On-Line Appendices to
Auctions with Almost Homogeneous Bidders

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Appendix I

We divide the proof of Theorem 3 into several lemmas. Throughout, we assume that F is extended as in D (i). The proof extends straightforwardly to the case of a mass point at c (see Lebrun, 2006b).

Lemma AI-1: *For all* $\boldsymbol{\tau} = (\tau^1, ..., \tau^n)$ *in* $(-\rho, \rho)^{nm}$*:*

(i) There exists a unique[1] *Bayesian equilibrium* $(\beta_1(.;\tau), ..., \beta_n(.;\tau))$ *of the FPA with value distributions* $F_1 = F(.;\tau^1), ..., F_n = F(.;\tau^n)$*. This equilibrium is pure and there exists* $c < \eta < d$ *such that the inverse bid functions* $\alpha_1 = \beta_1^{-1}, ..., \alpha_n = \beta_n^{-1}$ *exist, are strictly increasing, and form a solution over* $(c, \eta]$ *of the system of differential equations (AI.1) below–considered in the domain* $D = \{(b, \alpha_1, .., \alpha_n) \in R^{n+1} | c, b < \alpha_i \leq d, \text{ for all } 1 \leq i \leq n\}$*–and satisfy the boundary conditions (AI.2, AI.3):*

$$\frac{d \ln F\left(\alpha_k(b;\boldsymbol{\tau});\tau^k\right)}{db} = \frac{1}{n-1}\left\{\frac{-(n-2)}{\alpha_k(b;\boldsymbol{\tau}) - b} + \sum_{\substack{l=1 \\ l \neq k}}^{n} \frac{1}{\alpha_l(b;\boldsymbol{\tau}) - b}\right\}, \text{ (AI.1)}$$

$$\alpha_k(\eta) = d, \text{ (AI.2)}$$

$$\alpha_k(c) = c, \text{ (AI.3)}$$

[1] The equilibrium is unique when participation is mandatory. If participation is voluntary, the only indeterminacy is at the lowest value c, where bidders may not take part, submit c, or randomize between the two.

for all $1 \leq k \leq n$. *Moreover,* $\frac{d}{db}\alpha_k(b;\boldsymbol{\tau}) > 0$, *for all* $1 \leq k \leq n$ *and all* b *in* $(c,\eta]$.

(ii) For all $1 \leq i \leq n$, *the functions* $\lambda_{ji}(.;\boldsymbol{\tau}) = F(.;\tau^j) \circ \alpha_j(.;\boldsymbol{\tau}) \circ \beta_i(.;\boldsymbol{\tau}) \circ F(.;\tau^i)^{-1}$, *with* $1 \leq j \leq n$ *and* $j \neq i$, *and* $\gamma_i(.;\boldsymbol{\tau}) = \beta_i(.;\boldsymbol{\tau}) \circ F(.;\tau^i)^{-1}$ *are differentiable (with respect to the first argument) over* $(0,1]$ *and form a solution of the system (AI.4, AI.5)–considered in the domain* D_i*–with initial condition (AI.6):*

$$D_i = \left\{ \begin{array}{c} \left(q, (\lambda_{ji})_{j\neq i}, \gamma_i\right) | 0 < q, \lambda_{ji} \leq 1, \\ \text{and } \gamma_i < F^{-1}(q;\tau^i), F^{-1}(\lambda_{ji};\tau^j), \text{ for all } j \neq i \end{array} \right\}$$

$$\frac{d}{dq}\lambda_{ji}(q;\boldsymbol{\tau}) = \frac{\lambda_{ji}(q;\boldsymbol{\tau})}{q} \frac{\frac{-(n-2)}{F^{-1}(\lambda_{ji}(q;\boldsymbol{\tau});\tau^j)-\gamma_i(q;\boldsymbol{\tau})} + \frac{1}{F^{-1}(q;\tau^i)-\gamma_i(q;\boldsymbol{\tau})} + \sum_{\substack{l=1\\ l\neq j,i}}^n \frac{1}{F^{-1}\left(\lambda_{li}(q;\boldsymbol{\tau});\tau^l\right)-\gamma_i(q;\boldsymbol{\tau})}}{\frac{-(n-2)}{F^{-1}(q;\tau^i)-\gamma_i(q;\boldsymbol{\tau})} + \frac{1}{F^{-1}(\lambda_{ji}(q;\boldsymbol{\tau});\tau^j)-\gamma_i(q;\boldsymbol{\tau})} + \sum_{\substack{l=1\\ l\neq j,i}}^n \frac{1}{F^{-1}\left(\lambda_{li}(q;\boldsymbol{\tau});\tau^l\right)-\gamma_i(q;\boldsymbol{\tau})}} \text{(AI.4)}$$

$$\frac{d}{dq}\gamma_i(q;\boldsymbol{\tau}) = \frac{1}{q} \frac{n-1}{\frac{-(n-2)}{F^{-1}(q;\tau^i)-\gamma_i(q;\boldsymbol{\tau})} + \frac{1}{F^{-1}(\lambda_{ji}(q;\boldsymbol{\tau});\tau^j)-\gamma_i(q;\boldsymbol{\tau})} + \sum_{\substack{l=1\\ l\neq j,i}}^n \frac{1}{F^{-1}\left(\lambda_{li}(q;\boldsymbol{\tau});\tau^l\right)-\gamma_i(q;\boldsymbol{\tau})}};\text{(AI.5)}$$

$$\lambda_{ji}(1;\boldsymbol{\tau}) = 1, \gamma_i(1;\boldsymbol{\tau}) = \eta.\text{(AI.6)}$$

(iii) For all $1 \leq i \leq n$ *and* v *in* $(c,d]$, *(AI.7) below holds true:*

$$\beta_i(v;\boldsymbol{\tau}) = v - \frac{\int_c^v \prod_{\substack{k=1\\ k\neq i}}^n F\left(\varphi_{ki}(w;\boldsymbol{\tau});\tau^k\right) dw}{\prod_{\substack{k=1\\ k\neq i}}^n F\left(\varphi_{ki}(v;\boldsymbol{\tau});\tau^k\right)}, \text{(AI.7)}$$

where $\varphi_{ki}(.;\boldsymbol{\tau}) = \alpha_k(.;\boldsymbol{\tau}) \circ \beta_i(.;\boldsymbol{\tau})$.

