Uniqueness of the Equilibrium in First-Price Auctions[1]

by

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Abstract

If the value cumulative distribution functions are strictly log-concave at the highest lower extremity of their supports, a simple geometric argument establishes the uniqueness of the equilibrium of the first-price auction in the asymmetric independent private values model.

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1. Introduction

The analysis of the first-price auction becomes quite complex as soon as one departs from the, often unrealistic, assumption that the bidders' valuations are distributed identically. Ties may have to be broken according to a rule different from the fair tie-breaking rule in order for an equilibrium to exist (see Lebrun 1996, 2002, Jackson *et al.* 2002, Maskin and Riley 2000b). Moreover, when an equilibrium exists, it may not be possible to obtain an explicit mathematical formula for it. Nevertheless, under some regularity assumptions on the valuation distributions, there exists an implicit characterization of the equilibrium (see Lebrun 1999a and Maskin and Riley 2000a).

In order to gain some insights into the properties of the equilibrium, some authors have tried to overcome the lack of a general formula by computing numerical estimates of the equilibria (see Athey 2001, Bajari 2001, Dalkir *et al.* 2000, Li and Riley 1997, Marshall *et al.* 1994, , Maskin and Riley 2000a, Marshall and Shulenberg 2000). The uniqueness of the equilibrium would save these authors the trouble of looking for other equilibria, once they have found one. The uniqueness would also be useful to experimental researchers. Indeed, they would have to compare their subjects' bidding behaviors with only one equilibrium.

Lebrun (2002) proves that the Nash equilibrium correspondence is upper-hemicontinuous with respect to the valuation distributions. Assumptions under which the equilibrium is unique would thus imply the continuity of this correspondence. In turn, this continuity would bring robustness to the numerical simulations. Properties of the equilibrium would not be particular to the precise examples the researchers have solved numerically, but would be robust to small changes in the valuation distributions. Similarly, the uniqueness of the equilibrium would bring robustness to some theoretical results, which would hold true for larger classes of valuation distributions than those for which they were proved[2].

[2] As Lebrun (2002) shows, the continuity, implied by the uniqueness, can also be used to disprove existing conjectures.

Maskin and Riley (1996) and Bajari (2001) consider the uniqueness issue when the valuation distributions are absolutely continuous with density functions continuous and strictly positive everywhere over the supports[3]. Even in natural examples where the valuations are distributed over the same interval, density functions may vanish at its lower extremity. Assume that $m > 1$ bidders whose valuations are distributed identically according to the cumulative distribution function F collude into one cartel. Assume further that, inside the cartel, the information about the members' valuations and the monitoring of the members' bids are perfect and that all allocations and side-payments are possible. Then, the cartel will maximize the total surplus of its members and, when it wins the item, will allocate it to its member with the highest valuation. The cartel will, thus, behave as a single bidder with valuation cumulative distribution function F^m. The density function of F^m vanishes at the lower extremity of the support, even if the density function of F does not[4]. There has been great interest in the literature for this case (see Marshall *et al.* 1994, Thomas 1997, Dalkir *et al.* 2000, Whaerer 1999). Considering the likely prevalence of collusion in auctions (see, for example, Graham and Marshall 1985, Hendricks and Porter 1989, Baldwin *et al.* 1997, Persendorfer 2000, Porter and Zona 1999), this interest is well deserved.

Lebrun (1999) proves uniqueness in the common-support case when the valuation distributions have a mass point at the lower extremity of the support. If the reserve price is binding, this result implies uniqueness even when the valuation distributions are atomless. However, it does not apply to atomless distributions when the reserve price is not binding. Lebrun (1999a) also proves uniqueness in the symmetric case, when all valuation distributions are identical, and in the case where the set of valuation distributions reduces to a pair of stochastically ranked distributions[5].

Here, we prove uniqueness without requiring mass points, nor the existence of strictly positive continuous extensions of the density functions. Furthermore, we do not require any relation of stochastic dominance, in particular, any equality, between valuation distributions. Rather, as is com-

[3]These authors do not offer any justification for their use of L'Hospital's rule at a point of singularity, where the derivatives may not be defined. See footnote 9.

[4]Our main uniqueness result is particularly appropriate in this case since our assumption of strict logconcavity of the cumulative distribution function F^m arising from collusion is equivalent to the strict logconcavity of the "primitive" cumulative distribution function F.

[5]In Appendix 6, we extend this last result.

mon in economic theory (see Bagnoli and Bergstrom 1989 and An 1998), we impose a condition of log-concavity on the valuation cumulative distribution functions. Although many standard parametric distributions are strictly log-concave over their whole supports, we require only local strict log-concavity at the highest lower extremity of the supports. This assumption is equivalent to the assumption of decreasing reverse hazard rates in some, arbitrarily small, interval to the right of this highest lower extremity.

Our proof consists manly in a simple geometric argument. In Section 2, we first consider the common-support case, where this argument is most easily explained. In Section 3, we extend our result from the common-support case to the case of possibly different lower extremities and identical upper extremities by truncating the valuation distributions from below. We next extend, in Section 4, our result to the case where even the upper extremities may differ by considering the equilibrium strategies as restrictions over the valuation supports of the best reply functions. These best reply functions are defined over larger intervals and we apply to these best reply functions the arguments we previously applied to the equilibrium strategies in the common-support case. Section 5 concludes. Details of our proofs can be found in Appendices 1 to 6.

2. Statement of the Main Result

Consider the standard independent private values model with n risk-neutral bidders, a reserve price r, the fair tie-breaking rule, and possibly different valuation probability distributions F_1, ..., F_n, with, possibly different, interval supports $[c_i, d_i]$, with $c_i < d_i$. We use the same notation for a probability distribution and its cumulative distribution function that is continuous from the right. We now describe our basic set of regularity assumptions on the valuation distributions.

Assumptions A.1:

(A.1) For all $i = 1, ..., n$, the support of F_i is an interval $[c_i, d_i]$, with $c_i < d_i$.

(A.2) For all $i = 1, ..., n$, the cumulative function F_i is differentiable over $(c_i, d_i]$ with a derivative f_i locally bounded away from zero over this

interval[6].

Under Assumptions A.1, the valuation interval $[c_i, d_i]$ may differ accross bidders and there may not exist any strictly positive and continuous extension of f_i over the whole interval $[c_i, d_i]$. Without loss of generality, we can assume that the support of F_1 has the largest lower extremity and the support of F_2 has the second largest, that is, $c_i \leq c_2 \leq c_1$, for all $i \geq 2$. Although we will allow c_i to be a mass point of F_i in some of our intermediate results, we will mainly focus on the atomless case, which is more natural in many applications[7]. Theorem 1 below is our main result.

Theorem 1: *Let Assumptions A.1 be satisfied. Assume also $F_i(c_i) = 0$, for all $i \geq 1$. Without loss of generality, assume $c_1 \geq c_2 \geq c_i$, for all $i \geq 2$. Then, under any of the following additional assumptions (i), (ii), or (iii), the first-price auction with reserve price r has one and only one Bayesian Nash equilibrium where bidders bid at most their valuations:*

(i) $r > c_1$

(ii) $c_1 > c_2$

(iii) *there exists $\delta > 0$ such that F_i is strictly log-concave over $(c_1, c_1 + \delta) \cap (c_i, d_i)$, for all $i \geq 1$.*

The cumulative function F_i is strictly log-concave over $(c_1, c_1 + \delta) \cap (c_i, d_i)$ if and only if $\ln F_i$ is strictly concave, that is, its derivative $\frac{f_i}{F_i}$–the reverse hazard rate–is strictly decreasing over this interval. In the next section, we prove Theorem 1 in the case of a common support. We then extend our proof to the general case in the following sections.

3. Relevant Existing Results

Assume that the valuation supports are identical, that is, that $c_i = c$ and $d_i = d$, for all i. In this section, we allow the valuation distributions to have a mass point at c. Since the case where the bidders' valuations are larger than the reserve price with probability zero is uninteresting, we assume that $r < d$. When the reserve price is nonbinding, that is, $r \leq c$, the proof in

[6]That is, for all v in $(c_i, d_i]$, there exists $\varepsilon > 0$ such that $f_i(w) > \varepsilon$, for all w in $(v - \varepsilon, v + \varepsilon)$.

[7]We could easily extend Theorem 1 to the case where c_i is a mass point of F_i, for some i.

Lebrun (1999a or 1997) of the existence of a Bayesian Nash equilibrium, in a model where c may be a mass point of some valuation distributions, extends easily to our model. Lebrun (1999a) also provides a characterization of the equilibria and, when c is a mass point of all valuation distributions, proves the uniqueness of the equilibrium . We have the results C.1 and U.1 below, which, along with their extensions and the methodology of their proofs, we will use in proving Theorem 1 (Section 2).

Existing Results I (Lebrun 1999a): *Let Assumptions A.1 be satisfied. Assume further $c_i = c$, $d_i = d$, for all i, and $r \leq c$.*

C.1: Characterization of the Equilibria

There exists a Bayesian Nash equilibrium. In every equilibrium the bidders follow nondecreasing bid functions $\beta_1, ..., \beta_n$ that are not smaller than c over $(c, d]$ and that are strictly increasing and differentiable when their values are strictly larger than c. Moreover, for every equilibrium $(\beta_1, ..., \beta_n)$ there exists η in (c, d) such that the inverse bid functions $\alpha_1 = \beta_1^{-1}, ..., \alpha_n = \beta_n^{-1}$ are solutions of the system of differential equations (1)–considered over the domain $D = \{(b, \alpha_1, ..., \alpha_n) \mid c, b < \alpha_i \leq d, \text{ for all } 1 \leq i \leq n\}$–with boundary conditions (2) and (3):

$$\frac{d}{db} \ln F_i(\alpha_i(b)) = \frac{1}{n-1} \left\{ -\frac{n-2}{\alpha_i(b) - b} + \sum_{j \neq i} \frac{1}{\alpha_j(b) - b} \right\}, \quad (1)$$

for all $i = 1, ..., n$ and b in $(c, d]$,

$$\alpha_i(c) = c, \text{ for all, except possibly one, } i \text{ between } 1 \text{ and } n, \quad (2)$$

$$\alpha_1(\eta) = ...\alpha_n(\eta) = d \quad (3).$$

U.1: Uniqueness of the Equilibrium

If $F_1(c), ..., F_n(c) > 0$, there exists one and only one equilibrium.

In the second initial condition (3) in C.1 above, η is the common maximum of all bid functions. Summing the equations in (1) for all $i \neq j$ and rearranging give:

$$\frac{d}{db}\sum_{i\neq j}\ln F_i\left(\alpha_i\left(b\right)\right)=\frac{1}{\alpha_j\left(b\right)-b}.(4)$$

In order to prove (1), Lebrun (1999a) actually obtained (4) first. Once the regularity conditions have been proved, (4) is the first-order condition from bidder j's maximization problem. In fact, the maximum of the expected payoff $(v_j-b)\prod_{i\neq j}F_j\left(\alpha_j\left(b\right)\right)$ of bidder j with valuation v_j must be reached at $b=\beta_j\left(v_j\right)$. The logarithmic derivative $\frac{-1}{v_j-b}+\frac{d}{db}\sum_{i\neq j}\ln F_i\left(\alpha_i\left(b\right)\right)$ of the expected payoff must thus vanish at $b=\beta_j\left(v_j\right)$ or, equivalently, at $v_j=\alpha_j\left(b\right)$, and (4) follows.

From (2), if there exists j such that $\alpha_j\left(c\right)>c$, then $\alpha_i\left(c\right)=c$, for all $i\neq j$. From (4), if $\alpha_j\left(c\right)>c$, then $\frac{d}{db}\ln F_i\left(\alpha_i\left(b\right)\right)$ is bounded when b tends towards c, and, thus, $\ln F_i\left(\alpha_i\left(c\right)\right)=\ln F_i\left(c\right)$ cannot be infinite, that is, $F_i\left(c\right)>0$, for all $i\neq j$. The only time a bidder's inverse bid function takes at c a value strictly larger than c or, equivalently, the only time a bidder's bid function can take the value c everywhere over a nondegenerate interval is when all the other bidders' valuation distributions have a mass point at c. We have proved Lemma 1 below.

Lemma 1: *Let Assumptions A.1 be satisfied. Assume further $c_i=c$, $d_i=d$, for all i, and $r\leq c$. Let $(\alpha_1,...,\alpha_n)$ be a solution over $(c,\eta]$, with $c<\eta$, of (2) and (1)–considered in D–and let j be in $\{1,..,n\}$. If $\alpha_j\left(c\right)>c$, then $F_i\left(c\right)>0$, for all $i\neq j$.*

When we consider, as in Lebrun (1999a), the unknown functions of the equations (1) to be $\psi_1=F_1\alpha_1,...,\psi_n=F_n\alpha_n$, the R.H.S.'s of these equations are not locally Lipschitz at $b=c$ since, at this point, denominators vanish and $\alpha_i=F_i^{-1}\left(\psi_i\left(b\right)\right)$ may not be locally Lipschitz, if f_i vanishes or is not defined. Even if the first initial condition (2) was completely determined, we could, thus, not infer the uniqueness of the solution of (1), (2), and (3) from the theory of ordinary differential equations.

Lebrun (1999a) chooses rather to consider the solutions of the system the differential equations (1) form and of the second initial condition (3), which satisfies the assumptions of the standard theorems (again, for the unknown functions $\psi_1=F_1\alpha_1,...,\psi_n=F_n\alpha_n$). However, in this initial condition, η is an unknown parameter. Lebrun (1999a) proves the existence and uniqueness of a solution to (1), (2), and (3) by proving the existence and uniqueness of a value of the parameter η such that the solution of (1) and (3) also satisfies (2). In his proof, Lebrun (1999a) establishes and uses the following important property of strict monotonicity over $(c, d]$ of the solutions of (1) and (3) with respect to η.

Lemma 2: *Let Assumptions A.1 be satisfied. Assume further $c_i = c$, $d_i = d$, for all i, and $r \leq c$. Let η and $\widetilde{\eta}$ be in (c, d) such that $\eta < \widetilde{\eta}$. Let $(\alpha_1, ..., \alpha_n)$ be the solution over $(\gamma, \eta]$, with $\gamma < \eta$, of (1) and (3) for the value η of the parameter and let $(\widetilde{\alpha}_1, ..., \widetilde{\alpha}_n)$ be the solution over $(\widetilde{\gamma}, \widetilde{\eta}]$, with $\widetilde{\gamma} < \widetilde{\eta}$, of (1) and (3) for the value $\widetilde{\eta}$ of the parameter. Then, $\widetilde{\alpha}_i(b) < \alpha_i(b)$, for all b in $(\max(\gamma, \widetilde{\gamma}), \eta]$ and all $i = 1, ..., n$.*

For the sake of completeness and because we will use Lemma 2 several times in this paper, we provide its proof in Appendix 1. In Section 4 and Appendix 5.2 (Lemma A5.2-2), we extend Lemma 2 to the case of different supports.

