = 2(px(1-x))^{1/2}/(q^{1/2}(1-b))$ and $I^*(b)H^*(b) = 2px(1-x)^{1/2}(q^{*1/2}(1-b)(x-b)) > I(b)H(b) = 2px(1-x)^{1/2}/(q^{1/2}(1-b)(x-b))$, for all b in $(0,\underset{\sim}{b} = \frac{q-p}{q}x)$ and we thus have an example where the probability distributions of the bid from another bidder and of the highest bid from the other bidders do not stochastically increase after a stochastic increase of the valuation distribution of bidder 2. The bid probability distributions of bidder 3 and of the highest bid from bidders 1 and 3 before and after the change cannot be stochastically ranked.

In order to prove the existence of such an example with valuation distributions satisfying Assumption A, all with mass points at c, thus satisfying Assumption B, and with the relation of stochastic dominance introduced in (2) between F_2 and F_2^*, it suffices to construct sequences $(F_{1n})_{n\geq 1}$, $(F_{2n})_{n\geq 1}$, $(F^*_{2n})_{n\geq 1}$, and $(F_{3n})_{n\geq 1}$ of valuation distributions with these properties and which converge towards the distributions F_1, F_2, F_2^*, and F_3 (respectively) of the previous paragraph. First take a sequence of functions g_n from [0,1] to (0,1] which converges towards F_2^*/F_2 and such that g_n is differentiable with a strictly positive derivative over [0,1], for all $n \geq 1$. Then take sequences $(F_{1n})_{n\geq 1}$, $(F_{2n})_{n\geq 1}$, and $(F_{3n})_{n\geq 1}$ of distributions satisfying assumption A with $[0,1] = [c,d]$ such that $F_{1n}(0) > 0$, $F_{2n}(0) > 0$, and $F_{3n}(0) > 0$, for all n, and thus satisfying Assumption B. Then, the sequences $(F_{1n})_{n\geq 1}$, $(F_{2n})_{n\geq 1}$, $(F_{2n}^* = F_{2n}g_n)_{n\geq 1}$, and $(F_{3n})_{n\geq 1}$ will display the required properties. Let b be a bid in $(0,\underset{\sim}{b} = \frac{q-p}{q}x)$. Let I_n, J_n, H_n be the bid cumulative distribution functions at the unique equilibrium when the valuations distributions are F_{1n}, F_{2n}, and F_{3n} and let I_n^*, J_n^*, H_n^* be the bid cumulative distribution functions at the unique equilibrium when the valuations distributions are F_{1n}, F_{2n}^*, and F_{3n}, for all $n \geq 1$. Then, from the upper hemicontinuity of $\widetilde{\mathcal{N}}$ stated in Corollary 1 and thus its continuity over $\mathbb{S} = \{ (F_{1n}, F_{2n}^*, F_{3n}), (F_{1n}, F_{2n}, F_{3n}) \mid n \geq 1 \} \cup \{ (F_1, F_2, F_3), (F_1, F_2^*, F_3) \}$ there exists $m \geq 1$ such that $H_n^*(b) > H_n(b)$ and $I_n^*(b)H_n^*(b) > I_n(b)H_n(b)$, for all $n \geq m$. Remark that for $n \geq m$, the first inequality implies the existence of a neighborhood of b over which $\beta_{3n}^{*-1} > \beta_{3n}^{-1}$ and thus of a neighborhood of $\beta_{3n}^{*-1}(b)$ over which $\beta_{3n}^* < \beta_{3n}$, where β_{3n} and β_{3n}^* are bidder 3's equilibrium bid functions when the valuations distributions are F_{1n}, F_{2n}, F_{3n} and F_{1n}, F_{2n}^*, F_{3n}, respectively.

6. Conclusion

We proved the continuity of the Nash equilibrium of the first price auction in the independent private value model with respect to the valuation distributions and thus established the robustness of theoretical results and numerical investigations about this auction procedure. We studied the relationships among several variants of the first price auction and applied the continuity of the equilibrium to disprove a conjecture of comparative statics.

Appendix 1.

Lemma 1: *Let* $(F_1,\ldots,F_n)$ *be a n-tuple of distributions in* $\mathbb{M}([c,L])^n$. *Let* $\mu = (\mu_1, \ldots, \mu_n)$ *be a Nash equilibrium of* $\Gamma(F_1,\ldots,F_n)$, *where* $\Gamma \in \{$ *FPA, FPA', F*$\widetilde{P}$*A, F*$\overline{P}$*A* $\}$. *Let b' be a mass point of the probability distribution of the highest submitted bid when the bidders bid according to* μ. *Without loss of generality we can assume that there exists* $1 \leq l \leq n$ *such that*

$$\mu_{12}(\{b'\}) > 0, \ldots, \mu_{l2}(\{b'\}) > 0,$$

$$\mu_{(l+1)2}(\{b'\}) = \ldots = \mu_{n2}(\{b'\}) = 0, \mu_{(l+1)2}([c,b']) > 0, \ldots, \mu_{n2}([c,b']) > 0.$$

Then for all $1 \leq i \leq l$ *such that* $marg \circ \mu_i([c,b') \times \{b'\}) > 0$ *, we have*

(A1) Prob (bidder i wins $|$ *bidder i bids b' and* $v_i < b'$ *)* $= 0$.

If $l > 1$ *and if* $marg \circ \mu_i((b',L] \times \{b'\}) > 0$, *we have*

(A2) Prob (bidder i wins $|$ *b' is the highest bid, bidder i bids b' and* $v_i > b'$ *)* $= 1$.

There exists j such that $1 \leq j \leq l$ *and*

(A3) $marg \circ \mu_i((b',L] \times \{b'\}) = 0$, *for all* $1 \leq i \leq l$ *and* $i \neq j$.

When $l > 1$, *there exists* $1 \leq j \leq l$ *such that (A3) holds true and such that*

(A4) $marg \circ \mu_j([b',L] \times \{b'\}) > 0$.

Proof: If (A1) did not hold, bidding according to μ and thus bidding b' for a set of strictly positive measure of $v_i < b'$ would contribute negatively to bidder i's expected payoff. Submitting, for example, $b = v_i$ would strictly increase his expected payoff and μ could not be an equilibrium.

If (A2) was not true when $l > 1$, then for a strictly positive measure set of v_i, submitting a bid b slightly larger than b' rather than b' itself would increase bidder i's probability of winning discontinuously and would decrease his potential payment only continuously. His expected payoff would then be strictly increased (for similar reasoning see Griesmer, Levitan and Shubik 1961).

If $l = 1$, (A3) is immediate. Consider the case $l > 1$. The property (A2) implies that there exists at most one bidder j, with $1 \leq j \leq l$, such that $\text{marg} \circ \mu_j((b',L] \times \{b'\}) > 0$. Consequently at least $(l - 1)$ bidders bid b' for valuations not larger than b' and (A3) is proved.

Assume (A4) is not true. Then for each j verifying (A3) (there is at least one such j), we have $\text{marg} \circ \mu_j([b',L] \times \{b'\}) = 0$. As a consequence we have $\text{marg} \circ \mu_i([b',L] \times \{b'\}) = 0$ and $\text{marg} \circ \mu_i([c,b') \times \{b'\}) > 0$, for all $1 \leq i \leq l$. This, however, contradicts (A1) and we have proved that there exists j such that (A4) holds true. ||

Lemma 2: Under the assumptions of Lemma 1, if l > 1 we have

$$b' = \max_{1 \leq i \leq n} \min \mathit{Supp}\,\mu_{i2}.$$

Proof: We define $\underline{b}$ as follows

$$\underline{b} = \max_{1 \leq i \leq n} \min \text{Supp}\,\mu_{i2}.$$

$\underline{b}$ is the minimum of the support of the highest bid. Let b' and $l > 1$ as in Lemma 1. We must prove that b' $=$ $\underline{b}$. Since b' is a mass point of the distribution of the highest bid, we have b' $\geq$ $\underline{b}$. We must thus prove the reverse inequality. When $l > 1$, we know from (A4) in Lemma 1 that there exists $1 \leq j \leq l$ such that (A3) applies. There also exist $(l-1)$ other indices i $\neq$ j such that

(A5) $\qquad$ $\text{marg} \circ \mu_i([\text{c,b'}] \times \{\text{b'}\}) > 0.$

We denote by V_i the set of valuations v_i not larger than b' for which bidder i bids b' with a strictly positive probability. V_i has a strictly positive F_i-measure.