(iv) If $\tau^1 = ... = \tau^n$, *then* $\beta_i(v;\boldsymbol{\tau}) = v - \frac{\int_c^v F\left(w;\tau^i\right)^{n-1} dw}{F(v;\tau^i)^{n-1}}$, *for all* $1 \leq i \leq n$

and v in $(c,d]$.

Proof: The existence of an equilibrium in (i) follows from Theorem 2 in Lebrun (1999) and its characterization from Theorem 1 in Lebrun (1999). The uniqueness in (i) follows from Corollary 1 in Lebrun (1999).

(ii) follows from Lemma A2-5 in Lebrun (1997) or from Lemma A1-1 in Lebrun (2006a). An application, standard in auction theory, of the envelope theorem gives (iii) ((iii) also follows from Lemma A2-6 in Lebrun 1997). (iv) follows from Corollary 3 (v) in Lebrun (1999). ||

Lemma AI-2[2]: *For all τ in $(-\rho,\rho)^{nm}$, all v in $(c,d]$, and all $1 \leq i,j \leq n$, we have:*

$$F\left(v;\tau^i\right)\min_{w\in[v,d]}\frac{F\left(w;\tau^j\right)}{F\left(w;\tau^i\right)} \leq F\left(\varphi_{ji}\left(v;\boldsymbol{\tau}\right);\tau^j\right) \leq F\left(v;\tau^i\right)\max_{w\in[v,d]}\frac{F\left(w;\tau^j\right)}{F\left(w;\tau^i\right)},$$

where $\varphi_{ji}\left(v;\boldsymbol{\tau}\right)$ is equal to $\alpha_j\left(\beta_i\left(v;\boldsymbol{\tau}\right);\boldsymbol{\tau}\right)$.

Proof: Subtracting the equation in (AI.1) for $\frac{d\ln F\left(\alpha_i(b;\boldsymbol{\tau});\tau^i\right)}{db}$ from the equation for $\frac{d\ln F\left(\alpha_j(b;\boldsymbol{\tau});\tau^j\right)}{db}$, we find:

$$\frac{d\ln F\left(\alpha_j\left(b;\boldsymbol{\tau}\right);\tau^j\right)}{db}-\frac{d\ln F\left(\alpha_i\left(b;\boldsymbol{\tau}\right);\tau^i\right)}{db}=\frac{1}{\alpha_i\left(b;\boldsymbol{\tau}\right)-b}-\frac{1}{\alpha_j\left(b;\boldsymbol{\tau}\right)-b}. \quad \text{(AI.8)}$$

Let u be in $(c,d]$ and let $z>0$ such that $z<\min_{w\in[u,d]}\frac{F\left(w;\tau^j\right)}{F\left(w;\tau^i\right)}$.

Define y in $[u,d]$ as follows: $y=\inf\left\{w \text{ in } [u,d]\,|zF\left(w;\tau^i\right)\geq F\left(c;\tau^j\right)\right\}$, with the convention $d=\inf\varnothing$. Since $\varphi_{ji}\left(w;\tau\right)\geq c$, for all w, we have $zF\left(w;\tau^i\right)\leq F\left(\varphi_{ji}\left(w;\boldsymbol{\tau}\right);\tau^j\right)$, for all w in (u,y). Suppose v in $(y,d]$ is such that $zF\left(v;\tau^i\right)=F\left(\varphi_{ji}\left(v;\boldsymbol{\tau}\right);\tau^j\right)$. Then, $\varphi_{ji}\left(v;\boldsymbol{\tau}\right)>c$. From (AI.8), we

[2] Although not explicitly proved in Lebrun (1997), Lemma AI-2 can be derived from the proof of its Lemma A2-3.

have:

$$\begin{aligned}\frac{d\ln F\left(\varphi_{ji}\left(v;\boldsymbol{\tau}\right);\tau^{j}\right)}{dv} &= \\ \frac{d\ln F\left(v;\tau^{i}\right)}{dv} &+\frac{d}{dv}\beta_{i}\left(v;\boldsymbol{\tau}\right)\left\{\frac{1}{v-\beta_{i}\left(v;\boldsymbol{\tau}\right)}-\frac{1}{\varphi_{ji}\left(v;\boldsymbol{\tau}\right)-\beta_{i}\left(v;\boldsymbol{\tau}\right)}\right\}.\text{(AI.9)}\end{aligned}$$

By definition of z, we have $z<\frac{F\left(v;\tau^{j}\right)}{F\left(v;\tau^{i}\right)}$ and thus $zF\left(v;\tau^{i}\right)<F\left(v;\tau^{j}\right)$. Consequently, $\varphi_{ji}\left(v;\boldsymbol{\tau}\right)<v$. Since $\frac{d}{dv}\ln zF\left(v;\tau^{i}\right)=\frac{d}{dv}\ln F\left(v;\tau^{i}\right)$, (AI.9) then implies $\frac{d\ln F\left(\varphi_{ji}\left(v;\boldsymbol{\tau}\right);\tau^{j}\right)}{dv}<\frac{d}{dv}\ln zF\left(v;\tau^{i}\right)$. Moreover, from the definition of z, $zF\left(d;\tau^{i}\right)=z<1=F\left(d;\tau^{j}\right)=F\left(\varphi_{ji}\left(d;\boldsymbol{\tau}\right);\tau^{j}\right)$. From a variant of Lemma 2 in Milgrom and Weber (1982), we obtain $zF\left(w;\tau^{i}\right)\leq F\left(\varphi_{ji}\left(w;\boldsymbol{\tau}\right);\tau^{j}\right)$, for all w in $[y,d]$, hence in $[u,d]$, and, in particular, $zF\left(u;\tau^{i}\right)\leq F\left(\varphi_{ji}\left(u;\boldsymbol{\tau}\right);\tau^{j}\right)$. Finally, making z tend towards $\min_{w\in[u,d]}\frac{F\left(w;\tau^{j}\right)}{F\left(w;\tau^{i}\right)}$, we find $F\left(u;\tau^{i}\right)\min_{w\in[u,d]}\frac{F\left(w;\tau^{j}\right)}{F\left(w;\tau^{i}\right)}\leq F\left(\varphi_{ji}\left(u;\boldsymbol{\tau}\right);\tau^{j}\right)$. The other inequality can be proved similarly. ||