The uniqueness result U.1 actually follows easily from (4) and Lemma 2. In fact, suppose that there exist two equilibria and thus two different values η and $\widetilde{\eta}$ such that the respective solutions $(\alpha_1, ..., \alpha_n)$ and $(\widetilde{\alpha}_1, ..., \widetilde{\alpha}_n)$ of (1) and (3) are also solutions of (2). Let j be an integer between 1 and n such that $\alpha_i(c) = c$, for all $i \neq j$ (from (2) such an integer exists). Without loss of generality, we can assume that $\eta < \widetilde{\eta}$. The value[8] of $\ln \prod_{i \neq j} F_i(\alpha_i(b))$ at $b = \eta$ is thus strictly larger than the value of $\ln \prod_{i \neq j} F_i(\widetilde{\alpha}_i(b))$ at the same point. From Lemma 2, we have $\widetilde{\alpha}_i(b) < \alpha_i(b)$, for all b in $(c, \eta]$ and all $i = 1, ..., n$, and, by taking the limit for b tending towards c, $\widetilde{\alpha}_i(c) = c$, for all $i \neq j$. From (4), the derivative of $\ln \prod_{i \neq j} F_i(\alpha_i(b))$ is thus strictly smaller than the derivative of $\ln \prod_{i \neq j} F_i(\widetilde{\alpha}_i(b))$ over $(c, \eta]$. Consequently, the difference between these two logarithms increases as b decreases towards c and they cannot both be equal to the same finite value $\ln F_i(c)^{n-1}$ at $b = c$.

As is apparent from the previous paragraph and from Lemma 2, all that is needed to ensure the uniqueness of the equilibrium is that, among $F_1, ..., F_n$, at least $n - 1$ valuation distributions have a mass point at c[9]. Observe also

[8] This value is 0.

[9] The proof of uniqueness does not carry over to the atomless case with nonbinding reserve price since, in this case, $\ln \prod_{i \neq j} F_i(\alpha_i(b))$ takes the infinite value $-\infty = \ln 0$ at $b = c$. A priori, the difference between the two logarithms $\ln \prod_{i \neq j} F_i(\alpha_i(b))$ and $\ln \prod_{i \neq j} F_i(\widetilde{\alpha}_i(b))$ could increase as b tends towards c, while, at the same time, both

that if the reserve price is binding, that is, if $r > c$, the equations characterizing the equilibrium are unchanged if we modify the valuation distributions by concentrating at r the probabilities spread over $[c, r]$. By doing so, r becomes the lower extremity of the common support and all valuation distributions have a mass point at r. Result U.1 can then be applied and one and only one equilibrium exists. We have C.2 and U.2 below.

From Lemma 1, the condition (2) in the characterization C.1 can be replaced by the condition (2") in C.3 below when at least two valuation distributions are atomless and the reserve price is not binding, that is, $r \leq c$.

Extension of Results I: *Let Assumptions A.1 be satisfied. Assume further $c_i = c$ and $d_i = d$, for all i.*

C.2: Characterization of the Equilibria

C.1 holds true, even when $r > c$, if (2) is replaced by (2') below:

$$\alpha_i\left(\max\left(c, r\right)\right) = \max\left(c, r\right), \textit{ for all, except possibly one, } i \textit{between } 1 \textit{and } n. \quad (2')$$

C.3: Characterization with a Nonbinding Reserve Price and At Least Two Atomless Distributions: *Assume that $r \leq c$ and that there exist at least two different values of the index k such that $F_k\left(c\right) = 0$.*

C.1 holds true even if (2) is replaced by (2") below:

$$\alpha_1\left(c\right) = ... = \alpha_n\left(c\right) = c. (2")$$

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logarithms could tend towards $-\infty$.

A common mistake here is to make b tend towards c in (1) and "apply" L'Hospital's rule to "find" that (1) and (2')–the boundary condition at c that holds true in the atomless case (see C.3 below)–determine the values of the derivatives $\frac{d}{db}\alpha_1\left(c\right), ..., \frac{d}{db}\alpha_n\left(c\right)$. In the case of density functions strictly positive everywhere, by "applying" L'Hospital's rule again (whether implicitly or explicitly), as b tends towards c, to $\frac{F_i(\alpha_i(b))}{F_i(\widetilde{\alpha_i}(b))}$ and by dividing numerator and denominator by $\frac{d}{db}\alpha_i\left(c\right) = \frac{d}{db}\widetilde{\alpha}_i\left(c\right)$ and $f_i\left(c\right)$, one "finds" that the ratio $\frac{F_i(\alpha_i(b))}{F_i(\widetilde{\alpha_i}(b))}$ and thus the product $\prod_{i \neq j} \frac{F_i(\alpha_i(b))}{F_i(\widetilde{\alpha_i}(b))}$ tend towards one and one "rules out" an increasing difference $\ln \prod_{i \neq j} F_i\left(\alpha_i\left(b\right)\right) - \ln \prod_{i \neq j} F_i\left(\widetilde{\alpha_i}\left(b\right)\right)$. This procedure is obviously flawed since it uses the derivatives $\frac{d}{db}\alpha_1\left(c\right), ..., \frac{d}{db}\alpha_n\left(c\right)$ without proving that they exist. For a published instance of this oversight, see pp 202-203 in Bajari (2001). However, a similar approach can work in some cases such as, under some assumptions, when there are only two valuation distributions.

U.2: Uniqueness of the Equilibrium[10]

If $F_i(c) > 0$, for at least $(n-1)$ values in $\{1, ..., n\}$ of the index i, or if $r > c$, then there exists one and only one equilibrium.

4. The Common-Support Case

4.1 The Symmetric Case

As in the previous section, we assume in this section that $c_i = c$, $d_i = d$, $r < d$, and we allow the valuation distributions to have a mass point at c. Lebrun (1999a), in the general n bidder case, and Maskin and Riley (2000a), in the two bidder case, prove the property P.1 below of the equilibrium, according to which the same relations of first order stochastic dominance pass through from the valuation distributions to the bid distributions. From P.1, if two distributions F_i and F_j are equal, bidders i and j use the same equilibrium bid function, that is, $\beta_i = \beta_j$. Thus, in the symmetric case, where the bidders' valuations are distributed identically, all bidders use the same bid function β. By transforming any equation in (1) as an equation in the only unknown β and solving this differential equation as in Riley and Samuelson (1981), Lebrun (1999a) proves U.3 below–the uniqueness in the symmetric case.

Existing Results II (Lebrun 1999a): *Let Assumptions A be satisfied. Assume further $c_i = c$ and $d_i = d$, for all i.*

P.1: Properties of the Equilibria

(i) If $F_i \leq F_j$, then, for all Bayesian Nash equilibrium $(\beta_1, ..., \beta_n)$, we have $F_i\alpha_i \leq F_j\alpha_j$ over $[r, \eta]$, where $\eta = \beta_1(d) = ... = \beta_n(d)$.

(ii) If $F_i = F_j$, then, for all Bayesian Nash equilibrium $(\beta_1, ..., \beta_n)$, we have $\beta_i = \beta_j$.

[10] Here, as in the rest of the paper, uniqueness refers to the uniqueness of the equilibrium strategies over the domains of valuations where the bidders submit "serious bids," that is, bids that win with strictly positive probability.

U.3: Uniqueness of the Equilibrium in the Symmetric Case

If $F_1 = ... = F_n$, the equilibrium $(\beta_1, ..., \beta_n)$ is unique and symmetric, that is, $\beta_1 = ... = \beta_n$.

From U.3 above, the equilibrium is unique in the symmetric case where $F_i = F$, for all i, even when, contrary to the assumptions of U.2, F is atomless and r is nonbinding, that is, $r \leq c$. Consequently, we do not need to consider the symmetric case in our proof of Theorem 1 (Section 2). However, because this case is particularly simple, we use it in this subsection to introduce and illustrate our main argument of proof.

The symmetric case with an atomless valuation distribution F and a nonbinding reserve price satisfies the assumptions of Theorem 1 (Section 2) if and only if there exists $\delta > 0$ such that F is strictly log-concave over an interval $(c, c+\delta)$, with $\delta > 0$. Let $(\beta_1, ..., \beta_n)$ be an equilibrium for such a symmetric case. From P.1 (ii), this equilibrium is symmetric and $\beta_1 = ... = \beta_n = \beta$. The system of differential equations (1) in the characterization C.1 reduces to the single equation (5) and the boundary conditions (2) and (3) reduce to (6) and (7) below:

$$\frac{d}{db}\alpha(b) = \frac{F(\alpha(b))}{f(\alpha(b))}\frac{1}{\alpha(b) - b}, (5)$$

$$\begin{aligned} \alpha(c) &= c \ (6) \\ \alpha(\eta) &= d \ (7). \end{aligned}$$

where $\eta = \beta(d)$ belongs to (c, d) and α is the inverse of β.

Figure 1 depicts a solution of (5), (6), and (7) as well as the "direction field" defined by the equation (5) in the space of couples (b, v). The graph of any solution of (5) must be tangent to the line segment of slope $\frac{F(v)}{f(v)}\frac{1}{v-b}$ through everyone of its points (b, v). Since the difference between v and b is fixed along any line parallel to the 45 degree-line, the ratio $\frac{1}{v-b}$ is also fixed along such a line. From our assumption of strict log-concavity of F over $(c, c+\delta)$, the ratio $\frac{F(v)}{f(v)}$ is increasing in v over this interval. Consequently, near (c, c), the slope defined by (5) at a point on a line parallel to the 45 degree-line is steeper the further this point is to the right of c. If there existed two different solutions of (5) and (6), the points of their graphs on

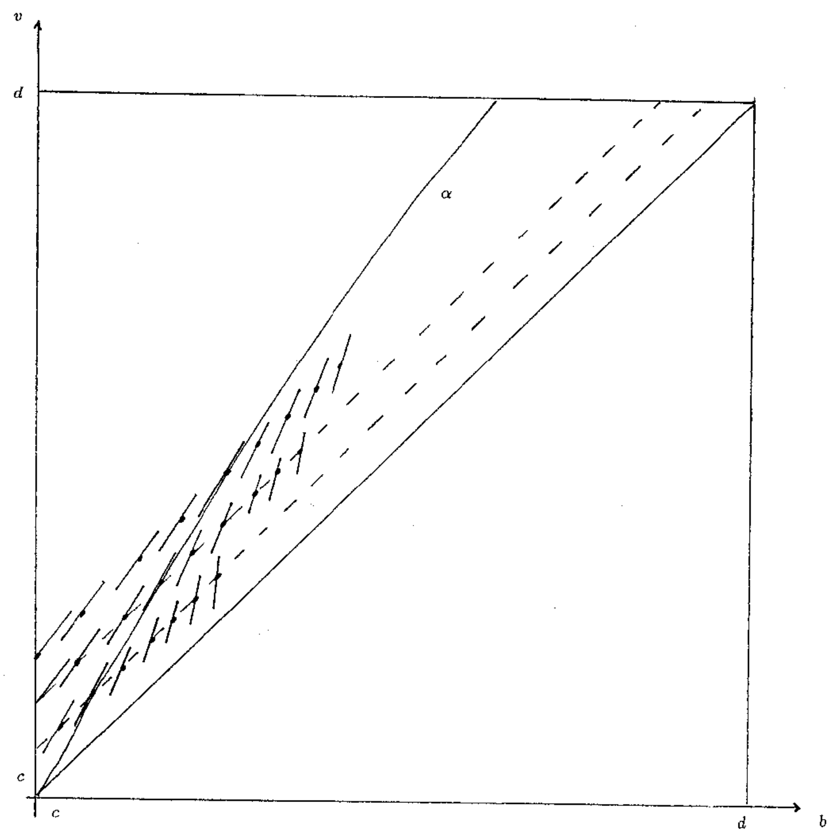

Figure 1:

a line parallel to and above the 45 degree-line would spread further apart as this line gets closer to the 45 degree-line. Thus, both their graphs could not get closer to the point (c, c) on the 45 degree-line as b tends towards c.

One way to make the argument at the end of the previous paragraph more formal is to first suppose that there exist two different solutions of (5) and (6) and thus two different values η and $\widetilde{\eta}$ of the parameter such that the corresponding solutions α and $\widetilde{\alpha}$ of (5) and (7) are also solutions of (6). Assume that $\widetilde{\eta}$ is the strictly smaller value. As depicted in Figure 2, the graph of the corresponding solution $\widetilde{\alpha}$ thus lies above the graph of the solution α. If we slide the graph of the function α down along the

45 degree-line by a small $\varepsilon > 0$, we obtain the graph of the new function $\widehat{\alpha}$, such that $\widehat{\alpha}(b) = \alpha(b+\varepsilon) - \varepsilon$ and such that $\widehat{\alpha}(c-\varepsilon) = c - \varepsilon$. The derivative $\frac{d}{db}\widehat{\alpha}(b)$ of this function at b is equal to the derivative $\frac{d}{db}\alpha(b+\varepsilon)$ of the initial function α at $b+\varepsilon$ and, from (5), is equal to $\frac{F(\alpha(b+\varepsilon))}{f(\alpha(b+\varepsilon))}\frac{1}{\alpha(b+\varepsilon)-b-\varepsilon}$ and, thus, to $\frac{F(\widehat{\alpha}(b)+\varepsilon)}{f(\widehat{\alpha}(b)+\varepsilon)}\frac{1}{\widehat{\alpha}(b)-b}$. Since $F(v)/f(v)$ is strictly increasing for v close to c, the derivative $\frac{d}{db}\widehat{\alpha}(b)$ is strictly larger than $\frac{F(\widehat{\alpha}(b))}{f(\widehat{\alpha}(b))}\frac{1}{\widehat{\alpha}(b)-b}$, for all b close to c. Consequently, at any intersection point near (c, c), the graph of the function $\widehat{\alpha}$ must be strictly steeper than the graph of any solution of (5) and, in particular, than the graph of $\widetilde{\alpha}$. The graph of this latter function $\widetilde{\alpha}$ could, thus, not cross from the right and from above the graph of the former function $\widehat{\alpha}$ in order to reach the point (c, c), and $\widetilde{\alpha}$ could not be a solution of (6). There cannot be two distinct solutions of (5), (6), and (7) and the equilibrium is unique. In the next subsection, we apply this argument to the general, possibly asymmetric, common-support case.

4.2 The General Common-Support Case

Let the assumptions of Theorem 1 (Section 2) be satisfied. The valuation distributions are, thus, atomless. From U.2 (Section 3), we can assume, in our proof of Theorem 1 for the common-support case, that the reserve price is not binding, that is, $r \leq c$. The characterization C.3 (Section 3) then applies.