Assume first that there exist such an index i $\neq$ j and a subset W_i of V_i, of strictly positive F_i-measure and such that $v_i <$ b', for all v_i in W_i. Then, from (A4) we know that the probability that bidder i wins if he submits b' and if his valuation belongs to W_i is equal to zero. Since a lower bound of this probability is given by Prob $(\max_{k \neq i} b_k < \text{b'}) = \prod_{k \neq i}$ Prob $(b_k < \text{b'})$ $= \prod_{k \neq i} \mu_{k2}([\text{c, b'}))$, we see that there exists $k \neq$ i such that $\mu_{k2}([\text{c, b'})) = 0$, that is b' $\leq$ min Supp μ_{k2} and thus b' $\leq$ $\underline{b}$, and b' $= \underline{b}$.

Assume next that there does not exist an index i as in the previous paragraph. As a consequence, any index i verifying (A5) is such that $\text{marg} \circ \mu_i(\{\text{b'}\} \times \{\text{b'}\}) > 0.$. If it was the case that b' $>$ $\underline{b}$, we would have $\prod_{k \neq i} \mu_{k2}([\text{c, b'})) \geq \prod_{1 \leq k \leq n} \mu_{k2}([\text{c, b'})) > 0$ and there would exist $\epsilon > 0$ such that Prob $(\max_{k \neq i} b_k < \text{b'} - \epsilon) = \prod_{k \neq i} \mu_{k2}([\text{c, b'} - \epsilon)) > 0$. Bidder i's payoff when b' is his valuation and when he submits b' $- \epsilon$ would then be strictly positive and would thus be strictly larger than what he obtains if he submits b' as his equilibrium strategy prescribes. This is impossible and thus b' $\leq$ $\underline{b}$, and b' $= \underline{b}$. ||

Lemma *3*: *Under the assumptions of Lemma 1, if* $l > 1$ *and if there exists* $1 \leq i \leq l$ *such that* $\mu_i([c,\underline{b}\,) \times \{\underline{b}\,\}) > 0$ *then there exists* $1 \leq j \leq n$ *such that* $j \neq i$ *and (A4) holds true and such that* $\mu_{j2}([c,\underline{b}\,)) = 0$ *and thus such that* $\underline{b} = \min Supp\mu_{j2}$, *where* $\underline{b}$ *is as defined in Lemma 2.*

Proof: Since l $>$ 1, there is a strictly positive probability of a tie. From Lemma 2, we know that the tie can only happen at $\underline{\text{b}} = \max\limits_{1 \leq i \leq n} \min \text{Supp}\mu_{i2}$. Assume that for all j $\neq$ i such that (A4) holds true we have $\mu_{j2}([\text{c},\underline{\text{b}}\,)) > 0$. Then there is a strictly positive probability that these bidders j will not be involved in the tie. In fact, there is a strictly positive probability that, for all such j, bidder j bids strictly less than $\underline{\text{b}}$. Consequently, there is a strictly positive probability that the bidders k involved in the tie will be such that (A4) does not hold true, that is, that $\text{marg} \circ \mu_k([\text{b}',\text{L}] \times \{\text{b}'\}) = 0$. Since $\mu_{k2}(\{\text{b}'\}) > 0$, all bidders k involved in the tie would be such that $\mu_k([\text{c},\text{b}') \times \{\text{b}'\}) > 0$. However, this contradicts (A1) and Lemma 3 is proved. $\|$

Lemma *4*: *Let* $(F_1,\ldots,F_n)$ *be a n-tuple of distributions in* $\mathbb{M}([c,K])^n$. *Let* $\mu = (\mu_1, \ldots, \mu_n)$ *be a Nash equilibrium of* $\Gamma(F_1,\ldots,F_n)$, *where* $\Gamma = (\Sigma,\Pi) \in \{ FPA, FPA', F\widetilde{P}A, F\overline{P}A \}$. *Then* μ *is a Nash equilibrium of* $F\widetilde{P}A(F_1,\ldots,F_n)$ *and* $\Pi(\mu) = \widetilde{\text{P}}\,(marg\mu)$.

Proof: Let $\mu = (\mu_1, \ldots, \mu_n)$ be a Nash equilibrium of $\Gamma(\text{F}_1,\ldots,\text{F}_n)$. We first show that $\Pi_i(\mu) = \widetilde{\text{P}}_i(\text{marg}\mu)$, for all $1 \leq \text{i} \leq \text{n}$. Consider first the case where, in the notation of Lemma 1, $\text{l} = 1$ for all mass point b' of the probability distribution of the highest bid when the bidders follow μ. Then, according to μ there is a zero probability of a tie. Since π and $\widetilde{\text{p}}$ agree outside ties, we have $\Pi_i(\mu) = \widetilde{\text{P}}_i(\text{marg}\mu)$, for all $1 \leq \text{i} \leq \text{n}$. Assume now that if μ is followed there exists a bid b' where there is a strictly positive probability of a tie, that is, where (in the notation of Lemma 2) l $>$ 1. From Lemma 2 we know that b' $= \underline{\text{b}}$ where $\underline{\text{b}}$ is the minimum of the support of the highest bid i.e. $\underline{\text{b}} = \max\limits_{1 \leq i \leq n} \min \text{Supp}\mu_{i2}$. If there is a strictly positive probability that bidder i is involved in a tie at $\underline{\text{b}}$, breaking the tie will have the same result on his payoff in all games. In fact, if bidder i bids $\underline{b}$ with a strictly positive probability for a set of strictly positive probability of valuations $v_i < \underline{b}$ Lemma 3 implies that with probability one a bidder j will be involved in the tie with a valuation at least as large as $\underline{b}$. From Lemma 1 for almost all valuations $v_i < \underline{b}$ such that bidder i bids $\underline{b}$ with a strictly positive probability he wins the tie in Γ with a probability zero. The same outcome takes place in F$\widetilde{\text{P}}$A since bidder j has almost surely a strictly larger valuation. If bidder i bids $\underline{b}$ for his valuation $v_i = \underline{b}$, any way the tie is broken in any game will give a zero payoff to bidder i. If bidder i bids $\underline{b}$ with a strictly positive probability for a set of strictly positive probability of valuations $v_i > \underline{b}$, from Lemma 1 his probability of wining the tie in those cases in Γ is equal to 1. From Lemma 1, any other bidder involved in the tie has almost surely a valuation not larger than $\underline{b}$. Consequently, in the game ($\mathcal{S}$, $\widetilde{\text{P}}$) he will also win the tie with probability one. Since the payoff $\widetilde{\text{p}}$ and π in both games only differ at the ties, we have proved $\Pi_i(\mu) = \widetilde{\text{P}}_i(\text{marg}\mu)$, for all $1 \leq i \leq n$.