Lemma AI-3: *Let $\eta\left(\boldsymbol{\tau}\right)$ be the common maximum of the equilibrium bid functions $\beta_{1}\left(.;\boldsymbol{\tau}\right),...,\beta_{n}\left(.;\boldsymbol{\tau}\right)$, for all $\boldsymbol{\tau}$ in $\left(-\rho,\rho\right)^{nm}$. Then, there exists K such that $\frac{|\eta(0)-\eta(\boldsymbol{\tau})|}{|\boldsymbol{\tau}|}\leq K$, for all $\boldsymbol{\tau}$ in $\left(-\rho,\rho\right)^{nm}$.*

Proof: From Lemma AI-2, we have:

$$F\left(v;\tau^{i}\right)\min_{w\in[v,d]}\frac{F\left(w;\tau^{j}\right)}{F\left(w;\tau^{i}\right)}\leq F\left(\varphi_{ji}\left(v;\boldsymbol{\tau}\right);\tau^{j}\right)\leq F\left(v;\tau^{i}\right)\max_{w\in[v,d]}\frac{F\left(w;\tau^{j}\right)}{F\left(w;\tau^{i}\right)},$$

where $\varphi_{ji}\left(v;\boldsymbol{\tau}\right)$ is equal to $\alpha_{j}\left(\beta_{i}\left(v;\boldsymbol{\tau}\right);\boldsymbol{\tau}\right)$, for all $1\leq i,j\leq n$, $\boldsymbol{\tau}$ in $\left(-\rho,\rho\right)^{nm}$, and v in $(c,d]$. From D (iii), we then have:

$$\begin{aligned}F\left(v;\tau_{i}\right)\left(1-\frac{M\left|\tau_{j}-\tau_{i}\right|}{F\left(v;\tau_{i}\right)}\right) &= F\left(v;\tau_{i}\right)\min_{w\in[v,d]}\left(1-\frac{M\left|\tau_{j}-\tau_{i}\right|}{F\left(w;\tau_{i}\right)}\right)\\ &\leq F\left(\varphi_{ji}\left(v;\boldsymbol{\tau}\right);\tau_{j}\right)\leq\\ F\left(v;\tau_{i}\right)\max_{w\in[v,d]}\left(1+\frac{M\left|\tau_{j}-\tau_{i}\right|}{F\left(w;\tau_{i}\right)}\right) &= F\left(v;\tau_{i}\right)\left(1+\frac{M\left|\tau_{j}-\tau_{i}\right|}{F\left(v;\tau_{i}\right)}\right),\text{(AI.10)}\end{aligned}$$

and thus:

$$F(v;0) - M(|\tau_j| + 2|\tau_i|) \le F\left(\varphi_{ji}(v;\boldsymbol{\tau});\tau_j\right) \le F(v;0) + M(|\tau_j| + 2|\tau_i|),$$

where M is an upper bound of $\frac{\partial}{\partial\tau_1}F(v;\tau), ..., \frac{\partial}{\partial\tau_m}F(v;\tau)$ over $(c,d]\times(-\rho,\rho)^m$, for all $1 \le i, j \le n$, $\boldsymbol{\tau}$ in $(-\rho,\rho)^{nm}$, and v in $(c,d]$.

From (AI.7), we have, for $1 \le i \le n$, $\eta(\boldsymbol{\tau}) = d - \int_c^d \prod_{\substack{j=1 \\ j\ne i}}^{n} F\left(\varphi_{ji}(v;\boldsymbol{\tau});\tau_j\right) dv$ and hence, from (AI.10):

$$\begin{aligned}
& \int_c^d \frac{F(v;0)^{n-1} - (F(v;0) + M(|\tau_j| + 2|\tau_i|))^{n-1}}{|\boldsymbol{\tau}|} dv \\
\le\ & \frac{\eta(0) - \eta(\boldsymbol{\tau})}{|\boldsymbol{\tau}|} \\
\le\ & \int_c^d \frac{F(v;0)^{n-1} - \max(0, F(v;0) - M(|\tau_j| + 2|\tau_i|))^{n-1}}{|\boldsymbol{\tau}|} dv, \text{(AI.11)}
\end{aligned}$$

for all $(v;\boldsymbol{\tau})$ in $(c,d]\times(-\rho,\rho)^{nm}$. From the mean value theorem, for all v in $(c,d]$, there exists x between $F(v;0)$ and $F(v;0)+M(|\tau_j| + 2|\tau_i|)$, such that $\frac{F(v;0)^{n-1} - (F(v;0)+M(|\tau_j|+2|\tau_i|))^{n-1}}{|\boldsymbol{\tau}|}$ is equal to $-(n-1)x^{n-2}\frac{M(|\tau_j|+2|\tau_i|)}{|\boldsymbol{\tau}|}$. Since $0 \le x \le 1 + 3\rho M$ and $0 \le \frac{|\tau_j|+2|\tau_i|}{|\boldsymbol{\tau}|} \le 3$, there exists a finite K' such that the L.H.S. of the first inequality in (AI.11) is not smaller than K'. Similarly, there exists a finite K'' such that the R.H.S of the second inequality is not larger than K'', for all $\boldsymbol{\tau}$ in $(-\rho,\rho)^{nm}$. The lemma follows. ||