For the general asymmetric case, where the valuation distributions may be different, it is difficult to depict the direction field as we have done in Figure 1 for the symmetric case. In fact, according to (1), the slope of any component α_i depends not only on this component and on b, but also on all the other components α_j, $j \neq i$. Nevertheless, it is straightforward to apply the "sliding" argument of the previous subsection to the general asymmetric case. According to Lemma 3 below, if we slide a solution of (1) down the 45 degree-line, we obtain a solution, not of the system of differential equations (1), but rather of the system of differential strict inequations (10). At any meeting point between a solution of the system of differential equations (1) and a not larger solution of the system (10) of differential inequations, the solution of the system of differential inequations (10) is steeper. As we show in Lemma 4 below, no solution of the system of the differential equations (1) can, thus, cross from the right and from above a solution of the

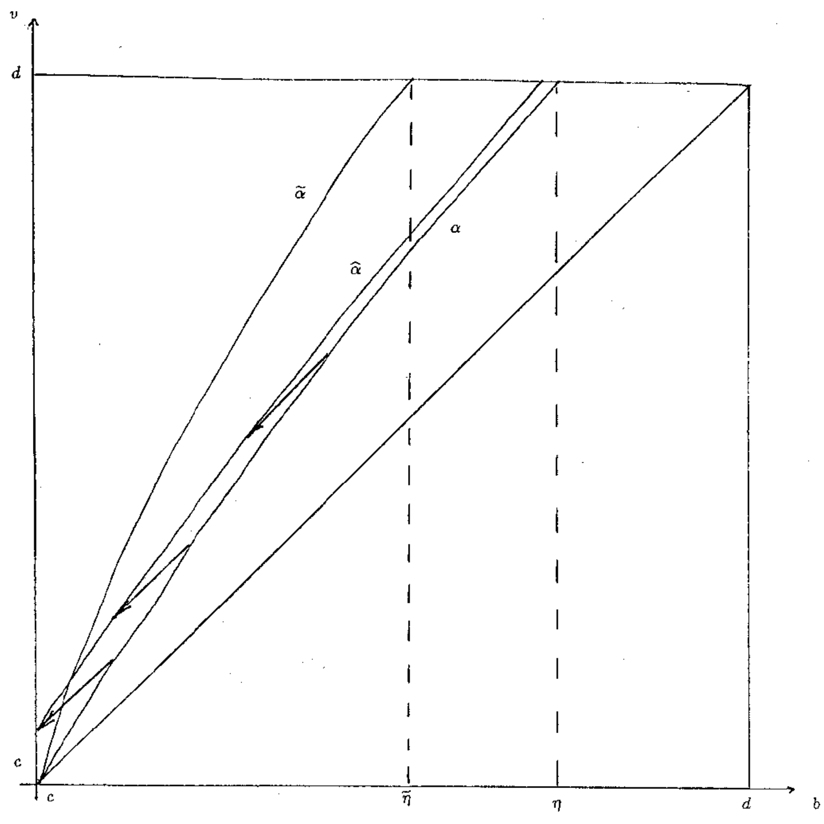

Figure 2:

system of differential inequations (10). The proof (by *reductio ab absurdum)*, illustrated in Figure 2 for the symmetric case, goes through to the asymmetric case.

Lemma 3: *Let Assumptions A.1 be satisfied. Assume $c_i = c$ and $d_i = d$, for all i. Assume also that there exists $\delta > 0$ such that $F_1, ..., F_n$ are strictly log-concave over $(c, c+\delta)$. Let $(\alpha_1, ..., \alpha_n)$ be a solution of (1) with strictly positive derivatives*[11] *over an interval $(c, c+\gamma]$ such that $\gamma > 0$ and $\alpha_i(c+\gamma) < c+\delta$. Let ε be a strictly positive number strictly smaller than γ, that is, $0 < \varepsilon < \gamma$. Let $\widehat{\alpha}_1, ..., \widehat{\alpha}_n$ be defined as follows:*

$$\widehat{\alpha}_i(b) = \alpha_i(b+\varepsilon) - \varepsilon, \text{(8)}$$

for all b in $(c-\varepsilon, c+\gamma-\varepsilon]$ and all $1 \leq i \leq n$. Then,

$$c < \widehat{\alpha}_i(b) < c+\delta-\varepsilon, \text{(9)}$$

for all b in $(c, c+\gamma-\varepsilon]$, and $(\widehat{\alpha}_1, ..., \widehat{\alpha}_n)$ is a solution over $(c, c+\gamma-\varepsilon]$ of the system of differential inequations (10)–considered in the same domain D as the system (1)–below:

$$\frac{d}{db}\widehat{\alpha}_i(b) > \frac{F_i(\widehat{\alpha}_i(b))}{f_i(\widehat{\alpha}_i(b))}\frac{1}{n-1}\left\{-\frac{n-2}{\widehat{\alpha}_i(b)-b}+\sum_{j\neq i}\frac{1}{\widehat{\alpha}_j(b)-b}\right\}, \text{(10)}$$

$1 \leq i \leq n$.

Proof: See Appendix 2.

Lemma 4: *Let Assumptions A.1 be satisfied. Assume $c_i = c$ and $d_i = d$, for all i. Assume also that there exists $\delta > 0$ such that $F_1, ..., F_n$ are strictly log-concave over $(c, c+\delta)$. Let $(\alpha_1, ..., \alpha_n)$, γ, ε, and $(\widehat{\alpha}_1, ..., \widehat{\alpha}_n)$ be as in Lemma 3. Let $(\widetilde{\alpha}_1, ..., \widetilde{\alpha}_n)$ be another solution of the system (1) over the interval $(c, \gamma]$ as in Lemma 3, that is, $(\widetilde{\alpha}_1, ..., \widetilde{\alpha}_n)$ is a solution of*

[11] This will be the case of any inverse equilibrium bid function, since, from C.1 (Section 3), any direct equilibrium bid function is differentiable when its value is strictly larger than c. Actually, as shown in Lebrun (1999a and 1997), any solution of (1) and (3) over an interval $(\gamma, \eta]$ has strictly positive derivatives. For the sake of completeness, we replicate the proof of this property in Appendix 1 (Lemma A1-2). In Section 5 and Appendix 5.1 (Lemma A5.1-4), we extend this property to the general case with possibly different supports.

(1)–considered in the domain D–over $(c, \gamma]$ *with strictly positive derivatives over this interval and such that* $\widetilde{\alpha}_i(c+\gamma) < c+\delta$*, for all* $1 \leq i \leq n$*. If*

$$\widehat{\alpha}_i(c+\gamma-\varepsilon) < \widetilde{\alpha}_i(c+\gamma-\varepsilon), \text{(11)}$$

for all $1 \leq i \leq n$*, then*

$$\widehat{\alpha}_i(b) < \widetilde{\alpha}_i(b), \text{(12)}$$

for all b *in* $(c, c+\gamma-\varepsilon]$ *and all* $1 \leq i \leq n$*.*

Proof: See Appendix 2.

Proof of Theorem 1 (Section 2) in the Common-Support Case: As we explain above, we can assume that the assumptions of C.3 are satisfied and we may replace (2) by (2"). Suppose that there exist two equilibria $(\beta_1, ..., \beta_n)$ and $\left(\widetilde{\beta}_1, ..., \widetilde{\beta}_n\right)$. The inverse bid functions $(\alpha_1, ..., \alpha_n)$ and $(\widetilde{\alpha}_1, ..., \widetilde{\alpha}_n)$ are solutions, with strictly positive derivatives, of the differential system (1) with boundary conditions (2") and (3) for different values η and $\widetilde{\eta}$ of the parameter. Without loss of generality, we can assume that $\widetilde{\eta} < \eta$.

From (2"), there exists $0 < \gamma < \widetilde{\eta}$ such that

$$\widetilde{\alpha}_i(c+\gamma) < c+\delta, \text{ (13)}$$

for all $1 \leq i \leq n$. From Lemma 2 (Section 2), we know that $\alpha_i(b) < \widetilde{\alpha}_i(b)$, for all $1 \leq i \leq n$ and for all b in $(c, \widetilde{\eta}]$, and thus that $\alpha_i(c+\gamma) < \widetilde{\alpha}_i(c+\gamma)$, for all $1 \leq i \leq n$. Since $\widetilde{\alpha}_i(c+\gamma-\varepsilon)$ tends towards $\widetilde{\alpha}_i(c+\gamma)$ when ε tends towards zero, there exists $0 < \varepsilon < \gamma$ such that

$$\alpha_i(c+\gamma) - \varepsilon < \widetilde{\alpha}_i(c+\gamma-\varepsilon), \text{(14)}$$

for all $1 \leq i \leq n$.

For all $1 \leq i \leq n$, define $\widehat{\alpha}_i$ as follows:

$$\widehat{\alpha}_i(b) = \alpha_i(b+\varepsilon) - \varepsilon,$$

for all b in $(c-\varepsilon, \eta-\varepsilon]$. From (14), we have (15) below:

$$\widehat{\alpha}_i(c+\gamma-\varepsilon) < \widetilde{\alpha}_i(c+\gamma-\varepsilon), \text{ (15)}$$

for all $1 \leq i \leq n$.

From (13), (15), and Lemma 4, we have:

$$\widehat{\alpha}_i (b) < \widetilde{\alpha}_i (b) , (16)$$

for all b in $(c, c + \gamma - \varepsilon]$ and for all $1 \leq i \leq n$. Making b in (16) tend towards c, we find, from (2”), $\widehat{\alpha}_i (c) = \alpha_i (c + \varepsilon) - \varepsilon \leq \widetilde{\alpha} (c) = c$, for all $1 \leq i \leq n$. However, since $(c + \varepsilon, \alpha_1 (c + \varepsilon), ..., \alpha_n (c + \varepsilon))$ belongs to the domain D of (1), we have $\alpha_i (c + \varepsilon) > c + \varepsilon$ and, thus, $\widehat{\alpha}_i (c) = \alpha_i (c + \varepsilon) - \varepsilon > c$, for all $1 \leq i \leq n$. We obtain a contradiction and we have proved Theorem 1. ||

5. Extension to Possibly Different Supports

5.1 Extension to Possibly Different Lower Extremities

In this section, we extend our proof of Theorem 1 (Section 2) from the common-support case (Section 4) to the general case, with possibly different supports. Consider first the case where only the lower extremities of the supports may differ. The upper extremities are identical and we still denote by d the common upper extremity, that is, $d_1 = ... = d_n = d$. We can again assume that the reserve price is strictly smaller than d, that is, $r < d$. Let the assumptions of Theorem 1 (Section 2) be satisfied[12]. Thus, the valuation distributions are atomless, the support of F_1 has the largest lower extremity, and the support of F_2 has the second largest, that is, $c_i \leq c_2 \leq c_1$, for all $i \geq 2$.

The case where the reserve price r is not smaller than c_1, that is, $r \geq c_1$, can easily fit in the common-support case, which we addressed in the previous section. It suffices, for example, to concentrate at r the probabilities spread by the valuation distributions over the intervals $[c_i, r]$ in order to obtain valuation distributions with the same support $[r, d]$ that will give rise to the same equilibria as the initial distributions. The results of the previous section thus apply to this case. As in the previous section, when (i) in Theorem 1 (Section 2) holds true, that is, $r > c_1$, the existence and uniqueness of the equilibrium follow from U.2 (Section 3) and log-concavity is unnecessary.

[12] All our existence and uniqueness results actually hold true even when lower extremities of the valuation intervals are mass points. However, contrary to the atomless case, the equilibrium may involve a mixed component when the highest lower extremity $c_1 > c_2$ is a mass point.

Assume next $r < c_1$. The results in Lebrun (1999a) according to which every equilibrium is pure and such that its inverse bid functions satisfy the differential system (1) and the initial condition (3) in the characterization C.1 (Section 3) go through to this case. However, the initial condition (2) has to be changed. In general, there will exist an infinity of possible substitutes for this initial condition and, thus, an infinity of equilibria. Focusing only on the equilibria where the bids are not strictly larger than the valuations, that is, eliminating weakly dominated strategies, allows to determine a unique initial condition and to obtain uniqueness of the equilibrium. Maskin and Riley (1996) and Lebrun (1999b) obtain this new initial condition (2"') below, which applies also to the case $r \geq c_1$, mainly by ruling out deviations by bidder 1 with valuation $\max(r, c_1)$ from $\beta_1(\max(r, c_1))$, which we denote by $\underline{v}$.

Definition 1: *Let (A1) in Assumptions A.1 be satisfied. Assume $d_i > r$, for all i. Let $c_{(1)}$ be the largest lower extremity and $c_{(2)}$ the second largest lower extremity. Then, $\underline{v}$ is the element of $\left[c_{(2)}, c_{(1)}\right]$ that is defined in (17) below:*

$$\underline{v} = \max \arg \max_{b \in \left[\max\left(r, c_{(2)}\right), \max\left(r, c_{(1)}\right)\right]} \left(c_{(1)} - b\right) \prod_{i>1} F_i(b) \quad (17).$$

C.4 Characterization with Possibly Different Lower Extremities: *Let Assumptions A.1 be satisfied. Assume $F_i(c_i) = 0$ and $d_i = d > r$, for all i. Without loss of generality, assume also $c_1 \geq c_2 \geq c_i$, for all $i \geq 2$.*

The characterization C.1 holds true for equilibria where bidders submit at most their valuations if[13] *(2) is replaced by (2"') below:*

$$\alpha_i(\underline{v}) = \underline{v}, \text{ for all, except possibly one, } i \text{ between 1 and } n \quad (2"')$$

where $\underline{v}$ is as in Definition 1.

According to (17), $\underline{v}$ is the maximum of the arguments b that would maximize bidder 1's expected payoff $(c_1 - b)\prod_{i>1} F_i(b)$ if his valuation was

[13]Obviously, the domain of the system (1) is here $\{(b, \alpha_1, ..., \alpha_n) \,|c_i, b < \alpha_i \leq d, \text{ for all } 1 \leq i \leq n\}$.

c_1 and if the other bidders bid their valuations. For the sake of completeness, we provide a proof of (2"') in Appendix 3 (see Lemma A3.2-2).

Under (ii) in Theorem 1 (Section 2), that is, if $c_1 > c_2$, Definition 1 of $\underline{v}$ implies $F_i(\underline{v}) > 0$, for all $i > 1$. In this case, uniqueness follows from U.2 (Section 3) by concentrating at $\underline{v}$ the probabilities $F_i(\underline{v})$, for all $i > 1$.

If (i) and (ii) in Theorem 2 are not satisfied, that is, if $c_1 = c_2 \geq r$, Definition 1 of $\underline{v}$ implies $c_2 = c_1 = \underline{v}$. From Lemma 1 (Section 3) and from, since the valuation distributions are atomless, $F_k(\underline{v}) = F_k(c_k) = 0$, for $k = 1, 2$, the initial condition (2"') reduces to (2") in C.3 (Section 3) where c has been replaced by $\underline{v} = c_1$. Under the assumption (iii) in Theorem 1 (Section 2) of strict log-concavity in an interval $(\underline{v}, \underline{v} + \delta) = (c_1, c_1 + \delta)$, with $\delta > 0$, the proof, from the previous section, for a common support goes through to this case. Theorem 1 thus holds true when the upper extremities of the supports are identical.