Suppose that there exists $\nu_i \in \mathcal{S}_i$ such that $\widetilde{\text{P}}_i(\nu_i,\text{marg}\mu_{-i}) > \widetilde{\text{P}}_i(\text{marg}\mu)$. Then, by reasoning as in the proofs of Theorems 1 and 2 in Lebrun (1996) it is possible to show that for all $\epsilon > 0$ there exists $\lambda_i \in \mathcal{S}_i$ such that $\widetilde{\text{P}}_i(\lambda_i,\text{marg}\mu_{-i}) > \widetilde{\text{P}}_i(\nu_i,\text{marg}\mu_{-i}) - \epsilon$ and following λ_i bidder i does not bid with a strictly positive probability any mass point of the distribution of the highest bid from the other bidders if they follow $\text{marg} \circ \mu_{-i}$. For all bid b' where there is a strictly positive probability of a tie, it suffices to alter slightly ν_i by submitting a smaller bid when bidder i's valuation is smaller than b' and by submitting a larger bid when bidder i's valuation is larger. More precisely, let $\{b_h \mid h \geq 1\}$ be the set of mass points of the distribution of the highest bid from $\text{marg}\mu_{-i}$. This set is at most countable. For every b_h in this set which ν_i submits with a strictly positive probability and for almost every valuation $v_i \neq b_h$ for which ν_i submits b_h, it suffices to bid slightly above or under b_h depending on whether $v_i > b_h$ or $v_i < b_h$, respectively, to a bid which does not belong to $\{b_h \mid h \geq 1\}$ (this change can be done in a measurable way). By taking $\epsilon > 0$ small enough, we have $\widetilde{\text{P}}_i(\lambda_i,\text{marg}\mu_{-i}) > \widetilde{\text{P}}_i(\text{marg}\mu)$. Since π and $\widetilde{\text{p}}$ differ only at the ties, we would have $\Pi_i(\lambda_i^*,\mu_{-i}) = \widetilde{\text{P}}_i(\lambda_i,\text{marg}\mu_{-i}) > \Pi_i(\mu) = \widetilde{\text{P}}_i(\text{marg}\mu)$, where

λ_i^* is any strategy in Σ_i whose marginal over $[c,L]_1 \times [c,L]_2$ is equal to λ_i. However, μ is an equilibrium of Γ and this inequality is impossible and we have proved Lemma 4. ||

Lemma 5: Let $(F_1,\ldots,F_n)$ be a n-tuple of distributions in $\mathbb{M}([c,L])^n$. Let $\mu = (\mu_1, \ldots, \mu_n)$ be a Nash equilibrium of FPA$(F_1,\ldots,F_n)$. If μ' is a n-tuple of strategies in FPA'$(F_1,\ldots,F_n)$ such that marg$\mu' = \mu$, then μ' is a Nash equilibrium of FPA'$(F_1,\ldots,F_n)$ and P'(μ') = P(μ).

Proof: Let μ and μ' be as in the statement of the lemma. We first show that $P_i'(\mu') = P_i(\mu)$, for all $1 \leq i \leq n$. Consider first the case where, in the notation of Lemma 1, $l = 1$ for all mass point b' of the probability distribution of the highest bid when the bidders follow μ. Then, according to μ there is a zero probability of a tie. Since p' and p agree outside ties, we have $P_i'(\mu') = P_i(\mu)$, for all $1 \leq i \leq n$. Assume now that if μ is followed there exists a bid b' where there is a strictly positive probability of a tie, that is, where (in the notation of Lemma 1) $l > 1$. From Lemma 2 we know that $b' = \underline{b}$ where $\underline{b}$ is the minimum of the support of the highest bid i.e. $\underline{b} = \max_{1 \leq i \leq n} \min \mathrm{Supp}\mu_{i2}$. If there is a strictly positive probability that bidder i is involved in a tie at $\underline{b}$, breaking the tie will have the same result on his payoff in all games. In fact, from Lemma 1 if bidder i is involved in the tie with a strictly positive probability it is almost surely when his valuation $v_i = \underline{b}$ and any way the tie is resolved in any game will give a zero payoff to bidder i. Since the payoffs p' and p in both games only differ at the ties, we have proved $P'_i(\mu') = P_i(\mu)$, for all $1 \leq i \leq n$.

Suppose that there exists $\nu_i' \in \mathcal{S}_i'$ such that $P_i'(\nu_i',\mu_{-i}') > P'_i(\mu') = P_i(\mu)$. Then, by proceeding as in the proof of Lemma 4 it is possible to show that there exists $\lambda_i' \in \mathcal{S}_i'$ with the same property and such that following λ_i' bidder i does not bid with a strictly positive probability any mass point of the distribution of the highest bid from the other bidders if they follow μ_{-i}. Since p' and p differ only at the ties, we would have $P_i(\mathrm{marg}\lambda_i',\mu_{-i}) = P_i'(\lambda_i',\mu_{-i}') >$

$P'_i(\mu') = P_i(\mu)$. However, μ is an equilibrium of FPA and this inequality is impossible and we have proved Lemma 5. ||

Lemma 6: *For all* $(F_1,\ldots,F_n)$ *in* $\mathbb{M}([c,L])^n$, *we have* $\widetilde{\mathcal{N}}(F_1,\ldots,F_n) \bigcap \mathcal{U}^n \subseteq$ *marg* $\circ\, \mathcal{N}'(F_1,\ldots,F_n)$.

Proof: Let μ be an element of $\widetilde{\mathcal{N}}(F_1,\ldots,F_n)$ such that $\mu_i(\{(v,b) \in [c,L]^2 \mid b \leq v\}) = 1$, for all $1 \leq i \leq n$. First consider the case where under μ every mass point of the distribution of the highest bid is played with strictly positive probability by no more than one (and thus just one) bidder. That is, according to the notation of Lemma 1, $l = 1$ for all mass point of the distribution of the highest bid. For all $1 \leq i \leq n$, let $\overline{\mu}_i$ be any strategy of bidder i in FPA' which induces the same distribution over $[c,L]^2$ as μ_i does. Then proceeding as in te proof of Lemma 4 it is simple to prove that $\overline{\mu} = (\overline{\mu}_1, \ldots, \overline{\mu}_n)$ is a equilibrium of FPA'.

Consider now the case when under μ there exists a bid b' where there is a strictly positive probability of a tie, that is, where (in the notation of Lemma 1) $l > 1$. From Lemma 2, we know that $b' = \underline{b}$ where $\underline{b}$ is the minimum of the support of the highest bid, that is, $\underline{b} = \max_{1 \leq i \leq n} \min \mathrm{Supp}\mu_{i2}$. From Lemma 1, there exists $1 \leq j \leq l$ such that (A3) and (A4) hold true. Moreover, here $\mu_i(\{(v,b) \in [c,L]^2 \mid b \leq v\}) = 1$, for all i, and thus $\mu_i([c,\underline{b}) \times \{\underline{b}\}) = 0$, for all $1 \leq i \leq l$, and $\mu_i(([c,L]\backslash \{\underline{b}\}) \times \{\underline{b}\}) = 0$, for all $1 \leq i \leq l$ and $i \neq j$.

For all $i \neq j$, let μ_i' be the element of $\mathcal{S}'_i$ which induces the same measure over the valuation-bid space $[c,L]^2$ and which always send the message 0, that is, $\mathrm{marg}\mu_i' = \mu_i$ and $\mu'_i([c,L]^2 \times \{1\}) = 0$. Let μ'_j be the element of the element of $\mathcal{S}'_j$ which induces the same measure over the valuation-bid space $[c,L]^2$ and which always send the message 1, that is, $\mathrm{marg}\mu_j' = \mu_j$ and $\mu'_j([c,L]^2 \times \{0\}) = 1$. Then $\widetilde{P}(\mu) = P'(\mu')$. In fact, p and p' agree outside ties. If there is a tie, it is almost surely at $b = \underline{b}$. For $1 \leq i \leq l$ and $i \neq j$, if bidder i bids $\underline{b}$ it is almost surely for $v_i = \underline{b}$ and bidder i's payoff is the same no matter how the tie is solved. If bidder j is involved in the tie, if $v_j = \underline{b}$ the way the tie is solved does not matter. If $v_j \neq \underline{b}$, with probability 1 we have $v_j > \underline{b}$ and bidder j wins the tie with probability 1 in both games. Consequently, we have $\widetilde{P}(\mu) = P'(\mu')$. Then showing that μ' is a Nash equilibrium of $FPA'(F_1,\ldots,F_n)$ can proceed as in the proof of Lemma 4[15]. ||

Lemma 7: Let $(F_1,\ldots,F_n)$ be an element of $\mathbb{M}([c,L])^n$ such that $c \in SuppF_i$, for all $1 \leq i \leq n$, and let μ be an element of $\widetilde{\mathcal{N}}(F_1,\ldots,F_n)$. Then $c \in Supp\mu_{i2}$, for all $1 \leq i \leq n$.