Lemma AI-4: *$\alpha_i(c;\boldsymbol{\tau})$ is continuous with respect to $\boldsymbol{\tau}$ at $\boldsymbol{\tau} = \mathbf{0}$.*

Proof: From Lemma AI-3, $\eta(\boldsymbol{\tau})$ is a continuous function of $\boldsymbol{\tau}$ at $\boldsymbol{\tau} = \mathbf{0}$. From Lemma AI-1 (i) and from the continuity, under our assumptions, of the solution of a differential system with respect to the parameters and to the value of the solution at the initial condition, we know that for all b in the interior $(c,\eta(\mathbf{0}))$ of the definition domain of $\alpha_1(.,\mathbf{0}) = ... = \alpha_n(.,\mathbf{0})$ and for all $\varepsilon > 0$, there exists $\delta > 0$ such that, for all $1 \le i \le n$, $\alpha_i(.,\boldsymbol{\tau})$ is defined at b and $|\alpha_i(b,\boldsymbol{\tau}) - \alpha_i(b,\mathbf{0})| \leq \varepsilon$ if $|\boldsymbol{\tau}| < \delta$. Consequently, for all b in $(c, \eta(\mathbf{0}))$, $\limsup_{\boldsymbol{\tau}\to\mathbf{0}} \alpha_i(c,\boldsymbol{\tau}) \leq \alpha_i(b,\mathbf{0}) \leq b$. By making b tend towards c, we find $\limsup_{\boldsymbol{\tau}\to\mathbf{0}} \alpha_i(c,\boldsymbol{\tau}) \leq c$. Since $\alpha_i(c,\boldsymbol{\tau})$ is never smaller than c, we have $\lim_{\boldsymbol{\tau}\to\mathbf{0}} \alpha_i(c,\boldsymbol{\tau}) = c$ and Lemma AI-4 is proved. $||$

Lemma AI-5: *There exists $\zeta' > 0$ such that $F^{-1}(q;\tau)$ exists and is (jointly) continuously differentiable with respect to $(q;\tau)$ over $(0, 1+\zeta') \times (-\rho,\rho)^m$ and, for all (q,τ) in this set and all $1 \leq l \leq m$, we have:*

$$
\begin{aligned}
\frac{\partial}{\partial q} F^{-1}(q;\tau) &= \frac{1}{f\left(F^{-1}(q;\tau);\tau\right)}, \\
\frac{\partial}{\partial \tau_l} F^{-1}(q;\tau) &= \frac{-\frac{\partial}{\partial \tau_l} F\left(F^{-1}(q;\tau);\tau\right)}{f\left(F^{-1}(q;\tau);\tau\right)}.
\end{aligned}
$$

Proof: It suffices to apply the inverse function theorem to the function $\mathcal{F}$ such that $\mathcal{F}(v,\tau) = (F(v;\tau),\tau)$, for all $(v;\tau)$ in $(c, d+\zeta) \times (-\rho,\rho)^m$. $||$

Lemma AI-6: *Let $\boldsymbol{\tau}(\pi)$ be a continuously differentiable function from $(-1,1)$ to $(-\rho,\rho)^{nm}$ such that $\boldsymbol{\tau}(0) = \mathbf{0}$. Then, for all sequence $(\pi_k)_{k\geq 1}$ of strictly positive numbers converging towards 0, there exists a subsequence $(\pi_{k_t})_{t\geq 1}$ such that, for all q in an interval $(0, 1+\zeta')$ with $\zeta' > 0$ and all $1 \leq i \neq j \leq n$, $\lim_{t\to+\infty} \frac{\lambda_{ji}(q;\mathbf{0}) - \lambda_{ji}\left(q;\boldsymbol{\tau}\left(\pi_{k_t}\right)\right)}{\pi_{k_t}}$ exists, is finite, and is equal to $\overline{\lambda}_{ji}(q)$ below:*

$$
\begin{aligned}
\overline{\lambda}_{ji}(q) &= q\left(\int_c^{F^{-1}(q;0)} F(w;0)^{n-1}\, dw\right)^{n-1}. \\
& \left\{ \begin{array}{c} \sum_{l=1}^{m} (n-1) \int_q^1 \frac{p^{n-2} \frac{\partial}{\partial \tau_l} F\left(F^{-1}(p;0);0\right)}{f(F^{-1}(p;0);0)\left(\int_c^{F^{-1}(q;0)} F(w;0)^{n-1} dw\right)^n} dp \\ \left(\frac{d}{d\pi}\tau_l^j(0) - \frac{d}{d\pi}\tau_l^i(0)\right). \end{array} \right\} . \quad \text{(AI.12)}
\end{aligned}
$$

Proof: For all $\boldsymbol{\tau}$ in $(-\rho,\rho)^{nm}$, let $\eta(\boldsymbol{\tau})$ be the common maximum of the equilibrium bid functions. From Lemma AI-3, there exists a subsequence $(\pi_{k_t})_{t\geq 1}$ such that $\lim_{t\to+\infty} \frac{\eta(\mathbf{0})-\eta\left(\boldsymbol{\tau}\left(\pi_{k_t}\right)\right)}{\pi_{k_t}}$ exists and is finite. Let χ be this limit. For all $1 \leq i \neq j \leq n$ and $q > 0$, we may assume, from Lemma AI-1 (ii) above and Lemma AII-2 in Appendix II, that $\overline{\lambda}_{ji}(q) = \lim_{t\to+\infty} \frac{\lambda_{ji}(q;\mathbf{0})-\lambda_{ji}\left(q;\boldsymbol{\tau}\left(\pi_{k_t}\right)\right)}{\pi_{k_t}}$ and $\overline{\gamma}_i(q) = \lim_{t\to+\infty} \frac{\gamma_i(q;\mathbf{0})-\gamma_i\left(q;\boldsymbol{\tau}\left(\pi_{k_t}\right)\right)}{\pi_{k_t}}$ exist and form a solution of the linear differential system obtained from (AI.4, AI.5) by differentiating it around its solution $\lambda_{ji}(.;\mathbf{0}), \gamma_i(.;\mathbf{0})$, and of the initial condition below:

$$\begin{aligned} \overline{\lambda}_{ji}(1) &= 0,\ j \neq i, (\text{AI.13}) \\ \overline{\gamma}_i(1) &= \chi. \end{aligned}$$