5.2 Extension to Possibly Different Lower and Upper Extremities

We now extend our proof of Theorem 1 (Section 2) to the case where even the upper extremities of the supports may differ. Let the assumptions of Theorem 1 (Section 2) be satisfied. Thus, $c_i \leq c_2 \leq c_1$, for all $i \geq 2$, and $F_i(c_i) = 0$, for all i. Let $d_{(i)}$ be the $(n-i+1)$th order statistics of $(d_1, ..., d_n)$, that is, the ith largest upper extremity. The n-tuple $\left(d_{(1)}, ..., d_{(n)}\right)$ is the n-tuple $(d_1, ..., d_n)$ rearranged by order of nonincreasing value. Since the case where only one bidder can have valuations strictly larger than r is simple and uninteresting[14], we assume that $d_{(2)} > r$.

We first extend the characterization C.4. The lowest serious bid $\underline{v}$ is as defined in Definition 1 in the previous subsection. In any equilibrium, the bidders whose valuations are never larger than $\underline{v}$ cannot obtain strictly positive payoffs and engage only in "nonserious" bidding. Thus, the characterization involves only those bidders who can have valuations strictly larger than $\underline{v}$. Let n' be the number of those bidders and let $J \subseteq I = \{1, ..., n\}$ be the set of their indices. We have Definition 2 below.

Definition 2: *Assume $c_1 \geq c_2 \geq c_i$, for all $i \geq 2$.*

[14] At the unique equilibrium, no bidder bids strictly above r.

(i) $\left(d_{(1)},...,d_{(n)}\right)$ *is the n-tuple such that* $d_{(1)} \geq ... \geq d_{(n)}$ *and there exists a permutation* $\pi : \{1,...,n\} \rightarrow \{1,...,n\}$ *such that* $d_{(i)} = d_{\pi(i)}$*, for all* i *in* $\{1,...,n\}$*.*

(ii) Assume $d_{(2)} > r$*. Then,* J *is the subset of* $\{1,...,n\}$ *such that*

$$J = \{j \text{ such that } 1 \leq j \leq n \text{ and } d_j > \underline{v}\}.$$

(iii) Assume $d_{(2)} > r$*. Then,* n' *is the number of elements of* J*, that is,* $n' = \#J$*.*

From the previous definition, if $n' < n$ then $d_{(n'+1)} \leq \underline{v} < d_{(n')}$. In order to describe the new boundary condition at the maximum bid, we define the integer-valued function $k(\eta)$ and the real-valued function $d(\eta)$ below.

Definition 3: *Assume* $c_1 \geq c_2 \geq c_i$*, for all* $i \geq 2$*, and* $d_{(2)} > r$*.*

(i) For all η *in* $\left(\underline{v}, d_{(2)}\right)$*, let* $k(\eta) \in \{2,...,n\}$ *be such that* $\eta < d_{(k(\eta))}$ *and:*

$$\frac{1}{d_{(k(\eta))} - \eta} \leq \frac{1}{k(\eta) - 1} \sum_{i=1}^{k(\eta)} \frac{1}{d_{(i)} - \eta},$$

$$\text{and if } \eta < d_{(k(\eta)+1)} \text{ then } \frac{1}{k(\eta) - 1} \sum_{i=1}^{k(\eta)} \frac{1}{d_{(i)} - \eta} < \frac{1}{d_{(k(\eta)+1)} - \eta}.$$

(ii) For all η *in* $\left(\underline{v}, d_{(2)}\right)$*, let* $d(\eta)$ *be as follows:*

$$d(\eta) = \eta + \frac{k(\eta) - 1}{\sum_{i=1}^{k(\eta)} \frac{1}{d_{(i)} - \eta}} (18).$$

We show in Lemma A4.1 in Appendix 4 that, for all η in $\left(\underline{v}, d_{(2)}\right)$, there exists one and only one $k(\eta)$ as in Definition 3 (i) above. From the definitions of $d(\eta)$ and $k(\eta)$ above, we have[15]

$$d_{k(\eta)+1} < d(\eta) \leq d_{k(\eta)}.$$

[15] The first inequality below applies only when $k(\eta) < n$.

C.5 below is our most general characterization.

C.5 Characterization with Possibly Different Lower and Upper Extremities: *Let Assumptions A.1 be satisfied. Assume* $F_i(c_i) = 0$*, for all* i*. Without loss of generality, assume* $c_1 \geq c_2 \geq c_i$*, for all* $i \geq 2$*. Let* $\underline{v}$ *be as in Definition 1 and* $\left(d_{(1)}, ..., d_{(n)}\right)$ *as in Definition 2. Assume* $d_{(2)} > r$*. Let* J, n'*,and* $d(.)$ *be as in Definitions 2 and 3.*

There exists a Bayesian Nash equilibrium where bidders submit at most their valuations. In every such equilibrium, bidder $i \in J$ *follows a bid function* β_i*, for all* $1 \leq i \leq n$*. Moreover, for every such equilibrium there exists* η *in*[16] $\left(\underline{v}, d_{(2)}\right)$ *such that, for all* $i \in J$*, there exists a continuous extension of* β_i *to the interval* $[\underline{v}, \max(d_i, d(\eta))]$ *that is differentiable with a strictly positive derivative everywhere over this interval, except possibly at* d_i *or when its value is equal to* $\underline{v}$*, and such that the inverses* $(\alpha_i)_{i \in J}$ *of these extensions, where differentiable, satisfy the system of differential equations (1)–considered over the domain* $D' = \{(b, \alpha_1, ..., \alpha_n) | c_i, b < \alpha_i, \text{ for all } 1 \leq i \leq n\}$*–in C.1 (Section 3), the boundary conditions (2"') in C.4, and (3') below:*

$$\alpha_i(\eta) = \max(d_i, d(\eta)), \text{ (3')}$$

for all $i \in J$*.*

The above characterization can easily be proved as Lebrun (1999a) proved the characterization C.1 (Section 3) with common supports[17].

In the general characterization C.5, if i is such that $\underline{v} < d_i < d(\eta)$, the bid function β_i is extended to the interval $[\underline{v}, d(\eta)]$, which is strictly larger than the actual support, truncated at $\underline{v}$, $[\underline{v}, d_i]$ of bidder i's valuation. We also denote this extension by β_i. When (1), (2"'), and (3') hold true, this extension is bidder i's best reply function. Even if, according to F_i, the valuation v_i belongs to $\left[d_i, d_{(\eta)}\right]$ with probability zero, $\beta_i(v_i)$ is a best response from bidder i with valuation v_i in $\left[d_i, d_{(\eta)}\right]$.

As in the previous subsection, η is the highest bid that can actually be submitted[18]. However, contrary to the previous subsection, only some bidders bid η at the upper extremities of their valuation supports. As we

[16] Note that, from our definitions and assumptions, $d_{(2)} > \underline{v}$.

[17] For the sake of completeness, we provide the main steps of this proof in Appendix 3.2.

[18] It is the maximum of the support of the distribution of the highest bid.

show in Appendix 3.2 (see also Maskin and Riley 1996), $d(\eta)$ defined in Definition 3 is the smallest upper extremity of the support at which a bidder bids the highest bid η. Since any k such that $d_k = d_{(1)}$ or $d_k = d_{(2)}$ satisfies the inequality $d_k \geq d(\eta)$, (3') implies that η is the actual maximum bid of any bidder whose extremity of his support is the largest or the second largest. On the other hand, if bidder k's upper extremity of the support $d_k > \underline{v}$ is strictly smaller than $d(\eta)$, bidder k's actual maximum bid $\beta_k(d_k)$ will be[19] strictly smaller than η.

Also as in the previous sections and subsection, no two solutions as in C.5 of (1) and (3') can correspond to the same value of the parameter η. For all i such that $\underline{v} < d_i < d(\eta)$, (3') requires a value of α_i at η that is outside the support of F_i. Outside its support, the cumulative function F_i is constant and the equation in (1) reduces to:

$$0 = -\frac{n' - 2}{\alpha_i(b) - b} + \sum_{\substack{j \neq i \\ j \in J}} \frac{1}{\alpha_j(b) - b}, (19)$$

for all i such that $\underline{v} < d_i < d(\eta)$. We obtain one such equation for all i such that $\underline{v} < d_i < d(\eta)$. Solving the system these equations form, we see that, for any such i, the function α_i is determined by the functions α_j, with j such that $d_j \geq d(\eta)$. Because this system is symmetric in α_i and since there are exactly $k(\eta)$ values of j such that $d_j \geq d(\eta)$, α_i is equal to the same function–the solution of the equation (20) below–for all i such that $\underline{v} < d_i < d(\eta)$:

$$\frac{1}{\alpha_i(b) - b} = \frac{1}{k(\eta) - 1} \sum_{\substack{j \\ d_j \geq d(\eta)}} \frac{1}{\alpha_j(b) - b} (20).$$

Replacing in the system (1), the functions α_i, i such that $\underline{v} < d_i < d(\eta)$, by their expressions (20) as functions of α_j, j such that $d_j \geq d(\eta)$, we obtain a system of differential equations with the only unknowns α_j, j such that $d_j \geq d(\eta)$. This system is actually the system (1) we obtain when only the $k(\eta)$ bidders j, j such that $d_j \geq d(\eta)$, are present. That is, it is the system

[19] Because the best reply function β_k will be strictly increasing between d_k and $d(\eta)$.

(21) below:

$$\frac{d}{db}\ln F_j\left(\alpha_j\left(b\right)\right)=\frac{1}{k\left(\eta\right)-1}\left\{-\frac{k\left(\eta\right)-2}{\alpha_j\left(b\right)-b}+\sum_{\substack{k\neq j\\ d_k\geq d(\eta)}}\frac{1}{\alpha_k\left(b\right)-b}\right\}, \quad (21)$$

for all j such that $d_j \geq d\left(\eta\right)$.

This last system (21) and the initial condition (3') satisfy the standard assumptions of the theory of ordinary differential equations and only one solution exists. The initial condition (3') thus locally determines α_j, for all j such that $d_j \geq d\left(\eta\right)$, and, through the equations (19), the function α_i, for all i such that $\underline{v} < d_i < d\left(\eta\right)$. We then extend these unique solutions below η until the common function α_i, i such that $d_i < d\left(\eta\right)$, takes as its value the highest upper extremity strictly smaller than $d\left(\eta\right)$. At the bid where this next higher upper extremity will be reached, we will add all functions α_i of the bidders with this upper extremity of their supports to the system (21), to which we may apply again the standard theory of differential equations. The remaining functions α_k will be determined through equations similar to (21). Continuing in this fashion, we see that the value of the parameter η determines the solution of (1) and (3').

Moreover, the property, stated in Lemma 2 (Section 3), of monotonicity of the solution of the differential system with respect to η extends to (1) and (3') (for a proof, see Lemma A5.2-2 in Appendix 5.2). The proof of Theorem 1 can then proceeds as in the case of common upper extremities, in the previous subsection.

6. Conclusion

We addressed the issue of uniqueness of the equilibrium in first-price auctions with independent private valuations in the general case of possibly different supports. By a simple geometric argument, consisting in "sliding" a solution of the differential system down the 45-degree line, we showed that the equilibrium is unique when the valuation cumulative distribution functions are strictly log-concave near the highest lower extremity of their supports.

Appendix 1

Lemma A1-1: *Let Asssumptions A.1 be satisfied. Assume further* $c_i = c$*,* $d_i = d$*, for all* i*, and* $r \leq c$*. Let* $(\alpha_1, ..., \alpha_n)$ *be a solution over an interval* $(\gamma, \gamma']$*, with* $c \leq \gamma < \gamma' < d$*, of the differential system (1) considered in the domain* D *and such that* $\frac{d}{db}\alpha_1(\gamma') > 0$*,...,* $\frac{d}{db}\alpha_n(\gamma') > 0$*. Then* $\frac{d}{db}\alpha_1(b) > 0, ..., \frac{d}{db}\alpha_n(b) > 0$*, for all* b *in* $(\gamma, \gamma']$*.*

Proof: For all $1 \leq i \leq n$, consider b'_i defined as follows:

$$b'_i = \inf\left\{b' \in [\gamma, \gamma'] \,|\, \frac{d}{db}\ln F_i(\alpha_i(b)) > 0, \text{ for all } b \text{ in } (b', \gamma']\right\}.$$

For $\alpha_i(b) \in (c, d]$, we have $\frac{d}{db}\ln F_i(\alpha_i(b)) > 0$ if and only if $\frac{d}{db}\alpha_i(b) > 0$ and $\frac{d}{db}\ln F_i(\alpha_i(b)) < 0$ if and only if $\frac{d}{db}\alpha_i(b) < 0$. From (1), we see that $\frac{d}{db}\ln F_1\alpha_1, ..., \frac{d}{db}\ln F_n\alpha_n$ are continuous over $(\gamma, \gamma']$. Since $\frac{d}{db}\alpha_i(\gamma') > 0$, we have $\frac{d}{db}\ln F_i(\alpha_i(\gamma')) > 0$ and $b'_i < \gamma'$, for all i. We want to prove that $b'_1, ..., b'_n = \gamma$. From their definitions, we know that $b'_1, ..., b'_n \geq \gamma$. We will thus have proved Lemma A1-1 if we prove that $\max_k b'_k \leq \gamma$.

Suppose that $\max_k b'_k > \gamma$. Let i be such that $b'_i = \max_k b'_k$. From the continuity of $\frac{d}{db}\ln F_i\alpha_i$, we have $\frac{d}{db}\ln F_i\alpha_i(b'_i) = 0$ and, thus, $\frac{d}{db}\alpha_i(b'_i) = 0$. Moreover, since $b'_i \geq b'_k$ we also have $\frac{d}{db}\ln F_k\alpha_k(b'_i) \geq 0$ and, thus, $\frac{d}{db}\alpha_k(b'_i) \geq 0$, for all $1 \leq k \leq n$. From (1), we have:

$$(\alpha_i(b) - b)\frac{d}{db}\ln F_i(\alpha_i(b)) = \frac{1}{n-1}\left\{-(n-2) + \sum_{k \neq i}\frac{\alpha_i(b) - b}{\alpha_k(b) - b}\right\},$$

for all b in $(\gamma, \gamma']$. Taking the derivative of the equation above, we obtain:

$$\begin{aligned}&\frac{d}{db}\left\{(\alpha_i(b) - b)\frac{d}{db}\ln F_i(\alpha_i(b))\right\}\\ = \; &\frac{1}{n-1}\left\{\sum_{k \neq i}\frac{1}{(\alpha_k(b) - b)^2}\left[\left(\frac{d}{db}\alpha_i(b) - 1\right)(\alpha_k(b) - b) - (\alpha_i(b) - b)\left(\frac{d}{db}\alpha_k(b) - 1\right)\right]\right\}, \text{(A1-1)}\end{aligned}$$

for all b in $(\gamma, \gamma']$.