Proof: Let $\underline{b}_i$ be the infimum of $Supp\mu_{i2}$, for all $1 \leq i \leq n$. From the definition of the game $\widetilde{FPA}$, we have $\underline{b}_i \geq c$, for all i. Assume that $\max_i \underline{b}_i > c$. Let J be the set of the indices of bidders whose lower extremities of their bid supports are equal to $\max_i \underline{b}_i$, that is, $J = \{ j \mid 1 \leq j \leq n, \underline{b}_j = \max_i \underline{b}_i \}$. Since a winner is always declared in the game $\widetilde{FPA}$, we have[16] Prob (the index of the winner belongs to J | $v_j \in [c,\max_i \underline{b}_i)$, for all j in J) > 0. Consequently, there exists $j \in J$ such that Prob (j wins the auction | $v_j \in [c,\max_i \underline{b}_i)$, for all j in J) > 0 and thus Prob (j wins the auction | $v_j \in [c,\max_i \underline{b}_i)$) > 0. However, bidder j bids at least $\max_i \underline{b}_i$ with probability 1. Consequently, his bidding for $v_j \in [c,\max_i \underline{b}_i)$ contributes strictly negatively to his expected payoff. Bidder j would then be strictly better off if he submitted, for example, a bid equal to his valuation with probability 1 for $v_j \in [c,\max_i \underline{b}_i)$. This is impossible at an equilibrium, and thus $\max_i \underline{b}_i = c$ and Lemma 7 is proved. ||

Lemma 8: Let $(F_1,\ldots,F_n)$ be an element of $\mathbb{M}([c,L])^n$ such that $c \in SuppF_i$, for all $1 \leq i \leq n$, and let μ be an element of $\widetilde{\mathcal{N}}(F_1,\ldots,F_n)$. Then $\mu_i(\{(v,b) \in [c,L]^2 \mid b \leq v\}) = 1$, for all $1 \leq i \leq n$.

Proof: For all $1 \leq i \leq n$, let $\mu_{i2}(.|v)$ be a conditional probability distribution of μ over the second component space. Then $\mu_i(\{(v,b) \in [c,L]^2 \mid b \leq v\}) = \int \mu_{i2}([c,v]|v)\, d\mu_{i1} = \int \mu_{i2}([c,v]|v)\, dF_i$. We show that $\mu_{i2}([c,v]|v) = 1$ for F_i-almost all v in [c,d]. Assume there exists a Borel subset B of [c,d] of strictly positive F_i- measure such that $\mu_{i2}([c,v]|v) < 1$ and thus $\mu_{i2}((v,d]|v) > 0$, for all v in B. From the previous lemma, bids in (c,d] have a strictly positive probability of winning. Consequently, bids in (v,d] contribute strictly negatively to bidder i's payoff if his valuation is equal to v. Bidder i's expected payoff would then be strictly increased if, for all v in B, he bid $b = v$ instead of bidding in (v,d] when his valuation is equal to v. This is impossible at an equilibrium and Lemma 8 is proved. ||

Proof of Theorem 1: The first inclusion in (i) follows from Lemma 5. The inclusion $\widetilde{\mathcal{N}} \subseteq \text{marg} \circ \overline{\mathcal{N}}$ is proved in Lebrun (1996). The inclusions $\text{marg} \circ \overline{\mathcal{N}} \subseteq \widetilde{\mathcal{N}}$ and $\text{marg} \circ \mathcal{N}' \subseteq \widetilde{\mathcal{N}}$ follow from Lemma 3. Statement (i) then follows. Lemma 3 implies (ii). Lemma 6 and (i) imply (iii). Lemma 8 implies (iv). ||

Appendix 2

This appendix deals only with the game $F\widetilde{P}A$. For the sake of convenience, we denote the value of $\widetilde{p}_i$ at $(v_1,\ldots,v_n,b_1,\ldots,b_n)$ by $\widetilde{p}_i(v_i,b_i,v_{-i},b_{-i})$ where the first arguments pertain to bidder i.

Lemma 9: Let δ be a strictly positive number. There exists a function ρ from $[c,L]_1 \times [c,L]_2$ to $[c,L]_2$ such that, , $\mu_{-i}(\{(v_{-i},b_{-i}) \mid \max b_{-i} = \rho(v, b)\}) = 0$ and $\int \widetilde{p}_i(v,\rho(v,b),v_{-i},b_{-i})\, d\mu_{-i}(v_{-i},b_{-i}) - \int \widetilde{p}_i(v,b,v_{-i},b_{-i})\, d\mu_{-i}(v_{-i},b_{-i}) \geq -\delta$, for all (v,b) in $[c,L]_1 \times [c,L]_2$.

Proof: It suffices to take $\rho(v,b) = b$ if b is not a mass point of the distribution of the highest bid form the bidders $j \neq i$ when they follow μ, $\rho(v,b)$ is slightly (by at most δ) above b if b is such a mass point and if v is strictly larger than b, and $\rho(v,b)$ is slightly larger (by at most δ) than v if v $<$ b and if b is such a mass point. It is then easy to check that the statement of the lemma holds true. $\|$

<u>Lemma</u> <u>10</u>: *Let μ_{-i} be an element of $\mathcal{S}_{-i}$. Then, for all $\epsilon > 0$ there exists a measurable function function ζ_i such that $\zeta_i(v)$ is an ϵ-best response in $[c,L]_2$ from bidder i with valuation v to μ_{-i} and $\mu_{-i}(\{(v_{-i},b_{-i}) \mid \max b_{-i} = \zeta_i(v)\}) = 0$, for all v in $[c,L]_1$, ζ_i takes only a finite number of values, and the set of discontinuity points of ζ_i in $[c,L]_1$ is a Borel set of F_i-measure 0.*

<u>Proof</u>: Let[17] $w_0, w_1, \ldots, w_m$ be a strictly increasing sequence in $[c,L]_1$ such that $w_0 = c$, $w_m = L$ and $|w_{k+1} - w_k| < \delta$, for all k, and such that $F_i(\{w_1, \ldots, w_m\}) = 0$. For all v in $[c,L]_1$, let $\beta(v)$ be a δ best response to μ_{-i} from bidder i when his valuation is v, that is, $\int \tilde{p}_i(v,\beta(v),v_{-i},b_{-i})\, d\mu_{-i}(v_{-i},b_{-i}) \geq \int \tilde{p}_i(v,b,v_{-i},b_{-i})\, d\mu_{-i}(v_{-i},b_{-i}) - \delta$, for all b in $[c,L]_2$. Let γ be the function from $[c,L]_1$ to $[c,L]_2$ such that $\gamma(v) = \rho(v,\beta(v))$, for all v in $[c,L]_1$, where ρ is the function defined in the statement of Lemma 9 for δ. Then from this lemma we have $\int \tilde{p}_i(v,\gamma(v),v_{-i},b_{-i})\, d\mu_{-i}(v_{-i},b_{-i}) \geq \int \tilde{p}_i(v,\beta(v),v_{-i},b_{-i})\, d\mu_{-i}(v_{-i},b_{-i}) - \delta$ and thus $\int \tilde{p}_i(v,\gamma(v),v_{-i},b_{-i})\, d\mu_{-i}(v_{-i},b_{-i}) \geq \int \tilde{p}_i(v,b,v_{-i},b_{-i})\, d\mu_{-i}(v_{-i},b_{-i}) - 2\delta$, for all b, that is, $\gamma(v)$ is a 2δ-best response and furthermore, from the definition of ρ in Lemma 9, we have $\mu_{-i}(\{(v_{-i},b_{-i}) \mid \max b_{-i} = \gamma(v)\}) = 0$, for all v. Consequently, in the range of γ there is a probability zero of a tie and thus Prob (bidder i wins | bidder i's valuation $=$ w, bidder i's bid $=$ b) is independent of w for all b in the range of γ.