Differentiating (AI.4) with respect to π, setting $\pi = 0$, using the equalities $\boldsymbol{\tau}(0) = \mathbf{0}$ and $\lambda_{ji}(q;\mathbf{0}) = q$, for all q in the interval $(0, 1+\zeta')$, where ζ' is from Lemma AI-5, and rearranging, we find that the coefficients of $\overline{\lambda}_{hi}$, $h \neq j, i$, and $\overline{\gamma}_i$ cancel out in $\overline{\lambda}_{ji}$'s equation, and we have:

$$\begin{aligned} \frac{d}{dq}\overline{\lambda}_{ji}(q) &= \frac{n-1}{F^{-1}(q;0)-\gamma_i(q;0)} \sum_{l=1}^{m} \frac{\partial}{\partial \tau_l} F^{-1}(q;0) \left(\frac{d}{d\pi}\tau_l^j(0) - \frac{d}{d\pi}\tau_l^i(0) \right) \\ &\quad + \left\{ \frac{\partial}{\partial q} F^{-1}(q;0) \frac{n-1}{F^{-1}(q;0)-\gamma_i(q;0)} + \frac{1}{q} \right\} \overline{\lambda}_{ji}(q). \end{aligned}$$

From Lemma AI-1 (iv)[3] and Lemma AI-5, we obtain:

$$\begin{aligned}\frac{d}{dq}\overline{\lambda}_{ji}(q) &= -\frac{(n-1)\,q^{n-1}}{f\left(F^{-1}(q;0);0\right)\int_c^{F^{-1}(q;0)} F(w;0)^{n-1}\,dw}. \\ &\quad \sum_{l=1}^{m}\frac{\partial}{\partial\tau_l}F\left(F^{-1}(q;0);0\right)\left(\frac{d}{d\pi}\tau_l^j(0)-\frac{d}{d\pi}\tau_l^i(0)\right) \\ &\quad +\left\{\frac{(n-1)\,q^{n-1}}{f\left(F^{-1}(q;0);0\right)\int_c^{F^{-1}(q;0)} F(w;0)^{n-1}\,dw}+\frac{1}{q}\right\}\overline{\lambda}_{ji}(q).\text{(AI.14)}\end{aligned}$$

Using, for example, the method of "variation of constants," we find that the unique solution of (AI.13) and (AI.14) is (AI.12). ||

Lemma AI-7: *For all* $1 \leq i \neq j \leq n$ *and* q *in an interval* $(0, 1+\zeta')$, *with* $\zeta' > 0$, $\lambda_{ji}(q;\boldsymbol{\tau})$ *is differentiable at* $(q;\mathbf{0})$.

Proof: Let $\boldsymbol{\tau}(\pi)$ be a continuously differentiable function from $(-1,1)$ to $(-\rho,\rho)^{nm}$ such that $\boldsymbol{\tau}(0)=\mathbf{0}$. From Lemma AI-6, $\lim_{\Delta\pi\to 0}\frac{\lambda_{ji}(q;\mathbf{0})-\lambda_{ji}(q;\boldsymbol{\tau}(\Delta\pi))}{\Delta\pi}$ exists and is equal to $\overline{\lambda}_{ji}(q)$ in (AI.12), for all q in an open interval $(0,1+\zeta')$, where $\zeta'>0$. In fact, otherwise there would exist a sequence $(\Delta\pi_k)_{k\geq 1}$ such that the difference ratio would be bounded away from $\overline{\lambda}_{ji}(q)$, which would contradict Lemma AI-5. Consequently, $\left(\frac{d}{d\pi}\lambda_{ji}(q;\boldsymbol{\tau}(\pi))\right)_{\pi=0}$ exists and is equal to $\overline{\lambda}_{ji}(q)$ in (AI.12), which is linear in $\overline{\tau}_l^k$.

The differentiability with respect to $\boldsymbol{\tau}$ at $(q,\mathbf{0})$ then follows from Lemma AII-3[4] in Appendix II. Finally, the joint differentiability with respect to $(q,\boldsymbol{\tau})$ follows from Lemma AII-4 in Appendix II. ||

[3] $\lambda_{ji}(q;\mathbf{0})=q$, with $j\neq i$, and $\gamma_i(q;\mathbf{0})=F^{-1}(q;0)-\frac{\int_c^{F^{-1}(q;0)}F(w;0)^{n-1}dw}{q^{n-1}}$, obtained from Lemma AI-1 (iv) for q in $(0,1]$, also describe the solution to (AI-4-AI-6) past 1 in $\left[1,1+\zeta'\right)$.

[4] Because we can solve the differential equations only at the symmetric setting, we need a local condition, such as Lemma AII-3, that is sufficient for differentiability. We could not have applied more familar, "global," conditions such as, for example, the existence and continuity of the partial derivatives everywhere in a neighborhood of the symmetric setting (we used this latter condition in the proof of Theorem 2).

Lemma AI-8:

(i) For all $1 \leq i \neq j \leq n$ and v in an interval $(c, d+\zeta'')$, with $\zeta'' > 0$, the function $\varphi_{ji}(v; \boldsymbol{\tau})$ and *the probability of winning $\prod_{l \neq i} F\left(\varphi_{ji}(v; \boldsymbol{\tau}); \tau^j\right)$ are differentiable with respect to $(v, \boldsymbol{\tau})$ at $(v, \mathbf{0})$.*

(ii) The revenues $R^F(\boldsymbol{\tau})$ are differentiable at $\boldsymbol{\tau} = \mathbf{0}$.