If we substitute b'_i to b in (A1-1), the expression between brackets in the sum in the R.H.S. is equal to $(\alpha_i(b'_i) - \alpha_k(b'_i)) - (\alpha_i(b'_i) - b'_i)\frac{d}{db}\alpha_k(b'_i)$. From

(1), we have $\frac{d}{db}\ln F_k\left(\alpha_k\left(b_i'\right)\right)-\frac{d}{db}\ln F_i\left(\alpha_i\left(b_i'\right)\right)=\frac{1}{\alpha_i\left(b_i'\right)-b_i'}-\frac{1}{\alpha_k\left(b_i'\right)-b_i'}$, for all k. Since $\frac{d}{db}\ln F_i\left(\alpha_i\left(b_i'\right)\right)=0$ and $\frac{d}{db}\ln F_k\left(\alpha_k\left(b_i'\right)\right)\geq 0$, we have $\alpha_i\left(b_i'\right)\leq\alpha_k\left(b_i'\right)$, for all k. The sum between brackets in the sum in the R.H.S. of (A1-1) is thus nonpositive. Moreover, there exists k such that the corresponding term is strictly negative. In fact, from equation (4) (with $j=i$) there exists $k\neq i$ such that $\frac{d}{db}\ln F_k\left(\alpha_k\left(b_i'\right)\right)>0$ and, thus, $\frac{d}{db}\alpha_k\left(b_i'\right)>0$. Consequently, from (A1-1) the derivative of the function $\left(\alpha_i\left(b\right)-b\right)\frac{d}{db}\ln F_i\left(\alpha_i\left(b\right)\right)$ at $b=b_i'$ is strictly negative and this function is strictly decreasing in a neighborhood of b_i'. However, since $\frac{d}{db}\ln F_i\left(\alpha_i\left(b_i'\right)\right)=0$, the value of this function at $b=b_i'$ is equal to zero. There thus exists $\varepsilon>0$ such that $\left(\alpha_i\left(b\right)-b\right)\frac{d}{db}\ln F_i\left(\alpha_i\left(b\right)\right)<0$, for all b in $\left(b_i',b_i'+\varepsilon\right)$. Since $\left(\alpha_i\left(b\right)-b\right)>0$, for all b in $\left(\gamma,\gamma'\right]$, we obtain $\frac{d}{db}\ln F_i\left(\alpha_i\left(b\right)\right)<0$, for all b in $\left(b_i',b_i'+\varepsilon\right)$. This contradicts the definition of b_i' and we have proved Lemma A1-1. ||

Lemma A1-2: *Let Assumptions A.1 be satisfied. Assume further $c_i=c$, $d_i=d$, for all i, and $r\leq c$. Let $\left(\alpha_1,...,\alpha_n\right)$ be a solution over $\left(\gamma,\eta\right]$ of (1) and (3) with $c<\eta<d$. Then, $\frac{d}{db}\alpha_1\left(b\right)>0,...,\frac{d}{db}\alpha_n\left(b\right)>0$, for all b in $\left(\gamma,\eta\right]$.*

Proof: By substituting in (1) η to b and d to $\alpha_i\left(b\right)$, for all $1\leq i\leq n$, we see that (3) implies $\frac{d}{db}\alpha_i\left(d\right)=\frac{1}{(n-1)f_i(d)(d-\eta)}>0$, for all $1\leq i\leq n$. Lemma A1-2 then follows from Lemma A1-1. ||

In Lemma A5.1-4, we extend Lemma A1-2 to the model with possibly different supports.

Proof of Lemma 2 (Section 3): From Lemma A1-2, $\alpha_1,...,\alpha_n$ are strictly increasing over $\left(\gamma,\eta\right]$ and thus $\alpha_i\left(\eta'\right)<\alpha_i^*\left(\eta'\right)=d$, for all $1\leq i\leq n$. Define g in $\left[\max\left(\gamma,\gamma'\right),\eta'\right]$ as follows:

$$g=\inf\left\{b\in\left[\max\left(\gamma,\gamma'\right),\eta'\right]\middle|\alpha_i^*\left(b'\right)>\alpha_i\left(b'\right),\text{ for all }1\leq i\leq n\text{ and all }b'\in\left(b,\eta'\right]\right\}.$$

We want to prove that $g=\max\left(\gamma,\gamma'\right)$. We already know that $g<\eta'$. Suppose that $g>\max\left(\gamma,\gamma'\right)$. By continuity, there exists $1\leq i\leq n$ such that $\alpha_i^*\left(g\right)=\alpha_i\left(g\right)$. From the definition of g, we also have $\alpha_j^*\left(g\right)\geq\alpha_j\left(g\right)$, for all $1\leq j\leq n$. Moreover, there exists $j\neq i$ such that $\alpha_j^*\left(g\right)>\alpha_j\left(g\right)$. Otherwise, the solutions $\left(\alpha_1,...,\alpha_n\right)$ and $\left(\alpha_1^*,...,\alpha_n^*\right)$ of the differential system (1) would coincide at g and, from the uniqueness of the solution of (1,3), over their common definition domain. However, this is impossible since $\alpha_k(\eta') < \alpha_k^*(\eta') = d$, for all $1 \leq k \leq n$.

From (1), $\frac{d}{db}\alpha_i(g)$ is a strictly decreasing function of $\alpha_j(g)$, for all $j \neq i$. Consequently, $\frac{d}{db}\alpha_i(g) > \frac{d}{db}\alpha_i^*(g)$. There thus exists $\delta > 0$ such that $\alpha_i(b) > \alpha_i^*(b)$, for all b in $(g, g+\delta)$. This contradicts the definition of g and Lemma 2 is proved. ||

In Lemma A5.2-2, we extend Lemma 2 to the model with possibly different supports.

Appendix 2

Proof of Lemma 3 (Section 4): Let i be between 1 and n. Since α_i is strictly increasing over $(c, c+\gamma]$, so is $\widehat{\alpha}_i$ over $(c-\varepsilon, c+\gamma-\varepsilon]$ and, thus, $\widehat{\alpha}_i(b) < \widehat{\alpha}_i(c+\gamma-\varepsilon) = \alpha_i(c+\gamma) - \varepsilon < c+\delta-\varepsilon$, for all b in $(c-\varepsilon, c+\gamma-\varepsilon]$ and, in particular, for all b in $(c, c+\gamma-\varepsilon]$. We have proved the second inequality in (9).

Since $(\alpha_1, ..., \alpha_n)$ is a solution of (1)–considered on the domain D–over the interval $(c, c+\gamma]$, $(b, \alpha_1(b), ..., \alpha_n(b))$ belongs to this domain, for all b in $(c, c+\gamma]$, and, thus, for $b = c+\varepsilon$. Consequently, $\alpha_i(c+\varepsilon) > c+\varepsilon$ and $\widehat{\alpha}_i(c) = \alpha_i(c+\varepsilon) - \varepsilon > c$, for all $1 \leq i \leq n$. Since $\widehat{\alpha}_i(b) > \widehat{\alpha}_i(c)$, for all b in $(c, c+\gamma-\varepsilon]$ and all $1 \leq i \leq n$, the first inequality in (9) follows.

From (8), we have $\frac{d}{db}\widehat{\alpha}_i(b) = \frac{d}{db}\alpha_i(b+\varepsilon)$ and, from (1) and the strict positivity of the derivative of α_i, we find $\frac{d}{db}\widehat{\alpha}_i(b) = \frac{F_i(\alpha_i(b+\varepsilon))}{f_i(\alpha_i(b+\varepsilon))}\frac{1}{n-1}\left\{-\frac{n-2}{\alpha_i(b+\varepsilon)-b-\varepsilon} + \sum_{j\neq i}\frac{1}{\alpha_j(b+\varepsilon)-b-\varepsilon}\right\}$ > 0, for all b in $(c, c+\gamma-\varepsilon]$ and all $1 \leq i \leq n$. From the definition (8) of $\widehat{\alpha}_i$, the difference $\alpha_i(b+\varepsilon) - b - \varepsilon$ is equal to the difference $\widehat{\alpha}_i(b) - b$, for all i. Consequently, $\frac{d}{db}\widehat{\alpha}_i(b) = \frac{F_i(\alpha_i(b+\varepsilon))}{f_i(\alpha_i(b+\varepsilon))}\frac{1}{n-1}\left\{-\frac{n-2}{\widehat{\alpha}_i(b)-b} + \sum_{j\neq i}\frac{1}{\widehat{\alpha}_j(b)-b}\right\}$. In the first ratio of the R.H.S., we can replace $\alpha_i(b+\varepsilon)$ by its value $\widehat{\alpha}_i(b)+\varepsilon$ and $(\widehat{\alpha}_1, ..., \widehat{\alpha}_n)$ is a solution of (A2-1) below:

$$\frac{d}{db}\widehat{\alpha}_i(b) = \frac{F_i(\widehat{\alpha}_i(b)+\varepsilon)}{f_i(\widehat{\alpha}_i(b)+\varepsilon)}\frac{1}{n-1}\left\{-\frac{n-2}{\widehat{\alpha}_i(b)-b} + \sum_{j\neq i}\frac{1}{\widehat{\alpha}_j(b)-b}\right\} > 0, \text{(A2-1)}$$

for all b in $(c, c+\gamma-\varepsilon]$ and all $1 \leq i \leq n$. Furthermore, from (9), $\widehat{\alpha}_i(b)$ and $\widehat{\alpha}_i(b)+\varepsilon$ belong to the interval $(c, c+\gamma]$ where F_i is strictly log-concave and, thus, where the ratio F_i/f_i is strictly increasing. Consequently, $\frac{F_i(\widehat{\alpha}_i(b)+\varepsilon)}{f_i(\widehat{\alpha}_i(b)+\varepsilon)} > \frac{F_i(\widehat{\alpha}_i(b))}{f_i(\widehat{\alpha}_i(b))}$, for all b in $(c, c+\gamma-\varepsilon]$ and all $1 \leq i \leq n$, and (A2-1) implies (10). ||

Proof of Lemma 4 (Section 4): Let b^* be the smallest argument b in $[c, c+\gamma-\varepsilon)$ such that (12) holds true everywhere in $(b, c+\gamma-\varepsilon]$, that is,

$$b^* = \inf\left\{b'' \in [c, c+\gamma-\varepsilon] \,|\widehat{\alpha}_i(b) < \widetilde{\alpha}_i(b), \text{ for all } b \text{ in } (b'', c+\gamma-\varepsilon] \text{ and all } 1 \leq i \leq n\right\}.$$

From (11), the set in this definition of b^* is not empty. Suppose that $b^* > c$. Then, there exists i between 1 and n such that $\widehat{\alpha}_i(b^*) = \widetilde{\alpha}_i(b^*)$ and $\widehat{\alpha}_j(b^*) \leq \widetilde{\alpha}_j(b^*)$, for all $j \neq i$. From Lemma 3, we have

$$\frac{d}{db}\widehat{\alpha}_i(b^*) > \frac{F_i(\widehat{\alpha}_i(b^*))}{f_i(\widehat{\alpha}_i(b^*))}\frac{1}{n-1}\left\{-\frac{n-2}{\widehat{\alpha}_i(b^*)-b^*} + \sum_{j\neq i}\frac{1}{\widehat{\alpha}_j(b^*)-b^*}\right\}.$$

From $\widehat{\alpha}_i(b^*) = \widetilde{\alpha}_i(b^*)$ and $\widehat{\alpha}_j(b^*) \leq \widetilde{\alpha}_j(b^*)$, for all $j \neq i$, the R.H.S. of the inequality above is not smaller than $\frac{F_i(\widetilde{\alpha}_i(b^*))}{f_i(\widetilde{\alpha}_i(b^*))}\frac{1}{n-1}\left\{-\frac{n-2}{\widetilde{\alpha}_i(b^*)-b^*} + \sum_{j\neq i}\frac{1}{\widetilde{\alpha}_j(b^*)-b^*}\right\}$. We thus obtain

$$\frac{d}{db}\widehat{\alpha}_i(b^*) > \frac{F_i(\widetilde{\alpha}_i(b^*))}{f_i(\widetilde{\alpha}_i(b^*))}\frac{1}{n-1}\left\{-\frac{n-2}{\widetilde{\alpha}_i(b^*)-b^*} + \sum_{j\neq i}\frac{1}{\widetilde{\alpha}_j(b^*)-b^*}\right\}.$$

However, from (1), the R.H.S. of this last inequality is $\frac{d}{db}\widetilde{\alpha}_i(b^*)$ and, consequently,

$$\frac{d}{db}\widehat{\alpha}_i(b^*) > \frac{d}{db}\widetilde{\alpha}_i(b^*).$$

Since $\widehat{\alpha}_i$ and $\widetilde{\alpha}_i$ coincide at b^* and since, at the same point, the derivative of $\widehat{\alpha}_i$ is strictly larger than the derivative of $\widetilde{\alpha}_i$, the function $\widehat{\alpha}_i$ must be larger than $\widetilde{\alpha}_i$ to the right of b^*, that is, there must exist $\zeta > 0$ such that $\widehat{\alpha}_i(b) > \widetilde{\alpha}_i(b)$, for all b in $(b^*, b^*+\zeta)$. However, this contradicts the definition of b^* and we have proved that b^* cannot be strictly larger than c and is, thus, equal to c. The inequalities (12) hold true for all b in $(c, c+\gamma-\varepsilon]$ and we have proved Lemma 4. ||

Appendix 3

In this appendix, we prove that the equilibrium strategies satisfy (2"') and (3'). We denote by G_i the continuous from the right cumulative distribution function of bidder i's marginal bid distribution. For any distribution H, we denote its support by $SuppH$.

A strategy β_i of bidder i defines[20] conditional bid probability distributions $\beta_i(.|v)$, for all v in the support $[c_i, d_i]$ of F_i. A n-tuple $(\beta_1, ..., \beta_n)$ of strategies is a Bayesian Nash equilibrium if and only if $\beta_i(.|v)$ is a best reply against the other bidders' strategies, for all bidder i and for all valuation v in the support of F_i.

For all i and v such that $1 \leq i \leq n$ and $v \geq r$, we consider the function $(v-b) I\{b \geq r\} \prod_{j \neq i} G_j(b)$. The function $I\{b \geq r\}$ is the indicator function of the set $\{b \geq r\}$. It is thus equal to 0 if $b < r$ and to 1 if $b \geq r$. We also consider the following maximization problem $\max_{b \in R}(v-b) I\{b \geq r\} \prod_{j \neq i} G_j(b)$. This would be the maximization problem of bidder i with valuation v if bidder i won every tie in which he was involved. In this case,

$$\mathcal{B}_i(v) = \arg\max_{b \in R}(v-b) I\{b \geq r\} \prod_{j \neq i} G_j(b) \text{ (A3-1)}$$

would be bidder i's "best bid correspondence." This correspondence $\mathcal{B}_i$ is nonempty valued since the maximization problem has always at least one solution. In fact, $(v-b) I\{b \geq r\} \prod_{j \neq i} G_j(b)$ is nonpositive if $b \geq v$, is equal to 0 if $b \leq r$, and is strictly positive and upper semi-continuous over $[r, v)$. We denote by $b_{iu}(v)$ the supremum of $\mathcal{B}_i(v)$ and by $b_{il}(v)$ its infimum. From its definition and following standard lines (or directly by log-super modularity, see Milgrom and Shannon 1994 or Theorems 2.8.6 and 2.8.7 pp82-83 in Topkis 1998), it is simple to prove (for a direct proof, see Appendix 6 in Appendix 1, Lebrun 1999b) that the correspondence $\mathcal{B}_i$ is nondecreasing over $\left[\max(\min Supp \prod_{j \neq i} G_j, r), +\infty\right)$, that is, that $b_{iu}(v) \leq b_{il}(v')$, for all $\max(\min Supp \prod_{j \neq i} G_j, r) \leq v < v'$. Moreover, b_{iu} and b_{il} are nondecreasing over $\mathcal{R}$.