Let k be such that $m \geq k \geq 1$ and v in $[w_{k-1},w_k]$. Using the observation in the end of the previous paragraph, we do not write the valuation as an argument of the probability of winning when the bid is in the range of γ. We then have

$$\int \tilde{p}_i(v,\gamma(w_{k-1}),v_{-i},b_{-i})d\mu_{-i}(v_{-i},b_{-i})$$
$$= (v - w_{k-1}) \text{ Prob (i wins} \mid \gamma(w_{k-1})) + (w_{k-1} - \gamma(w_{k-1})) \text{ Prob (i wins} \mid \gamma(w_{k-1}))$$

$\geq$ (v $-$ w_{k-1}) Prob (i wins | $\gamma(w_{k-1})$) $+$ ($w_{k-1} - \gamma$(v)) Prob (i wins | γ(v)) $-$ 2δ

$=$ (v $-$ w_{k-1}) (Prob (i wins | $\gamma(w_{k-1})$) $-$ Prob (i wins | γ(v))) $+$ (v $-$ γ(v)) Prob (i wins | γ(v)) $- 2\delta$

$\geq$ (v $-$ γ(v)) Prob (i wins | γ(v)) $-$ 3δ $=$ $\int \widetilde{p_i}$ (v,γ(v),v_{-i},b_{-i}) d$\mu_{-i}(v_{-i},b_{-i})$ $- 3\delta$.

The first inequality follows from the fact that $\gamma(w_{k-1})$ is a 2δ-best response of bidder i with valuation w_{k-1}. The second inequality follows from the fact that $|v - w_{k-1}| < \delta$, for all v in $[w_{k-1},w_k]$. We thus see that $\gamma(w_{k-1})$ is a 3δ-better response from bidder i with valuation v in $[w_{k-1},w_k]$ than γ(v). Since γ(v) is a 2δ-best response from bidder i with valuation v, we obtain that $\gamma(w_{k-1})$ is a 5δ-best response from bidder i with valuation v in $[w_{k-1},w_k]$.

To end the proof of Lemma 10, it suffices now to take $\delta = \epsilon/5$ and to define ζ over [c,L] as follows,

$$\zeta(v) = \gamma(w_{k-1}),$$

if and only if v belongs to $[w_{k-1},w_k)$, for k ≥ 1, and

$$\zeta(L) = \gamma(w_{m-1}).$$

In fact, ζ(v) is a $5\delta = \epsilon$ best response from bidder i with valuation equal to v, for all v in $[c,L]_1$ and the set of possible discontinuities is included in $\{w_1,\dots,w_m\}$ and $F_i(\{w_1,\dots,w_m\}) = 0$. ||

<u>*Lemma*</u> <u>*11*</u>*: Let* $(F_1^l,\dots,F_n^l)_{l\geq 1}$ *be a sequence in* $\mathbb{M}([c,L])^n$ *which converges weakly towards* $(F_1,\dots,F_n)$*. Let* $(\mu_{-i}^l)_{l\geq 1}$ *be a sequence in* $\mathcal{S}_{-i}(F_{-i}^l)$ *which converges weakly towards* μ_{-i} *in* $\mathcal{S}_{-i}(F_{-i})$*. Then for all* $\epsilon > 0$*, there exists a measurable function* ζ_i *from* $[c,L]_1$ *to* $[c,L]_2$ *such that*

the sequence $\int \widetilde{p}_i\ (v_i,b_i,v_{-i},b_{-i})\ d(\zeta_i * F_i^l)(v_i,b_i)\ d\mu_{-i}^l(v_{-i},b_{-i})$ *tends towards* $\int \widetilde{p}_i\ (v_i,b_i,v_{-i},b_{-i})$ $d(\zeta_i * F_i)(v_i,b_i)\ d\mu_{-i}(v_{-i},b_{-i})$ *and*

$$\int \widetilde{p}_i\ (v_i,b_i,v_{-i},b_{-i}) d(\zeta_i * F_i)(v_i,b_i) d\mu_{-i}(v_{-i},b_{-i}) \geq \int \widetilde{p}_i\ (v_i,b_i,v_{-i},b_{-i}) d\mu_i(v_i,b_i) d\mu_{-i}(v_{-i},b_{-i}) - \epsilon,$$

for all μ_i *in* $\mathcal{S}_i(F_i)$.

Proof: Let ϵ be strictly positive. Let ζ_i be the function as in the previous lemma corresponding to this ϵ. The expected payoff $\int \widetilde{p}_i\ (v_i,b_i,v_{-i},b_{-i}) d(\zeta_i * F_i^l)(v_i,b_i) d\mu_{-i}^l(v_{-i},b_{-i})$ can be rewritten as $\int \widetilde{p}_i\ (v_i,\zeta_i(v_i),v_{-i},b_{-i}) dF_i^l(v_i) d\mu_{-i}^l(v_{-i},b_{-i})$. The measures $F_i^l \otimes \mu_{-i}^l$ tend weakly towards $F_i \otimes \mu_{-i}$. Since the set of discontinuities of $\widetilde{p}_i\ (v_i,b_i,v_{-i},b_{-i})$ is included in $\{\ (v_i,b_i,v_{-i},b_{-i}) \mid b_i = \max b_{-i}\ \}$, we have that the set of 2n – 1 tuples (v_i,v_{-i},b_{-i}) where $\widetilde{p}_i\ (v_i,\zeta_i(v_i),v_{-i},b_{-i})$ is discontinuous is included in $\{\ (v_i,v_{-i},b_{-i}) \mid \zeta_i$ is discontinuous at v_i or $\zeta_i(v_i) = \max b_{-i}\ \}$. This last set can be rewritten as follows

$$\{\ (v_i,v_{-i},b_{-i}) \mid \zeta_i \text{ is discontinuous at } v_i\ \} \bigcup \{\ (v_i,v_{-i},b_{-i}) \mid \zeta_i(v_i) = \max b_{-i}\ \}.$$

The $F_i \otimes \mu_{-i}$-measure of the first set is equal to the F_i measure of $\{\ v_i \mid \zeta_i$ is discontinuous at $v_i\ \}$ and is thus equal to 0. The second set is included in the reunion $\bigcup_{k=1}^{m} \{\ (v_i,v_{-i},b_{-i}) \mid z_k = \max b_{-i}\ \}$, where $\{z_1, \ldots, z_m\}$ is the finite range of ζ_i. The $F_i \otimes \mu_{-i}$-measure of the set $\{\ (v_i,v_{-i},b_{-i}) \mid z_k = \max b_{-i}\ \}$ is the μ_{-i} measure of $\{\ (v_{-i},b_{-i}) \mid z_k = \max b_{-i}\ \}$ and , from Lemma 10, is equal to 0. Consequently, the $F_i \otimes \mu_{-i}$ measure of the set of discontinuities of $\widetilde{p}_i\ (v_i,\zeta_i(v_i),v_{-i},b_{-i})$ is equal to 0 and thus $\int \widetilde{p}_i\ (v_i,\zeta_i(v_i),v_{-i},b_{-i})\ dF_i^l(v_i)\ d\mu_{-i}^l(v_{-i},b_{-i})$ tends towards $\int \widetilde{p}_i\ (v_i,\zeta_i(v_i),v_{-i},b_{-i})\ dF_i(v_i)\ d\mu_{-i}(v_{-i},b_{-i})$. Since the last integral is equal to $\int \widetilde{p}_i\ (v_i,b_i,v_{-i},b_{-i})\ d(\zeta_i * F_i)(v_i,b_i)\ d\mu_{-i}(v_{-i},b_{-i})$, we have proved the first part of Lemma 11.