Proof: (i) From the definitions, we have $\varphi_{ji}(v; \boldsymbol{\tau}) = F^{-1}\left(\lambda_{ji}\left(F\left(v; \tau^i\right); \boldsymbol{\tau}\right); \tau^j\right)$, for all v. The differentiability of $\varphi_{ji}(v; \boldsymbol{\tau})$ and $\prod_{l \neq i} F\left(\varphi_{li}(v; \boldsymbol{\tau}); \tau^l\right)$ then follows from Lemmas AI-7, AI-5, and AI-4.

(ii) For example, from Myerson (1981), $R^F(\boldsymbol{\tau})$ is equal to the sum over $i = 1, ...n$ of the following integrals:

$$\int_c^d \left(\prod_{l \neq i} F\left(\varphi_{ji}(v; \boldsymbol{\tau}); \tau^j\right)\right)\left(\frac{\partial}{\partial v}\left(vF\left(v; \tau^i\right)\right) - 1\right) dv. \quad \text{(AI.15)}$$

We now prove that (AI.15) is differentiable, for all i. Let $\boldsymbol{\tau}(\pi)$ be a continuously differentiable function from $(-1, 1)$ to $(-\rho, \rho)^{nm}$ such that $\boldsymbol{\tau}(0) = \mathbf{0}$. The difference ratio in the definition of the derivative of (AI.15) can be broken down as the following sum:

$$\int_c^d \frac{\prod_{l \neq i} F\left(\varphi_{ji}(v; \boldsymbol{\tau}(\pi)); \tau^j(\pi)\right) - F(v; 0)^{n-1}}{\pi}\left(\frac{\partial}{\partial v}\left(vF\left(v; \tau^i(\pi)\right)\right) - 1\right) dv \quad \text{(AI.16)}$$

$$+ \int_c^d F(v; 0)^{n-1}\left(\frac{\frac{\partial}{\partial v}\left(vF\left(v; \tau^i(\pi)\right)\right) - \frac{\partial}{\partial v}\left(vF(v; 0)\right)}{\pi}\right) dv. \quad \text{(AI.17)}$$

Integrating (AI.17) by parts, we find that it is equal to (AI.18) below:

$$-\int_c^d \frac{F\left(v; \tau^i(\pi)\right) - F(v; 0)}{\pi} v \frac{\partial}{\partial v} F(v; 0)^{n-1} dv. \quad \text{(AI.18)}$$

From D (ii, iii) and proceeding as in the proof of Lemma AI-3, the absolute values of the integrands in (AI.16) and (AI.18) are bounded by an integrable function of v only. Consequently[5], the limits for π tending towards zero may be taken under the integral signs. From the linearity of the integral and the differentiability, from (i) above, of the integrands, these limits are linear functions of $\frac{d}{d\pi}\tau_k^l(0)$, $1 \leq l \leq n, 1 \leq k \leq m$. The differentiability of these two terms and hence of $R^F(\boldsymbol{\tau})$ then follows from Lemma AII-3 in Appendix II. ||

Proof of Theorem 3: Theorem 3 (i) follows from Lemma AI-1 (i). Lemma AI-8 (ii) implies Theorem 3 (ii). ||

Appendix II

Here, we prove the technical results we used in Appendix I: sufficient local conditions for differentiability and a result on the convergence of difference ratios of solutions to a system of differential equations.

Lemma AII-1: *Let $(\pi_k, \eta_k)_{k\geq 1}$ be a sequence in $\mathbb{R} \times \mathbb{R}^n$ converging towards $(\overline{\pi}, \overline{\eta})$ and such that $\pi_k \neq \overline{\pi}$, for all $k \geq 1$. If $\lim_{k\to+\infty} \frac{\eta_k - \overline{\eta}}{\pi_k - \overline{\pi}}$ exists and is finite, then there exists a subsequence $\left(\pi_{k_m}, \eta_{k_m}\right)_{m\geq 1}$ and a continuously differentiable function $\widetilde{\eta}$ from $(\overline{\pi}-1, \overline{\pi}+1)$ to $\mathbb{R}^n$, such that $\widetilde{\eta}(\overline{\pi}) = \overline{\eta}$ and $\widetilde{\eta}(\pi_{k_m}) = \eta_{k_m}$, for all $m \geq 1$ such that $\pi_{k_m} \in (\overline{\pi}-1, \overline{\pi}+1)$.*

Proof: By considering a subsequence if necessary, we may assume that $(\pi_k)_{k\geq 1}$ is strictly monotonic. Assume, for example, that it is strictly decreasing (the proof is similar when it is strictly increasing). We first prove the lemma for $(\overline{\pi}, \overline{\eta}) = 0$. Let χ be equal to $\lim_{k\to+\infty} \frac{\eta_k}{\pi_k}$. Let k_1 be a value of the index such that $\pi_{k_1} \leq 1$ and $\left|\chi - \frac{\eta_{k_1}}{\pi_{k_1}}\right| \leq 1$. Assume k_m has been defined and $\left|\chi - \frac{\eta_{k_m}}{\pi_{k_m}}\right| \leq 1/m$. Then k_{m+1} is a value of the index such that

[5] For example, from the Lebesgue Theorem of dominated convergence.

$k_{m+1} > k_m$, $\left|\chi - \frac{\eta_{k_{m+1}}}{\pi_{k_{m+1}}}\right| \leq 1/(m+1)$, and $\left|\chi - \frac{\eta_{k_m}-\eta_{k_{m+1}}}{\pi_{k_m}-\pi_{k_{m+1}}}\right| \leq 2/m$. The last requirement can be satisfied because $(\pi_k, \eta_k)_{k\geq 1}$ tends towards 0 and $\left|\chi - \frac{\eta_{k_m}}{\pi_{k_m}}\right| \leq 1/m$.