For a n-tuple $(\beta_1, ..., \beta_n)$ of strategies, we denote by $P_i(v)$ bidder i's (interim) expected payoff when his valuation is v, when he bids according to $\beta_i(.|v)$, and when the other bidders bid according to β_j, $j \neq i$. We also

[20] Technically, β_i is a regular conditional probability distribution or a stochastic kernel. See Lebrun 1999a and 1997.

denote by $P_i(v;b)$ and by $\Pr(i \text{ wins}|b)$, bidder i's expected payoff and probability of winning when his valuation is v, when he bids b, and when the other bidders bid according to β_j, $j \neq i$. Thus, $P_i(v;b) = (v-b)\Pr(i \text{ wins}|b)$ and the expected payoff $P_i(v)$ is the expectation of $P_i(v;b)$ when b is distributed according to $\beta_i(.|v)$.

For any bid $b \leq v$, bidder i with valuation v can obtain expected payoffs approaching the nonnegative payoff $(v-b)\, I\{b \geq r\} \prod_{j \neq i} G_j(b)$ by submitting bids approaching b from above. Consequently, at any equilibrium the correspondence $\mathcal{B}_i$ restricted to the support $[c_i, d_i]$ of F_i is actually the best bid correspondence of bidder i and the expected payoff $P_i(v)$ is equal to the value of the maximization problem in the definition (A3-1) of $\mathcal{B}_i(v)$, for all i and v such that $v \in [c_i, d_i]$. Moreover, $b_{iu}(v)$ is the supremum of the set of bids b that give bidder i the same expected payoff $P_i(v;b)$ as the distribution $\beta_i(.|v)$ does, that is, such that $P_i(v;b) = P_i(v)$, and $b_{il}(v)$ is the infimum of this set, for all i and v such that $v \in [c_i, d_i]$.

Let $\underline{v}$ be the maximum of the reserve price and the minimum of the support of the highest bid, that is, $\underline{v} = \max(\min Supp \prod_i G_i, r)$. It is straightforward to show that, at any equilibrium, no bidder bids a bid $b > \underline{v}$ with (ex-ante or marginal) strictly positive probability. Consequently, $\Pr(i \text{ wins}|b) = \prod_{j \neq i} G_j(b)$, for all i and $b > \underline{v}$. The probability $\Pr(i \text{ wins}|b)$ that a bidder i wins the auction with a bid b is continuous with respect to $b > \underline{v}$ and we have$P_i(v;b) = (v-b)\Pr(i \text{ wins}|b)$, for all i and $b > \underline{v}$, and $P_i(v) = (v - b_{iu}(v))\Pr(i \text{ wins}|b_{iu}(v))$, for all $v \in [c_i, d_i]$ such that $b_{iu}(v) > \underline{v}$.

Appendix 3.1

Lemma A3.1-1: *Let Assumptions A.1 be satisfied*[21]. *Assume also* $d_1, ..., d_n > r$. *Without loss of generality, assume* $c_1 \geq c_2 \geq c_i$, *for all* $i \geq 2$. *Let* $(\beta_1, ..., \beta_n)$ *be a Bayesian Nash equilibrium where bidders do not submit bids strictly larger than their valuations and let* $\underline{v}$ *be the minimum of the support of the maximum of* r *and the highest bid. Then, we have*

$$\underline{v} = \max \arg \max_{b \in [\max(r,c_2), \max(r,c_1)]} (c_1 - b) \prod_{i>1} F_i(b).$$

[21] Lemma A3.1-1 and its proof apply actually apply even if our only requirement on the valuation distributions is that their supports be compacts.

Proof: For all $1 \leq i \leq n$, let m_i be the maximum of c_i and r, that is, $m_i = \max(c_i, r)$ and let g_i be the minimum of the support of bidder i's marginal bid distribution G_i. Then, $\underline{v} = \max(\max_i g_i, r) = \max_i (g_i, r)$. Since bidder i does not bid strictly above his valuation, we have $g_i \leq c_i$ and, thus, $\max(g_i, r) \leq m_i$, for all i. Consequently, $\underline{v} \leq m_1 = \max_i m_i$.

Let j be such that $\underline{v} = \max(g_j, r)$. Since bidder j bids at most his valuation, we have $\underline{v} \leq m_j$. Suppose that there exists $i \neq j$ such that $\underline{v} < m_i$. If $\max(g_i, r) < \underline{v}$, we would have $\max(g_i, r) < \underline{v} < m_i = c_i$. However, the best response from bidder i with valuation $v_i \geq c_i$ cannot be strictly smaller than $\underline{v}$ since his payoff would be zero, while he could obtain a strictly positive payoff by submitting $(\underline{v} + c_i)/2$ instead. Consequently, $\max(g_i, r) < \underline{v}$ is impossible and we have $\underline{v} \leq \max(g_i, r)$. Since $\underline{v} < \max(g_i, r)$ would contradict the definition of $\underline{v}$, we must have $\underline{v} = \max(g_i, r)$. We have proved that $\underline{v} = \max(g_i, r)$, for all $i \neq j$ such that $\underline{v} < m_i$. From the definition of j, we therefore have $\underline{v} = \max(g_i, r)$, for all i such that $\underline{v} < m_i$.

Suppose that there exist i and k such that $\underline{v} < m_i$ and $\underline{v} < m_k$. In particular, we have $c_i = m_i$, $c_k = m_k > r$. Bidder i with valuation $v_i \geq c_i$ will not submit bids strictly below r, since such bids would bring him a zero payoff, while he can obtain strictly positive payoffs. Consequently, $g_i \geq r$. Similarly, $g_k \geq r$. From $m_i, m_k > \underline{v}$ and the result of the previous paragraph, we would have $\underline{v} = \max(g_i, r) = \max(g_k, r)$ and, thus, $\underline{v} = g_i = g_k$. At most one bidder among the bidders i and k can bid $\underline{v}$ with strictly positive marginal probability[22]. Without loss of generality, assume bidder k bids $\underline{v}$ with probability zero. Then bidder k submits bids strictly larger than $g_k = \underline{v}$ with probability one. Since $g_i = \underline{v}$, for all $\varepsilon > 0$ there exists a Borel set of strictly positive F_i-measure such that bidder i bids within $[\underline{v}, \underline{v} + \varepsilon)$ with a strictly positive probability for all valuation v_i in this set. However, as ε tends towards zero the probability of winning tends towards zero and so does the expected payoff for such bids, while by bidding instead $(v_i + \underline{v})/2$, for all such valuation v_i, bidder i can obtain a fixed strictly positive expected payoff. Consequently, there do not exist two such values i and k of the index and we must have $m_2 \leq \underline{v} \leq m_1$.

If $m_2 = m_1$, the lemma is immediate. For the rest of the proof, we will thus assume that $m_2 < m_1$. As a particular consequence, we have $r < m_1 = c_1$. From the definition of $\underline{v}$, if $\max(g_1, r) < \underline{v}$ then there exists

[22] Otherwise, it would be in the best interest of one of them to submit slightly larger bids.

$k > 1$ such that $\max(g_k, r) = \underline{v}$ and, since $\underline{v} > r$ in this case, $g_k = \underline{v}$. Since bidders do not bid strictly above their valuations, we have $\underline{v} \leq c_k = m_k$. However, $m_2 \leq \underline{v} \leq m_1$ and, thus, $\underline{v} = m_2$. The assumption $m_2 < m_1$ then implies $\underline{v} < m_1$. This last inequality and $\max(g_1, r) < \underline{v}$ contradict the result of the second paragraph of this proof and, thus, $\max(g_1, r) < \underline{v}$ is impossible. Consequently, $\underline{v} = \max(g_1, r)$. Since $\underline{v} \leq m_1 = c_1$, bidder 1 with valuation $v_1 > c_1$ does not bid strictly below r since it would give him a zero payoff, while he can obtain strictly positive payoff. Consequently, $g_1 \geq r$ and $\underline{v} = g_1$.

Since $\underline{v} = g_1$ belongs to the support of bidder 1's bid distribution and from the monotonicity of the best bid correspondence already mentioned above, there exists a sequence $(v_n, b_n)_{n \leq 1}$ such that b_n belongs to the support of the bidder 1's bid distribution $\beta_1(.|v_n)$ conditional on v_n , such that v_n tends towards m_1 and b_n tends towards $\underline{v}$ from above as n tends towards $+\infty$, and such that b_n is a best response from bidder 1 with valuation v_n, that is, $(v_n - b_n) \prod_{j \neq 1} G_j(b_n) \geq (v_n - b) \prod_{j \neq 1} G_j(b)$, for all $b \geq \max_{j \neq 1}(g_j, r)$. By making n tend towards $+\infty$ in the previous inequality, we find (A3.1-1):

$$(m_1 - \underline{v}) \prod_{j \neq 1} G_j(\underline{v}) \geq (m_1 - x) \prod_{j \neq 1} G_j(x), \text{ (A3.1-1)}$$

for all $x \geq \max_{j \neq 1}(g_j, r)$ and in particular for all x in[23] $[m_2, m_1]$.

The inequality (A3.1-1) already allows us to rule out the case $\underline{v} = m_1$. In fact, if $\underline{v} = m_1$ the L.H.S. of (A3.1-1) is equal to zero while the R.H.S. is strictly positive for $m_2 < x < m_1$, since $g_j \leq m_2$, for all $j \neq 1$. We thus have $\underline{v} < m_1$. Since bidders do not bid strictly higher than their valuations, we have $G_j(x) \geq F_j(x)$, for all x, and, thus:

$$(m_1 - \underline{v}) \prod_{j \neq 1} G_j(\underline{v}) \geq (m_1 - x) \prod_{j \neq 1} F_j(x), \text{(A3.1-2)}$$

for all x in $[m_2, m_1]$.

In this paragraph, we show that $G_j(\underline{v}) = F_j(\underline{v})$, for all $j \neq 1$. Since $g_1 = \underline{v}$, bids $b < \underline{v}$ from bidder $j \neq 1$ have a zero probability of winning (either $\underline{v} = r$ and b is strictly smaller than the reserve price or $\underline{v} = g_1 > r$ and b is strictly smaller than the bid from bidder 1 with probability one). Consequently, bidder $j \neq 1$ with valuation $v_j > \underline{v}$ will not submit a bid strictly smaller than

[23] $\max_{j \neq 1}(g_j, r) \leq \max_{j \neq 1}(c_j, r) = m_2$.

$\underline{v}$. Since bidder j does not bid strictly above his valuation, bidder j with valuation $v_j \leq \underline{v}$ will only submit bids $b \leq \underline{v}$. The only way $G_j(\underline{v})$ may thus be different from $F_j(\underline{v})$ is if, with strictly positive probability, bidder j with valuation $v_j > \underline{v}$ bids $\underline{v}$. If this is the case the bid $\underline{v}$ from bidder $j \neq 1$ must have a strictly positive probability of winning and thus $G_1(\underline{v}) = G_1(g_1) > 0$ and $\underline{v}$ must be a mass point of bidder 1's bid distribution. It is then in the best interest of bidder j with valuation $v_j > \underline{v}$ to bid slightly above $\underline{v}$ rather than at $\underline{v}$. This is impossible at an equilibrium and we find $G_j(\underline{v}) = F_j(\underline{v})$, for all $j \neq i$. Then (A3.1-2) implies (A3.1-3) below:

$$(m_1 - \underline{v}) \prod_{j \neq 1} F_j(\underline{v}) \geq (m_1 - x) \prod_{j \neq 1} F_j(x), \text{ (A3.1-3)}$$

for all x in $[m_2, m_1]$, and $\underline{v}$ belongs to $\max_{b \in [m_2, m_1]} (m_1 - x) \prod_{i>1} F_i(x)$.

Suppose next that there exists $x' > \underline{v}$ in $[m_2, m_1]$ such that $(m_1 - \underline{v}) \prod_{j \neq 1} F_j(\underline{v}) = (m_1 - x') \prod_{j \neq 1} F_j(x')$. From (A3.1-1) and $G_j(x) \geq F_j(x)$, for all x and j, we have

$$(m_1 - \underline{v}) \prod_{j \neq 1} F_j(\underline{v}) \geq (m_1 - x') \prod_{j \neq 1} G_j(x') \geq (m_1 - x') \prod_{j \neq 1} F_j(x').$$

Consequently, we have

$$(m_1 - \underline{v}) \prod_{j \neq 1} F_j(\underline{v}) = (m_1 - x') \prod_{j \neq 1} G_j(x') = (m_1 - x') \prod_{j \neq 1} F_j(x') \text{ (A3.1-4).}$$

From the second equation in (A3.1-4) and the strict positivity of the value of the maximization problem $\max_{x \in [m_2, m_1]} (m_1 - x) \prod_{i>1} F_i(x)$ (when $m_2 < m_1$), we obtain $\prod_{j \neq 1} G_j(x') = \prod_{j \neq 1} F_j(x') > 0$. Since $G_j(x') \geq F_j(x')$, for all j, we find $G_j(x') = F_j(x')$, for all $j \neq 1$. For all j and $x' > \underline{v}$, $b_{ju}(x') < x'$ (if $b_{ju}(x') \geq x'$, $b_{ju}(x') > \underline{v}$ and thus $\max_{b \in R} (x' - b) I\{b \geq r\} \prod_{k \neq j} G_k(b) = P_j(x'; b_{ju}(x')) \leq 0$, which is impossible since bidder j can obtain a strictly positive payoff by bidding strictly between $\underline{v}$ and x'). From the monotonicity of $\mathcal{B}_j$, bidder j with valuation $v_j \leq x'$ bids at most $b_{ju}(x')$. We thus have $F_j(x') \leq G_j(b_{ju}(x')) \leq G_j(x') = F_j(x')$, for all $j \neq 1$, and $G_j((b_{ju}(x'), x']) = 0$, for all $j \neq 1$. However, it is simple to show (for example, as in Lebrun 1999a or 1997), that at least two bidders bid with a strictly positive probability in the neighborhood of every bid between $\underline{v}$ and the maximum η of the support of the highest bid ($\eta = \max_{1 \leq i \leq n} SuppG_i$). Consequently, x' must be strictly larger than η. Take x'' such that $\eta < x'' < x'$. Then, from (A3.1-4) we have $(m_1 - \underline{v}) \prod_{j \neq 1} F_j(\underline{v}) < (m_1 - x'') \prod_{j \neq 1} G_j(x'') = (m_1 - x'')$. This inequality contradicts (A3.1-1) and the equality $(m_1 - \underline{v}) \prod_{j \neq 1} G_j(\underline{v}) = (m_1 - \underline{v}) \prod_{j \neq 1} F_j(\underline{v})$. There must thus not exist x' as in the top of this paragraph. We have proved that $\underline{v}$ is equal to max arg $\max_{b \in [\max(r,c_2), \max(r,c_1)]} (\max(r, c_1) - b) \prod_{i>1} F_i(b)$

The lemma then follows. $||$

Appendix 3.2

Once $\underline{v}$ is determined according to Lemma A3.1-1, in order to study the equilibrium strategies above $\underline{v}$ we can assume that $d_i > \underline{v}$, for all i, that is, that $J = \{1, ..., n\}$, where J is as defined in Definition 2 (Section 5). If d_i was not strictly larger than $\underline{v}$, then bidder i would never submit bids strictly above $\underline{v}$ at the equilibrium. The other bidder's "serious bidding behaviors," that is, their bids above $\underline{v}$ would not be affected if we simply dropped bidder i from the list of bidders. Since, in this subappendix, we will work more with the upper extremities $d_1, ..., d_n$ than with the lower extremities $c_1, ..., c_n$, we drop our assumption $c_1 \geq c_2 \geq c_i$, for all $i \geq 2$, and we assume rather that d_i is nonincreasing in i. Thus, we have $\underline{v} < d_n \leq d_{n-1} \leq ... \leq d_2 \leq d_1$, with $\underline{v}$ as in Definition 1 (Section 5) where c_1 and c_2 have been replaced by, respectively, the largest lower extremity $c_{(1)}$ and the second largest lower extremity $c_{(2)}$. We thus make the following assumptions.