The integrals $\int \widetilde{p}_i\,(v_i,b_i,v_{-i},b_{-i})\; d(\zeta_i * F_i)(v_i,b_i)\; d\mu_{-i}(v_{-i},b_{-i})$ and $\int \widetilde{p}_i\,(v_i,b_i,v_{-i},b_{-i})$ $d\mu_i(v_i,b_i)\;\; d\mu_{-i}(v_{-i},b_{-i})$ can respectively be rewritten as $\int(\int \;\; \widetilde{p}_i\,(v_i,\zeta_i(v_i),v_{-i},b_{-i})$ $d\mu_{-i}(v_{-i},b_{-i}))\; dF_i(v_i)$ and $\int\;\{\;\int\;(\int \widetilde{p}_i\,(v_i,b_i,v_{-i},b_{-i})\; d\mu_{-i}(v_{-i},b_{-i}))\; d\mu_i(b_i \mid v_i)\;\}\; dF_i(v_i)$, where $\mu_i(b_i \mid v_i)$ is a conditional distribution of μ_i with respect to v_i. The second part of Lemma 11 then follows from the fact that $\zeta_i(v_i)$ is a ϵ-best response of bidder i to μ_{-i} and thus $\int$ $\widetilde{p}_i\,(v_i,\zeta_i(v_i),v_{-i},b_{-i})\; d\mu_{-i}(v_{-i},b_{-i}) \;\geq\; \int \widetilde{p}_i\,(v_i,b_i,v_{-i},b_{-i})\; d\mu_{-i}(v_{-i},b_{-i}) \;-\; \epsilon$, for all b_i. ‖

<u>*Lemma*</u> <u>*12*</u>*: Let $(\mu^l)_{l\geq 1}$ be a sequence of Nash equilibria of $F\widetilde{P}A$ which converges weakly towards μ. Then μ is a Nash equilibrium of $F\widetilde{P}A$ and $(\widetilde{P}\;(\mu^l))_{l\geq 1}$ tends towards $\widetilde{P}\;(\mu)$.*

<u>Proof</u>: Let $(F_1^l,\ldots,F_n^l)_{l\geq 1}$ be the sequence in $\mathbb{M}([c,L]_1)^n$ of marginal distributions of $(\mu^l)_{l\geq 1}$ which converges weakly towards $(F_1,\ldots,F_n)$, the marginals of μ. Since the payoffs are bounded there exists a subsequence $(\mu^{l_t})_{t\geq 1}$, such that the limit of $(\widetilde{P}_1,\ldots,\widetilde{P}_n)(\mu^{l_t})$, for t tending towards $+\infty$, exists and is finite. We will first prove that

$$\widetilde{P}_i(\theta_i,\mu_{-i}) \;\leq\; \lim_{l\to+\infty} \widetilde{P}_i(\mu^l),$$

for all $1 \leq i \leq n$ and all $\theta_i \in \mathcal{S}_i(F_i)$. Suppose there exist $1 \leq i \leq n$ and $\theta_i \in \mathcal{S}_i(F_i)$ such that $\widetilde{P}_i(\theta_i,\mu_{-i}) \;>\; \lim_{t\to+\infty} \widetilde{P}_i(\mu^{l_t})$. There thus exists $\delta > 0$ and t' ≥ 1, such that

$$\widetilde{P}_i(\theta_i,\mu_{-i}) \;>\; \widetilde{P}_i(\mu^{l_t}) \;+\; \delta,$$

for all t $\geq$ t'. Since μ^{l_t} is a Nash equilibrium of $F\widetilde{P}A(F_1^{l_t},\ldots,F_n^{l_t})$, for all t ≥ 1, we also have

$$\text{(A7)} \quad \widetilde{P}_i(\theta_i,\mu_{-i}) \;>\; \widetilde{P}_i(\nu_i^{l_t},\mu_{-i}^{l_t}) \;+\; \delta,$$

or, equivalently,

$$\int\widetilde{\mathrm{p}}_{\,i}(\mathrm{v,b})\ \mathrm{d}(\nu_i^{l_t}\otimes\mu_{-i}^{l_t})\ <\ \int\widetilde{\mathrm{p}}_{\,i}(\mathrm{v,b})\ \mathrm{d}(\theta_i\otimes\mu_{-i})\ -\ \delta,$$

for all t $\geq$ t' and all $\nu_i^{l_t}$ in $\mathcal{S}(\mathrm{F}_i^{l_t})$. Let ζ_i be the function from the previous lemma corresponding to $\epsilon\ =\ \delta/2$ and consider the measures $\lambda_i^{l_t}\ =\ \zeta_i{*}\mathrm{F}_i^{l_t}$. We have $\lambda_i^{l_t}\ \in\ \mathcal{S}(\mathrm{F}_i^{l_t})$, and thus

(A8) $$\int\widetilde{\mathrm{p}}_{\,i}(\mathrm{v,b})\ \mathrm{d}(\lambda_i^{l_t}\otimes\mu_{-i}^{l_t})\ <\ \int\widetilde{\mathrm{p}}_{\,i}(\mathrm{v,b})\ \mathrm{d}(\theta_i\otimes\mu_{-i})\ -\ \delta,$$

for all t $\geq$ t'. From the previous lemma we also know that $\lim_{t\to+\infty}\int\widetilde{\mathrm{p}}_{\,i}(\mathrm{v,b})\ \mathrm{d}(\lambda_i^{l_t}\otimes\mu_{-i}^{l_t})$ exists, is equal to $\int\widetilde{\mathrm{p}}_{\,i}(\mathrm{v,b})\ \mathrm{d}(\lambda_i\otimes\mu_{-i}^{l_t})$, where $\lambda_i\ =\ \zeta_i{*}\mathrm{F}_i$, and is not smaller than $\int\widetilde{\mathrm{p}}_{\,i}(\mathrm{v,b})$ $\mathrm{d}(\nu_i\otimes\mu_{-i})\ -\ \delta/2$, for all ν_i in $\mathcal{S}_i(\mathrm{F}_i)$ and in particular for $\nu_i\ =\ \theta_i$. We thus have

$$\lim_{t\to+\infty}\int\widetilde{\mathrm{p}}_{\,i}(\mathrm{v,b})\ \mathrm{d}(\lambda_i^{l_t}\otimes\mu_{-i}^{l_t})\quad\geq\quad\int\widetilde{\mathrm{p}}_{\,i}(\mathrm{v,b})\ \mathrm{d}(\theta_i\otimes\mu_{-i})\ -\ \delta/2.$$

This last inequality however contradicts (A8), there exists no such θ_i and (A7) holds true for all $\theta_i\in\mathcal{S}_i(\mathrm{F}_i)$.