By extracting a subsequence as in the previous paragraph if necessary, we may assume that $(\pi_k, \eta_k)_{k\geq 1}$ is such that $\left|\chi - \frac{\eta_k}{\pi_k}\right| \leq 1/k$ and $\left|\chi - \frac{\eta_k-\eta_{k+1}}{\pi_k-\pi_{k+1}}\right| \leq 2/k$, for all $k \geq 1$. Consider a step function σ from $(-1, 1)$ to $\mathbb{R}^n$ such that $\sigma(\pi) = \frac{\eta_k-\eta_{k+1}}{\pi_k-\pi_{k+1}}$, for all π and k such that $\pi \in (\pi_{k+1}, \pi_k)$. Then approximate σ by a continuous function ζ from $(-1, 1)$ to $\mathbb{R}^n$ such that $\int_{\pi_{k+1}}^{\pi_k} (\zeta(\pi) - \sigma(\pi))\, d\pi = 0$, for all $k \geq 1$. Such a function exists. In fact, it suffices to consider a sequence $(\zeta_m)_{m\geq 1}$ of functions such that, for all $m \geq 1$: $\zeta_m(\pi) = \chi$, for all π in $[0, \pi_m]$; ζ_m is continuous over $(\pi_m, 1)$; $\zeta_m(\pi_k) = \frac{1}{2}\left(\frac{\eta_k-\eta_{k+1}}{\pi_k-\pi_{k+1}} + \frac{\eta_{k-1}-\eta_k}{\pi_{k-1}-\pi_k}\right)$, for all $m > k > 1$; $\left|\zeta_m(\pi) - \frac{\eta_k-\eta_{k+1}}{\pi_k-\pi_{k+1}}\right| \leq \frac{1}{2}\left(\frac{2}{k} + \frac{2}{k+1}\right) = \frac{1}{k} + \frac{1}{k+1}$, for all π and $k < m$ such that $\pi \in [\pi_{k+1}, \pi_k]$; $\int_{\pi_{k+1}}^{\pi_k} (\zeta_m(\pi) - \sigma(\pi))\, d\pi = 0$, for all $m > k \geq 1$; ζ_m is odd, that is, $\zeta_m(-\pi) = \zeta_m(\pi)$, for all π; ζ_{m+1} is equal to ζ_m over $(\pi_{m+1}, 1)$. The sequence $(\zeta_m)_{m\geq 1}$ is then a Cauchy sequence for the norm of the uniform convergence. As it can be easily shown, its limit ζ is continuous and satisfies our requirements. A function $\widetilde{\eta}$ can then be simply defined as follows: $\widetilde{\eta}(\pi) = \eta_1 - \int_{\pi}^{\pi_1} \zeta(\pi)\, d\pi$. We have proved the lemma for $(\overline{\pi}, \overline{\eta}) = 0$.

In the general case, it suffices to obtain the function $\widetilde{\eta}$ for the sequence $(\pi_k - \overline{\pi}, \eta_k - \overline{\eta})_{k\geq 1}$ and to define the new function $\widetilde{\eta}(\pi - \overline{\pi}) + \overline{\eta}$. ||

Lemma AII-2: *Consider a system of differential equations* $\frac{d}{dt}y(t) = h(t, y, \pi)$ *and an initial condition* $y(t_1) = \eta(\pi)$ *that depend on a parameter* π *and that are defined over an open subset* O *of* $\mathbb{R}^{n+2}$*, where* n *is the dimension of* y*. Assume that* h *is a continuous function from* O *to* $\mathbb{R}^n$ *such that* $\frac{\partial}{\partial y_i}h$*,* $1 \leq i \leq n$*, and* $\frac{\partial}{\partial \pi}h$ *exist and are continuous over* O*. Let* $(\pi_k)_{k\geq 1}$ *be a sequence in* $\mathbb{R}$ *such that* $(t_1, \eta(\pi_k), \pi_k)_{k\geq 1}$ *is a sequence in* O *that converges towards a point* $(t_1, \overline{\eta}, \overline{\pi})$ *in* O*. Assume also that* $\lim_{k\to+\infty} \frac{\eta(\pi_k)-\overline{\eta}}{\pi_k-\overline{\pi}}$ *exists and is finite. Let* χ *be this limit. Let* $y(., \pi)$ *be the solution of the differential system with the initial condition as a function of the parameter* π*.*

Then $\lim_{k\to+\infty}\frac{y(t,\pi_k)-y(t,\overline{\pi})}{\pi_k-\overline{\pi}}$ *exists, for all* t *in the maximal definition interval of the solution* $y(.,\overline{\pi})$, *and is equal to the solution* ρ *of the linear differential system* $\frac{d}{dt}\rho(t)=\sum_{i=1}^{n}\frac{\partial}{\partial y_i}h(t,y(t,\pi),\pi)\rho_i(t)+\frac{\partial}{\partial\pi}h(t,y(t,\pi),\pi)$ *with initial condition* $\rho(t_1)=\chi$.

Proof: The conclusion of the lemma will be proved if we prove it for all strictly monotonic subsequence of $(\pi_k)_{k\geq1}$. We may thus assume that $(\pi_k)_{k\geq1}$ is strictly monotonic. Through the change of variables $y=\eta(\pi)+z$, the initial system and initial condition are equivalent to $\frac{d}{dt}z(t)=h(t,\eta(\pi)+z,\pi)$ and $z(t_1)=0$. From Lemma AII-1, there exists a continuously differentiable function $\widetilde{\eta}$ over a neighborhood of $\overline{\pi}$ that coincides with η over $\{\pi_k|k\geq1\}\cup\{\overline{\pi}\}$. From the equality $\lim_{k\to+\infty}\frac{\eta(\pi_k)-\overline{\eta}}{\pi_k-\overline{\pi}}=\chi$, we have $\frac{d}{d\pi}\widetilde{\eta}(\overline{\pi})=\chi$. The lemma then follows from the application of the standard theorems of the theory of ordinary differential equations about the differentiability of the solution with respect to a parameter to the system $\frac{d}{dt}z(t)=g(t,z,\pi)$, where $g(t,z,\pi)=h(t,\widetilde{\eta}(\pi)+z,\pi)$, with initial condition $z(t_1)=0$. ||

Lemma AII-3: *Let* f *be a function from an open set* O *of* $\mathbb{R}^n$ *to* $\mathbb{R}$ *and let* ω *be an element of* O. *Assume that* f *is continuous at* ω *and that its partial derivatives* $\frac{\partial}{\partial\tau_i}f(\omega)$, $1\leq i\leq n$, *exist. Assume also that* $f\circ\tau$ *is differentiable at* 0 *and that* $\frac{d}{d\pi}f\circ\tau(0)=\sum_{i=1}^{n}\frac{\partial}{\partial\tau_i}f(\omega)\frac{d}{d\pi}\tau_i(0)$, *for all continuously differentiable function* $\tau(\pi)$ *from* $(-1,1)$ *to* O *such that* $\tau(0)=\omega$. *Then,* f *is differentiable at* ω.