Assumptions A.2:

$\underline{v} < d_n \leq d_{n-1} \leq ... \leq d_2 \leq d_1$, where $\underline{v}$ is as defined in Definition 1 (Section 5).

It can easily be shown that the distribution of the highest acceptable bid is not degenerate, that is, is not concentrated at $\underline{v}$. With some strictly positive probability, strictly higher bids are submitted.

Lemma A3.2-1: *Let Assumptions A.1 and A.2 be satisfied. Let $(\beta_1, ..., \beta_n)$ be a Bayesian Nash equilibrium where bidders bid at most their valuations and let η be the maximum of $b_{1u}(d_1)$,...,$b_{nu}(d_n)$. If $b_{iu}(d_i) = \eta$ and $d_j \geq d_i$, then $b_{ju}(d_j) = \eta$, for all i, j.*

Proof: Since, as we previously observed, the highest submitted bid is strictly larger than $\underline{v}$ with strictly positive probability, we must have $\eta > \underline{v}$.

Let i and j be such that $b_{iu}(d_i) = \eta$, $d_j \geq d_i$, and $j \neq i$. Since η wins with a strictly positive probability (equal to 1), we must have $d_i > \eta$ and, thus, $d_j > \eta$. Since $\eta > \underline{v}$, $b_{iu}(d_i) = \eta \in \mathcal{B}_i(d_i)$. There exists m such that:

$$P_j(d_j; m) = P_j(d_j).$$

Then, $m \in \mathcal{B}_j(d_j)$ and $b_{jl}(d_j) \leq m \leq b_{ju}(d_j)$. Moreover, m cannot be strictly smaller than $\underline{v}$. Otherwise, from the monotonicity of $\mathcal{B}_j(d_j)$ and from $F_j(\{d_j\}) = 0$, $\underline{v}$ would not belong to the support of G_j, which would contradict the definition of $\underline{v}$. Thus, $m \geq \underline{v}$. From the definitions of $\mathcal{B}_i(d_i)$ and η and from $G_j(m) = 1$, we find:

$$d_i - \eta = (d_i - \eta) \prod_{k \neq i} G_k(\eta) \geq (d_i - m) \prod_{k \neq i} G_k(m) = (d_i - m) \prod_{k \neq i,j} G_k(m).$$

An immediate consequence of the previous inequality and of the inequality $d_j \geq d_i$ is (A3.2-1) below:

$$d_j - \eta \geq (d_j - m) \prod_{k \neq i,j} G_k(m) \text{ (A3.2-1).}$$

From the continuity from the right of $\prod_{k \neq j} G_k$ and the definition of m, we have $(d_j - m) \prod_{k \neq j} G_k(m) = (d_j - m) G_i(m) \prod_{k \neq i,j} G_k(m) \geq P_j(d_j; m) = P_j(d_j)$. (A3.2-1) thus implies:

$$P_j(d_j; \eta) \geq P_j(d_j).$$

Thus, $P_j(d_j; \eta) = P_j(d_j)$, η is an optimal bid for bidder j with valuation d_j, and $\eta \leq b_{ju}(d_j)$. From the definition of η in the statement of the lemma, $\eta \geq b_{ju}(d_j)$ and, consequently, $\eta = b_{ju}(d_j)$. ||

Let η be the maximum of the support of the highest bid. Since no bid strictly larger than η can be an optimal bid, we have $b_{iu}(d_i) \leq \eta$, for all i. From the definition of η, it must belong to the support of at least one bid distribution and must thus be among the optimal bids of at least one bidder. From the monotonicity of b_{iu}, there must exist a bidder i such that $b_{iu}(d_i) = \eta$. If there was only one such bidder, all the other bidders would bid strictly below η with probability one, and η would not be an optimal bid for bidder i since a slightly smaller bid would give him a strictly higher expected payoff. There must thus exist at least two bidders i and j, $i \neq j$, such that $b_{iu}(d_i) = b_{ju}(d_j) = \eta$. From the previous lemma, for any other bidder k such that $d_k \geq d_i$ or $d_k \geq d_j$, we have $b_{ku}(d_k) = \eta$. In particular, $b_{ku}(d_k) = \eta$, for all $k = 1, 2$ and, thus, $\eta < d_2$.

As in Lebrun (1999a), we can show that the functions b_{iu} and b_{il} are identical between them, continuous, and strictly increasing where, over $[c_i, d_i]$, their values are strictly larger than $\underline{v}$. The equilibrium strategies thus never mix between several bids strictly larger than $\underline{v}$. At an equilibrium, a bidder i with valuation v bids $\beta_i(v) = b_{iu}(v) = b_{il}(v)$, for all v in $[c_i, d_i]$ such that $b_{iu}(v) > \underline{v}$.

If $r \geq c_1$, then $\underline{v}$ as defined in Lemma A3.1-1 is equal to r and, as in the common-support case, (2) and, thus, (2"') hold true. From the proof of Lemma A3.1-1, if $r < c_1$ then $g_1 = \underline{v}$, $\underline{v}$ is the minimum serious bid bidder 1 submits, and bidder 1 bids at least $\underline{v}$ with probability one, with $\underline{v}$ as in Lemma A3.1-1 or as Definition 1 (Section 5). Bidder j with valuation v_j bids at least $\underline{v}$ and has a strictly positive payoff instead of the zero payoff he would obtain by submitting a bid strictly smaller than $\underline{v}$, for all $j > 1$ and all $v_j > \underline{v}$. Thus, the minimum of the serious bids every bidder submits is $\underline{v}$. Of course, no two bidders can submit $\underline{v}$ with a (ex-ante or marginal) strictly positive probability, since otherwise there would be a strictly positive probability of a tie at $\underline{v}$ and it would be in the best interest of either of these two bidders to bid slightly higher. (2"') follows.

Furthermore, as in Lebrun (1999a), the inverses $\alpha_1, ..., \alpha_n$ of these bid functions can be shown to form a solution of the system (1) with initial condition (A3.2-2) below:

$$\begin{aligned} \alpha_i(\eta) &= d_i, \text{ for all } i \text{ such that } d_i \geq d_2, \\ \alpha_j(\eta_j) &= d_j, \text{ for all } j \text{ such that } d_j < d_2 \text{(A3.2-2)}. \end{aligned}$$

In (A3.2-2), η is the maximum of the support of the highest bid and the first part of (A3.2-2) follows from Lemma A3.2-1. η_j is the maximum of the support of bidder j's bid. The following lemma then follows.

Lemma A3.2-2: *Let Assumptions A.1 and A.2 be satisfied. Assume also $F_i(c_i) = 0$, for all i. Let $(\beta_1, ..., \beta_n)$ be a Bayesian Nash equilibrium where bidders bid at most their valuations. Then, the strategy β_i is a nondecreasing bid function not smaller than $\underline{v}$ over $(\underline{v}, d_i]$ and strictly increasing and differentiable when its value is strictly larger than $\underline{v}$, for all i. Moreover, there exists η in $(\underline{v}, d_2)$ and η_j in $(\underline{v}, d_j)$, for all j such that*

$d_j < d_2$, such that the inverse bid functions $\alpha_1 = \beta_1^{-1}, ..., \alpha_n = \beta_n^{-1}$ are solutions of the system (1) of differential equations–considered over the domain $\widetilde{D} = \{(b, \alpha_1, ..., \alpha_n) \,|\underline{v}, b < \alpha_i \leq d_i$, for all $1 \leq i \leq n\}$–with boundary conditions (2"') (in C.4 Section 5, with $\underline{v}$ defined in Definition 1) and (A3.2-2).

We next show that if bidder i submits η at his upper extremity of this support, that is, $\eta_i = \eta$, then d_i is not smaller than $d(\eta)$ defined in Definition 3 (Section 5).

Lemma A3.2-3: *Let Assumptions A.1 and A.2 be satisfied. Assume also $F_i(c_i) = 0$, for all i. Let η and ε be such that $\eta < d_i$, for all $1 \leq i \leq k$, and $\varepsilon > 0$. Let $(\alpha_1, ..., \alpha_k)$ be a strictly increasing solution over $[\eta - \varepsilon, \eta]$ with values in $\prod_{i=1}^{k}(c_i, d_i]$ of the differential system (A3.2-3) and initial condition (A3.2-4) below:*

$$\frac{d}{db}\ln F_i(\alpha_i(b)) = \frac{1}{k-1}\left\{-\frac{k-2}{\alpha_i(b)-b} + \sum_{k\neq i}\frac{1}{\alpha_j(b)-b}\right\}, \text{ for all } 1 \leq i \leq k, \text{ (A3.2-3)}$$

$$\alpha_i(\eta) = d_i, \text{ for all } 1 \leq i \leq k, \text{ (A3.2-4).}$$

Then, the following inequalities hold true:

$$d_i \geq d(k, \eta), \; 1 \leq i \leq k,$$

where

$$d(k, \eta) = \eta + \frac{k-1}{\sum_{i=1}^{k}\frac{1}{d_i - \eta}}$$

or, equivalently,

$$\frac{1}{d(k,\eta)-\eta} = \frac{1}{k-1}\sum_{i=1}^{k}\frac{1}{d_i-\eta} \text{ (A3.2-5).}$$

Proof: It suffices to prove

$$d_k \geq d(k, \eta) \;\; \text{(A3.2-6)}$$

Since $\ln F_i \alpha_i$ is strictly increasing over $(\eta - \varepsilon, \eta]$, we have $\frac{d}{db} \ln F_k (\alpha_k (\eta)) \geq 0$. From (A3.2-3) and (A3.2-4), we thus have

$$\frac{k-2}{d_k - \eta} \leq \sum_{j=1}^{k-1} \frac{1}{d_j - \eta}.$$

Adding $\frac{1}{d_k - \eta}$ to both sides of the previous inequality and dividing by $(k-1)$, we find:

$$\frac{1}{d_k - \eta} \leq \frac{1}{k-1} \sum_{j=1}^{k} \frac{1}{d_j - \eta}.$$

From the definition of $d(k, \eta)$, the R.H.S. of the last inequality is equal to $\frac{1}{d(k,\eta) - \eta}$. (A3.2-6) follows immediately. $||$

In the following proofs, the following notations will prove convenient.

Definition A3.2-1:

(i) For all integers n and k, we denote by $E_{n,k}$ the $n \times k$ matrix whose all components are equal to 1, that is:

$$E_{n,k} = i_{n,1} i'_{n,1} = \begin{pmatrix} 1 & 1 & ... & 1 \\ 1 & 1 & 1 & 1 \\ ... & ... & ... & ... \\ 1 & ... & 1 & 1 \end{pmatrix},$$

where $i_{n,1}$ is the column vector whose all n components are equal to 1.

(i) For all integer n, we denote the square $n \times n$ matrix $E_{n,n}$ simply by E_n, the square $n \times n$ identity matrix by I_n, and the square $n \times n$ matrix $E_n - I_n$ by K_n. We have:

$$E_n = E_{n,n}, I_n = \begin{pmatrix} 1 & 0 & ... & 0 \\ 0 & 1 & 0 & 0 \\ ... & ... & ... & ... \\ 0 & ... & 0 & 1 \end{pmatrix}, K_n = E_n - I_n = \begin{pmatrix} 0 & 1 & ... & 1 \\ 1 & 0 & 1 & 1 \\ ... & ... & ... & ... \\ 1 & ... & 1 & 0 \end{pmatrix}.$$

The matrix K_n above is, thus, the $n \times n$ matrix whose all components on the main diagonal are equal to 0 and all off-diagonal components are equal to 1. The technical lemma below follows easily from the equality $K_n^2 = (n-1) I_n + (n-2) K_n$.

Lemma A3.2-4: *For all integer $n \geq 1$ and for all real number x, the matrix $xI_n + K_n$, where I_n and K_n are as in Definition A3.2-1, is regular (invertible) if and only if $x \neq 1$ and $x \neq -(n-1)$, in which case we have:*

$$(xI_n + K_n)^{-1} = \frac{1}{(x-1)(x+n-1)} \{(x+n-2) I_n - K_n\}.$$

Lemma A3.2-5: *Let Assumptions A.1 and A.2 be satisfied. Assume also $F_i(c_i) = 0$, for all i. Let $(\beta_1, ..., \beta_n)$ be a Bayesian Nash equilibrium where bidders bid at most their valuations. Then, the strategies are pure for the valuations strictly larger than $\underline{v}$ and there exist η in $\left(\underline{v}, d_{(2)}\right)$ and $\eta_i < d_i, \eta$, for all i such that $d_i < d(\eta)$, such that $(\alpha_1, ..., \alpha_n)$ is a solution of the system (1) and the initial condition (A3.2-7) below:*

$$\begin{aligned} \alpha_i(\eta) &= d_i, \textit{ for all } i \textit{ such that } d_i \geq d(\eta), \textit{ or, equivalently, such that } i \leq k(\eta) \\ \alpha_i(\eta_i) &= d_i, \textit{ for all } i \textit{ such that } d_i < d(\eta), \textit{ or, equivalently, such that } i \geq k(\eta)+1. \quad \textit{(A3.2-7)} \end{aligned}$$

Proof: We know that the equilibrium is pure for the valuations strictly larger than $\underline{v}$ and that there exists $\eta_i < d_i$, for all $1 \leq i \leq n$, such that the inverse bid functions $\alpha_1, ..., \alpha_n$ are solutions of the system (1) and initial conditions below:

$$\alpha_i(\eta_i) = d_i,\ 1 \leq i \leq n.$$

Let η be equal to the maximum of $\eta_1, ..., \eta_n$. From (A3.2-2), $\eta_i = \eta < d_2$, for all i such that $d_i \geq d_2$. From Lemma A3.2-1, for all $1 \leq i \leq n$, if $\eta_i = \eta$, then $\eta_j = \eta$, for all j such that $j \leq i$. There thus exists $k \geq 2$ such that $\eta_i = \eta$, for all i such that $i \leq k$, and $\eta_i < \eta$, for all i such that $i > k$. From Lemma A3.2-3, $d_k \geq d(k, \eta)$. Consequently, there exists k such that $(\alpha_1, ..., \alpha_n)$ satisfies the initial conditions below:

$$\begin{aligned} \alpha_i(\eta) &= d_i, \text{ for all } i \text{ such that } 1 \leq i \leq k, \\ \alpha_i(\eta_i) &= d_i, \text{ for all } i \text{ such that } k+1 \leq i \leq n, \end{aligned}$$

with $d_k \geq d(k, \eta)$ and $d_i < d_k$ and $\eta_i < \eta$, for all i such that $i \geq k+1$.