In this second part of the proof of Lemma 12 we prove that, for all 1 $\leq$ i $\leq$ n, player i can obtain at least the limit of his equilibrium payoffs $\widetilde{\mathrm{P}}_{\,i}(\mu^{l_t})$ by playing the limit μ_i of his equilibrium strategies, that is, that

$$\widetilde{\mathrm{P}}_{\,i}(\mu)\ \geq\ \lim_{t\to+\infty}\widetilde{\mathrm{P}}_{\,i}(\mu^{l_t}).$$

Suppose there exists 1 $\leq$ i $\leq$ n such that the inequality above does not hold, that is, $\widetilde{\mathrm{P}}_{\,i}(\mu)\ <\ \lim_{t\to+\infty}\widetilde{\mathrm{P}}_{\,i}(\mu^{l_t})$. By taking a subsequence if necessary we can assume that $\sum_{i=1}^{n}\widetilde{\mathrm{P}}_{\,i}(\mu^{l_t})$ is convergent (since $\widetilde{\mathrm{P}}_i$ is bounded). Because the function $\sum_{i=1}^{n}\widetilde{\mathrm{p}}_i$ is upper semicontinuous, the integral $\int\sum_{i=1}^{n}\widetilde{\mathrm{p}}_i$ dν considered as a function of the probability measure ν is also upper semicontinuous. Consequently, we have

$$\lim_{t\to+\infty} \sum_{i=1}^{n}\widetilde{\mathrm{P}}_i(\mu^{l_t}) = \lim_{t\to+\infty} \int\sum_{i=1}^{n}\widetilde{\mathrm{p}}_i \, \mathrm{d}\mu^{l_t} \le \int \sum_{i=1}^{n}\widetilde{\mathrm{p}}_i \, \mathrm{d}\mu = \sum_{i=1}^{n}\widetilde{\mathrm{P}}_i(\mu).$$

There thus exist a convergent subsequence$\mu^{l_{t_1}}, \mu^{l_{t_2}}, \ldots$ and $1 \le \mathrm{j} \le \mathrm{n}$ with $\mathrm{j} \neq \mathrm{i}$ such that $\lim_{k\to+\infty} \widetilde{\mathrm{P}}_j(\mu^{l_{t_k}}) < \widetilde{\mathrm{P}}_j(\mu)$, which contradicts the result of the first part of the proof and thus the second part of the proof is finished. ||

<u>Proof of Theorem 2</u>: Immediate from Lemma 12. ||

<u>Proof of Corollary 2</u>: Let $(\mathrm{F}_1^l, \ldots, \mathrm{F}_n^l)_{l\ge1}$ be a sequence in $\mathbb{M}([\mathrm{c,L}])^n$ which converges weakly towards $(\mathrm{F}_1, \ldots, \mathrm{F}_n)$ and such that c $\in$ Supp F_i^l, for all $1 \le \mathrm{i} \le \mathrm{n}$. Let $(\mu^l)_{l\ge1}$ be a weakly convergent sequence such that $\mu^l \in \widetilde{\mathcal{N}}(\mathrm{F}_1^l,\ldots,\mathrm{F}_n^l)$, for all $\mathrm{l} \ge 1$. Let μ be its limit. From Corollary 1 or Theorem 2, μ belongs to $\widetilde{\mathcal{N}}(\mathrm{F}_1,\ldots,\mathrm{F}_n)$. From Theorem 1 (iv), we have $\mu_i^l(\{(\mathrm{v,b}) \in [\mathrm{c,L}]^2 \mid \mathrm{b} \le \mathrm{v}\}) = 1$, for all $1 \le \mathrm{i} \le \mathrm{n}$. Since $\mu_i(\{(\mathrm{v,b}) \in [\mathrm{c,L}]^2 \mid \mathrm{b} \le \mathrm{v}\}) \ge \limsup_{l\to+\infty} \mu_i^l(\{(\mathrm{v,b}) \in [\mathrm{c,L}]^2 \mid \mathrm{b} \le \mathrm{v}\})$, we have $\mu_i(\{(\mathrm{v,b}) \in [\mathrm{c,L}]^2 \mid \mathrm{b} \le \mathrm{v}\}) = 1$ and Corollary 2 is proved. ||

Appendix 3

<u>Proposition 2</u>: Under the assumptions of Proposition 1, if x $\le \frac{q}{p+q}$ the supports of the equilibrium bid distributions are as in Figure 2, and

$J(b) = I(b) = H(b) = 1$*, for all* $b \geq \eta$

$J(b) = I(b) = \frac{q(1-x)+px}{1-b}$ *and* $H(b) = 1$*, for b in Range A, that is, for b such that* $\widetilde{b} \leq b \leq \eta$,

$J(b) = q$, $I(b) = \frac{px}{x-b}$*, and* $H(b) = \frac{q(1-x)+px}{q(1-b)}$*, for b in Range C, that is,* $0 \leq b \leq \widetilde{b}$,

where

$\eta = 1 - q + (q-p)x$ *and* $\widetilde{b} = \frac{q-p}{q}x$.

[FIGURE 2]

Footnotes

[1]: Bidding is thus mandatory. However, we will assume in the next paragraph that the minimum allowable bid c is not larger than any possible valuation. No bidder is thus forced to a strictly negative payoff and any equilibrium would also be an equilibrium if bidding was only voluntary.

[2]: The assumption c $\geq$ 0 was unnecessary in Lebrun (1996) and we do not keep it here.

[3]: In Lebrun (1996), we assumed that the support of F_i was included in $[c, K-1]$, for all i, and, for the sake of convenience in the proofs where we shifted upwards some bid distributions, we required the strategies to define bid probability distributions in $[c, K]$. However, as it can be easily shown, no equilibrium involves strategies bidding above $K - 1$. Here, our bound L is equal to the bound $K - 1$ in Lebrun (1996).

[4]: Consider the three bidder example where bidder 1's valuation is equal to 1 with probability 1, bidder 2's valuation is equal to 3 with probability 1, and bidder 3's valuation is equal to 1 with probability 1/2 and 4 with probability 1/2. If μ_1 is bidder 1's strategy consisting in always bidding 3, μ_2 is bidder 2's strategy which always bids 3, and μ_3 is bidder 3's strategy consisting in bidding 0 if the valuation is equal to 1 and 3 if the valuation is 4. Then (μ_1,μ_2,μ_3) is a Nash equilibrium of F$\widetilde{\text{P}}$A but cannot be extended in an equilibrium of FPA'. In the example with two bidders from the introduction of Lebrun (1996) (where bidder 1's valuation is concentrated at 0 and bidder 2's valuation is uniformly distributed over $[0,1]$), there is no equilibrium of FPA but there exists one of FPA' (where both bidders submit 0 with probability 1).

[5]: In this example of footnote 4, bidder 2 has to loose with probability 1 as soon as bidder 3 is involved in the tie but has to win if he is involved in the tie with only bidder 1.

[6]: Since for any Nash equilibrium μ' of FPA' marg'μ' is a Nash equilibrium of F$\widetilde{\text{P}}$A, the same procedure to obtain an equilibrium of FPA' can be applied to margμ' and we see that any Nash equilibrium of FPA' is equivalent (induces the same distributions over the valuation-bid couples) to an equilibrium where one bidder always sends the message 1 and where the other bidders always send the message 0. For such an equilibrium, we can interpret the messages as bids in a secondary second price auction used to break possible ties. Notice that in this auction, the second highest bid and thus the bidders payments are always equal to 0. Maskin and Riley (1996a) call a similar tie breaking rule the Vickrey auction-tie breaking rule.

[7]: The value at L is equal to the value over $[w_{m-1},w_m)$.

[8]: At d, $F_1,\ldots,F_n$ are left-differentiable.

[9]: The requirement $\frac{d}{dv}\frac{G_1}{G_2}(v) < 0$, for all v in $(c,c + \delta]$, implies that G_1/G_2 is strictly increasing over $(c,c + \delta]$ and that conditionally on $v \in [c,e]$ the distribution G_1 first order stochastically dominates G_2 (strictly), for all e in $(c,c + \delta)$.