Proof: Suppose that f is not differentiable at ω. Then, there exists $\epsilon>0$ and a sequence $\left(\tau^k\right)_{k\geq1}$ converging towards ω such that $\tau^k\neq\omega$, for all k, and

$$\left|\frac{f\left(\tau^k\right)-f(\omega)}{|\tau^k-\omega|}-\sum_{i=1}^{n}\frac{\partial}{\partial\tau_i}f(\omega)\frac{\left(\tau_i^k-\omega_i\right)}{|\tau^k-\omega|}\right|>\epsilon, \text{ (AII.1)}$$

for all $1\leq k\leq n$. By extracting a subsequence, if necessary, we may assume that $\left(\left|\tau^k-\omega\right|\right)_{k\geq1}$ is strictly decreasing. Since the sequence $\left(\frac{\tau^k-\omega}{\left|\tau^k-\omega\right|}\right)_{k\geq1}$ is bounded, it admits a convergent subsequence. We may thus assume that this sequence itself is convergent. Let λ be its limit. Since every term of the sequence has a unit norm, this is also the case of the limit and we have $|\lambda| = 1$.

Applying Lemma AII-1 to $(\pi_k)_{k\geq 1} = \left(\left|\tau^k - \omega\right|\right)_{k\geq 1}$, $\overline{\pi} = 0$, $\left(\eta^k\right)_{k\geq 1} = \left(\tau^k\right)_{k\geq 1}$, and $\overline{\eta} = \omega$, we obtain the existence of a continuously differentiable function $\widetilde{\tau}$ from $(-1, 1)$ to R^n such that $\widetilde{\tau}(0) = \omega$ and $\widetilde{\tau}\left(\left|\tau^k - \omega\right|\right) = \tau^k$, for all $k \geq 1$ such that $\left|\tau^k - \omega\right| < 1$. Since $\lim_{k\to+\infty} \frac{\tau^k - \omega}{\left|\tau^k - \omega\right|} = \chi$, we have $\frac{d}{d\pi}\widetilde{\tau}(0) = \chi$. Then, from the assumptions of the lemma, $\frac{d}{d\pi} f \circ \tau(0)$ exists and is equal to $\sum_{i=1}^n \frac{\partial}{\partial \tau_i} f(\omega)\chi_i$. Consequently, the limit of the L.H.S. of (AII.1), for k tending towards infinity, exists and is equal to 0. This contradicts (AII.1) and the lemma is proved. $||$

Lemma AII-4: *Let f be a function from an open set O in $\mathbb{R} \times \mathbb{R}^n$ to $\mathbb{R}$ and let $(\overline{u}, \omega)$ be an element of O. Assume that the function $f(\overline{u}, .)$ from $\{\tau \in \mathbb{R}^n | (\overline{u}, \tau) \in O\}$ to $\mathbb{R}$ is differentiable at ω and that $\frac{\partial}{\partial u} f$ exists in O and is continuous at $(\overline{u}, \omega)$. Then, f is differentiable at $(\overline{u}, \omega)$.*

Proof: We will have proved the lemma if we prove that the limit of the ratio below for (u, τ) tending towards $(\overline{u}, \omega)$ exists and is equal to 0:

$$\frac{\left|f(u,\tau) - f(\overline{u},\omega) - \frac{\partial}{\partial u} f(\overline{u},\omega)(u - \overline{u}) - \sum_{i=1}^n \frac{\partial}{\partial \tau_i} f(\overline{u},\omega)(\tau_i - \omega_i)\right|}{|(u - \overline{u}, \tau - \omega)|}.$$

However, this ratio is not larger than (AII.2) below:

$$\left|\frac{f(u,\tau) - f(\overline{u},\tau)}{u - \overline{u}} - \frac{\partial}{\partial u} f(\overline{u},\omega)\right| \frac{|u - \overline{u}|}{|(u - \overline{u}, \tau - \omega)|} + \frac{\left|f(\overline{u},\tau) - \sum_{i=1}^n \frac{\partial}{\partial \tau_i} f(\overline{u},\omega)(\tau_i - \omega_i)\right|}{|\tau - \omega|} \frac{|\tau - \omega|}{|(u - \overline{u}, \tau - \omega)|}. \quad \text{(AII.2)}$$

Obviously, the two factors $|u - \overline{u}| \,/\, |(u - \overline{u}, \tau - \omega)|$ and $|\tau - \omega| \,/\, |(u - \overline{u}, \tau - \omega)|$

are not larger 1. From the mean value theorem, $\frac{f(u,\tau)-f(\overline{u},\tau)}{u-\overline{u}} = \frac{\partial}{\partial u} f\left(u', \omega\right)$, where u' lies strictly between u and $\overline{u}$. As (u, τ) tends towards $(\overline{u}, \omega)$, (u', τ) also tends towards $(\overline{u}, \omega)$ and, from the continuity of $\frac{\partial}{\partial u} f$ at $(\overline{u}, \omega)$, $\frac{\partial}{\partial u} f\left(u', \tau\right)$ tends towards $\frac{\partial}{\partial u} f\left(\overline{u}, \omega\right)$. Consequently, the first term in (AII.2) tends towards 0. From the differentiability (with respect to τ) of $f\left(\overline{u}, .\right)$ at ω, the second term also tends towards 0 and the lemma is proved. ||

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