[10]: Actually, in Lebrun (1996) it was already shown that this strict ranking holds true for an open set of parameters (γ,δ) including the set of couples of (γ,δ) such that $\gamma\delta \geq 1/2$.

[11]: Notice that the subset of couples of distributions satisfying B or C is everywhere dense in $\mathbb{M}([0,1])^2$ and thus that any open neighborhood of (F_1,F_2) has a non-empty intersection with this subset.

[12]: Here, we mean the natural variant for bidder i's expected payoff conditional on v_i, that is, the variant whose value at v_i is equal to the value of the variant of the conditional expected payoff if F_i was concentrated at v_i.

[13]: Under assumption A, if two valuation distributions are equal, so are the bid functions (see Corollary 3 (iv) in Lebrun 1997a or Corollary 4 (iv) in Lebrun 1997b).

[14]: Where, for example, $c = 0$ and $L = 2$.

[15]: As mentioned in the proof of Lemma 4, this method of proof is similar to the method followed in Lebrun (1996). A method identical to the method in Lebrun (1996) would here rely on (using the notation from the proof of Theorem 2 in Lebrun 1996) $\overline{N}_i$ equal to the set of strategies which do not submit any mass point $b_h \neq \underline{b}$ of the highest bid from the other biders following μ_{-i} for a strictly probability set of $v_i \neq b_h$ and which send the message 1 always except when the submitted bid is equal to $\underline{b}$, in which case the message sent is 0, for $i \neq j$, and on

$\overline{\mathrm{N}}_j$ equal to the set of strategies which do not submit any mass point different from $\underline{b}$ of the distribution of the highest bid from μ_{-j} for a strictly positive probability set of valuations different from this point and which always send the message 1. Here, we rather use a similar method which is equivalent to using (in the notation of the proof of Theorem 2 in Lebrun 1996) $\overline{\mathrm{N}}_i$ which is the set of strategies which always send the message 0, i $\neq$ j, and $\overline{\mathrm{N}}_j$ which is the set of strategies which always send the message 0 and to applying the obvious variant of Lemma 4 in Lebrun (1996) where assumption 1. is replaced by the assumption that, for all $1 \leq i \leq n$ and for all $\nu_i \in \mathrm{R}_i$, we have $\sup_{\eta_i \in \mathrm{A}_i} \mathrm{Q}_i(\eta_i,\mu_{-i}) \geq \mathrm{Q}_i(\nu_i,\mu_{-i})$.

[16]: If $J = \{1,\ldots,n\}$, the probability is equal to 1. If $J \neq \{1,\ldots,n\}$, the inequality follows from the fact that $b_i < b_j$ with a strictly positive probability, for all $i \notin J$ and $j \in J$.

[17]: Obviously, such a sequence exists.

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Figures

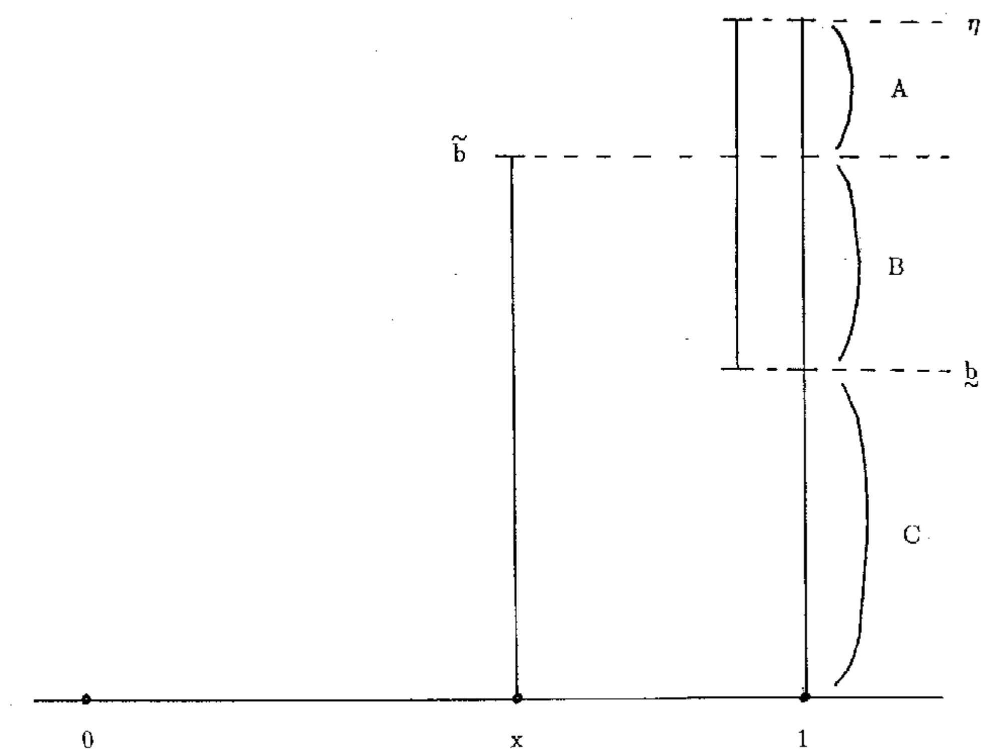

Figure 1

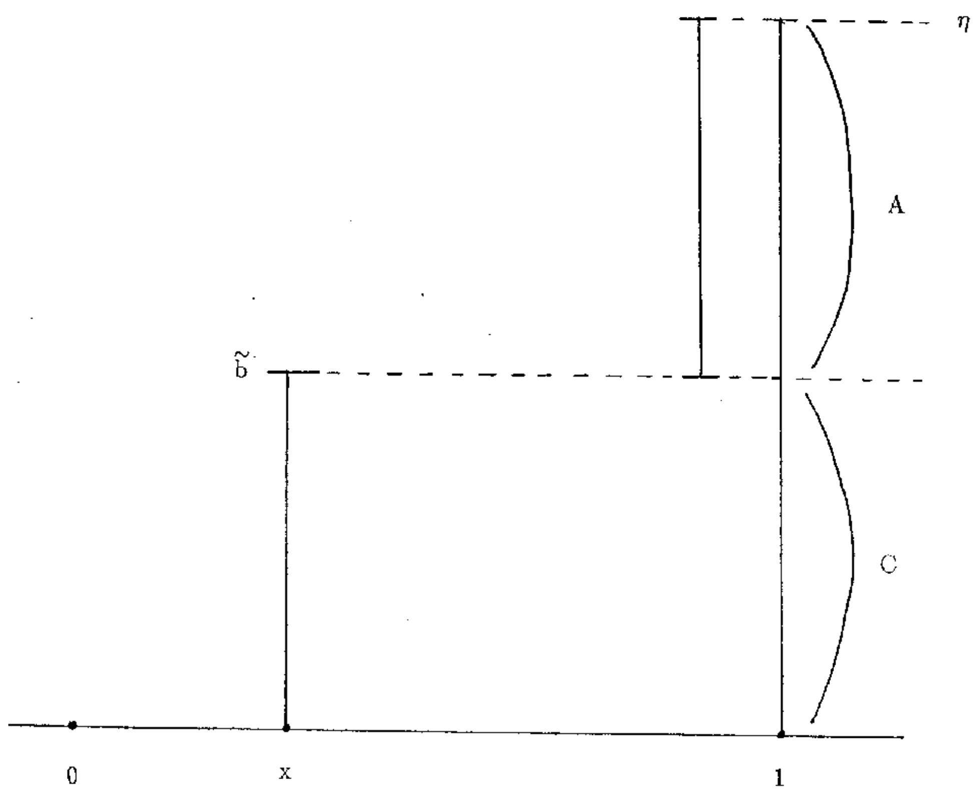

Figure 2