Let η' be the maximum of the η_i, for i such that $i \geq k+1$. Only the bidders $1, ..., k$ bid in $(\eta', \eta]$ and over this interval $(\alpha_i)_{1 \leq i \leq k}$ is a solution of the system below:

$$\frac{d}{db} \ln F_i (\alpha_i (b)) = \frac{1}{k-1} \left\{ -\frac{k-2}{\alpha_i (b) - b} + \sum_{k \neq i} \frac{1}{\alpha_j (b) - b} \right\}, \text{ for all } 1 \leq i \leq k. \text{(A3.2-8)}$$

For all $i \geq k+1$, define over $(\eta', \eta]$ the function α_i as the same unique solution of the equation below:

$$\frac{1}{\alpha_i (b) - b} = \frac{1}{k-1} \sum_{j=1}^{k} \frac{1}{\alpha_j (b) - b}. \text{(A3.2-9)}$$

As we have defined these functions, we have $\alpha_{k+1} = ... = \alpha_n$, over $(\eta', \eta]$, and $\alpha_{k+1} (\eta) = ... = \alpha_n \ (\eta) = d (k, \eta)$. (A3.2-9) can be rewritten in matrix form as (A3.2-10) below

$$\left(\frac{1}{\alpha_i (b) - b} \right)_{k+1 \leq i \leq n} = \frac{1}{k-1} E_{n-k,k} \left(\frac{1}{\alpha_j (b) - b} \right)_{1 \leq j \leq k}, \text{ (A3.2-10)}$$

where $E_{n-k,k}$ is as defined in Definition A3.2-1. As a simple computation shows, the equality below holds true:

$$\frac{1}{k-1} E_{n-k,k} = \frac{1}{(k-1)(n-1)} (k I_{n-k} + K_{n-k}) E_{n-k,k}, \text{ (A3.2-11)}$$

where I_{n-k} and K_{n-k} are as in Definition 3.2-1. Substituting its value from (A3.2-11) to $\frac{1}{k-1} E_{n-k,k}$ in (A3.2-10), we find that $\left(\frac{1}{\alpha_i (b) - b} \right)_{k+1 \leq i \leq n}$ is equal to $\frac{1}{(k-1)(n-1)} (k I_{n-k} + K_{n-k}) E_{n-k,k} \left(\frac{1}{\alpha_j (b) - b} \right)_{1 \leq j \leq k}$ and thus

$$\begin{aligned} & -(k-1)(n-1)(k I_{n-k} + K_{n-k})^{-1} \left(\frac{1}{\alpha_i (b) - b} \right)_{k+1 \leq i \leq n} + E_{n-k,k} \left(\frac{1}{\alpha_j (b) - b} \right)_{1 \leq j \leq k} \\ = & \ (0)_{k+1 \leq i \leq n}. \end{aligned}$$

Since, from Lemma A3.2-4, $(k-1)(n-1)(k I_{n-k} + K_{n-k})^{-1} = (n-2) I_{n-k} - K_{n-k}$, we have

$$\begin{aligned} & -((n-2) I_{n-k} - K_{n-k}) \left(\frac{1}{\alpha_i (b) - b} \right)_{k+1 \leq i \leq n} + E_{n-k,k} \left(\frac{1}{\alpha_j (b) - b} \right)_{1 \leq j \leq k} \\ = & \ (0)_{k+1 \leq i \leq n}, \end{aligned}$$

or, equivalently,

$$\frac{1}{n-1}\left\{-\frac{n-2}{\alpha_i(b)-b}+\sum_{j\neq i}\frac{1}{\alpha_j(b)-b}\right\}=0, \text{ (A3.2-12)}$$

for all $k+1 \leq i \leq n$. Since $F_i(\alpha_i(b)) = 0$, for all b in $(\eta', \eta]$ and all $k+1 \leq i \leq n$, we have $\frac{d}{db}\ln F_i(\alpha_i(b)) = 0$, for all b in $(\eta', \eta]$ and all $k+1 \leq i \leq n$. From (A3.2-12), the equations in (1) for $k+1 \leq i \leq n$ thus hold true over $(\eta', \eta]$.

The system (A3.2-8) can be rewritten under matrix form as (A3.2-13) below:

$$\left(\frac{d}{db}\ln F_i(\alpha_i(b))\right)_{1\leq i\leq k}=\frac{1}{k-1}C\left(\frac{1}{\alpha_j(b)-b}\right)_{1\leq j\leq k}, \text{ (A3.2-13)}$$

where $C = -(k-2)I_k + K_k$, with I_k and K_k as in Definition A3.2-1. A simple computation shows that the following equality between matrices holds true:

$$\frac{1}{k-1}C=\frac{1}{n-1}\left\{\frac{n-k}{k-1}E_k-(n-2)I_k+K_k\right\}, \text{ (A3.2-14)}$$

where E_k is as defined in Definition A3.2-1. From (A3.2-13) and (A3.2-14), we find:

$$\left(\frac{d}{db}\ln F_i(\alpha_i(b))\right)_{1\leq i\leq k}=\frac{1}{n-1}\left\{\frac{n-k}{k-1}E_k\left(\frac{1}{\alpha_j(b)-b}\right)_{1\leq j\leq k}+B\left(\frac{1}{\alpha_j(b)-b}\right)_{1\leq j\leq k}\right\}, \text{(A3.2-15)}$$

where $B = -(n-2)I_k + K_k$. Multiplying both sides of (A3.2-10) to the left by the transpose $E_{k,n-k}$ of $E_{n-k,k}$ and making use of the immediate equality $E_{k,n-k}.E_{n-k,k} = (n-k)E_k$, we obtain

$$E_{k,n-k}\left(\frac{1}{\alpha_i(b)-b}\right)_{k+1\leq i\leq n}=\frac{n-k}{k-1}E_k\left(\frac{1}{\alpha_j(b)-b}\right)_{1\leq j\leq k} \text{ (A3.2-16).}$$

Substituting its value from (A3.2-16) to $\frac{n-k}{k-1}E_k\left(\frac{1}{\alpha_j(b)-b}\right)_{1\leq j\leq k}$ in (A3.2-15), we find that $\left(\frac{d}{db}\ln F_i(\alpha_i(b))\right)_{1\leq i\leq k}$ is equal to $\frac{1}{n-1}\left\{E_{k,n-k}\left(\frac{1}{\alpha_i(b)-b}\right)_{k+1\leq i\leq n}+B\left(\frac{1}{\alpha_j(b)-b}\right)_{1\leq j\leq k}\right\}$ or, equivalently,

$$\frac{d}{db}\ln F_i(\alpha_i(b))=\frac{1}{n-1}\left\{-\frac{n-2}{\alpha_i(b)-b}+\sum_{j\neq i}\frac{1}{\alpha_j(b)-b}\right\},$$

for all b in $(\eta', \eta]$ and all $1 \leq i \leq k$. All equations in (1) thus hold true over $(\eta', \eta]$ and $(\alpha_1, ..., \alpha_n)$ is a solution over $(\eta', \eta]$ of the system (1). Moreover, from Lemma A5.1-4, $\frac{d}{db}\alpha_{k+1}(b) = ... = \frac{d}{db}\alpha_n(b) > 0$, for all b in (η', η).

Let i be not smaller than $k+1$ and let v be in $[d_i, d_1]$. If bidder i with valuation v submits b in $[\eta', \eta]$, his expected payoff is equal to $(v-b)\prod_{j=1}^{k} F_j(\alpha_j(b))$. The logarithmic derivative of this expected payoff is equal to $\frac{-1}{v-b}+\sum_{j=1}^{k}\frac{d}{db}\ln(F_j(\alpha_j(b)))$. However, by adding all the equations in (A3.2-8), we see that $\sum_{j=1}^{k}\frac{d}{db}\ln(F_j(\alpha_j(b)))$ is equal to $\frac{1}{k-1}\sum_{j=1}^{k}\frac{1}{\alpha_j(b)-b}$. Consequently, from the definition (A3.2-9), the logarithmic derivative of the expected payoff satisfies the equation below:

$$\frac{d}{db}\ln\left\{(v-b)\prod_{j=1}^{k} F_j(\alpha_j(b))\right\} = \frac{-1}{v-b} + \frac{1}{\alpha_j(b)-b} \text{ (A3.2-17).}$$

Assume that $d_i \geq d(k, \eta)$ with $i \geq k+1$. Then, since $\alpha_i(\eta) = d(k, \eta)$ and α_i is strictly increasing, (A3.2-17) implies that the derivative $\frac{d}{db}\ln\left\{(d_i - b)\prod_{i=1}^{k} F_i(\alpha_i(b))\right\}$ is strictly positive, for all b in $[\eta_i, \eta)$ and $v = d_i$. Since η gives a strictly higher expected payoff, $\eta_i < \eta$ is not a best reply from bidder i with valuation d_i, as it should at an equilibrium. This is impossible and, thus, $d_i < d(k, \eta)$, for all $i \geq k+1$. Since $d_{k+1} < d(k, \eta) \leq d_k$, we must have $k = k(\eta)$, $d_i < d(k(\eta), \eta)$, for all $i \geq k+1$, and $d_i \geq d(k(\eta), \eta)$, for all $i \leq k$. Since $d(\eta) = d(k(\eta), \eta)$, the lemma is proved. ||

Lemma A3.2-6: *Let Assumptions A.1 and A.2 be satisfied. Assume also $F_i(c_i) = 0$, for all i. Let $(\beta_1, ..., \beta_n)$ be an equilibrium. Then, the bid functions can be extended over larger intervals as best reply functions such that their inverses form a solution of the system (1) with initial condition (3') in C.4 (Section 5) for $J = \{1, ..., n\}$ and for a certain η in $(\underline{v}, d_{(2)})$.*

Proof: Let η' be the maximum of η_i, for $i > k(\eta)$. For all $i \geq k(\eta)+1$, we define α_i over $(\eta', \eta]$ as in (A3.2-9). From (A3.2-9), we have $\alpha_i(\eta) = d(\eta)$, for all $i \geq k(\eta)+1$. As in the proof of the previous lemma, we can show, from (A3.2-8) and (A3.2-9), that $(\alpha_1, ..., \alpha_n)$ is a solution of (1) over $(\eta', \eta]$. We now show that, if α_i is so defined, then $\lim_{b \rightarrow > \eta_i}\alpha_i(b) = d_i$, for all $i \geq k(\eta)+1$ such that $\eta_i = \eta'$. Let i be such an index. Since η_i must be a best response from bidder i with valuation d_i, the (right-hand) derivative (A3.2-17) at $v = d_i$ and $b = \eta_i$ must not be strictly positive. Thus, we find $\lim_{b \rightarrow > \eta_i}\alpha_i(b) \geq d_i$.

Let j be the largest value of the index such that $j \geq k(\eta)+1$ and $\eta_j = \eta'$. Over an interval $[\eta' - \varepsilon, \eta']$, with $\varepsilon > 0$, the inverses of the bid functions $(\beta_l)_{l \text{ s.t.} \eta_l \geq \eta'}$ satisfy the system below, similar to (A3.2-8):

$$\frac{d}{db} \ln F_i(\alpha_i(b)) = \frac{1}{s-1} \left\{ -\frac{s-2}{\alpha_i(b) - b} + \sum_{\substack{k \neq i \\ \eta_k \geq \eta'}} \frac{1}{\alpha_j(b) - b} \right\}, \text{ (A3.2-18)}$$

for all i such that $\eta_i \geq \eta'$ and all b in $[\eta' - \varepsilon, \eta']$. In (A3.2-18), s is the number of index values l such that $\eta_l \geq \eta'$ and α_i is the inverse of bidder i's equilibrium bid function. Since β_j is strictly increasing, $\frac{d_l}{db} \ln F_j(\alpha_j(\eta')) \geq 0$, and we find[24] $-\frac{s-2}{\alpha_j(\eta')-\eta'} + \sum_{\substack{i \neq j \\ \eta_i \geq \eta'}} \frac{1}{\alpha_i(\eta')-\eta'} \geq 0$. From (A3.2-9), $\frac{1}{k(\eta)-1} \sum_{j=1}^{k(\eta)} \frac{1}{\alpha_j(\eta')-\eta'} = \frac{1}{\lim_{b \to > \eta'} \alpha_j(b) - \eta'}$. We obtain:

$$-\frac{s-2}{d_j - \eta'} + \sum_{\substack{i \geq k(\eta)+1 \\ i \neq j \\ \eta_i \geq \eta'}} \frac{1}{d_i - \eta'} + \frac{k(\eta)-1}{\lim_{b \to > \eta'} \alpha_j(b) - \eta'} \geq 0.$$

Since $\eta_i < \eta$, for all $i \geq k(\eta)+1$, and $d_i \geq d_j$, for all $j \geq i$, we have $\sum_{\substack{i \geq k(\eta)+1 \\ i \neq j \\ \eta_i \geq \eta'}} \frac{1}{d_i - \eta'} = \sum_{\substack{i \geq k(\eta)+1 \\ i \neq j \\ \eta_i = \eta'}} \frac{1}{d_i - \eta'} \leq \sum_{\substack{i \geq k(\eta)+1 \\ i \neq j \\ \eta_i = \eta'}} \frac{1}{d_j - \eta'} = \frac{s-k(\eta)-1}{d_j - \eta'}$. We thus find:

$$-\frac{k(\eta)-1}{d_j - \eta'} + \frac{k(\eta)-1}{\lim_{b \to > \eta'} \alpha_j(b) - \eta'} \geq 0.$$

Consequently, $\lim_{b \to > \eta'} \alpha_j(b) \leq d_j$. From our definition of j, we thus have $\lim_{b \to > \eta'} \alpha_i(b) \leq d_i$, for all $i \geq k(\eta)+1$ such that $\eta_i = \eta'$.

We have proved that $\lim_{b \to > \eta'} \alpha_i(b) = d_i$, for all $i \geq k(\eta)+1$ such that $\eta_i = \eta'$. As a particular consequence, we have $d_i = d_j$, for all i and j such that $\eta_i = \eta_j = \eta'$.

For all $i \geq k(\eta)+1$ such that $\eta_i = \eta'$, we can thus define over $[\underline{v}, d(\eta)]$ the continuous function α_i that is equal to the inverse of the bid function β_i over $[\underline{v}, d_i]$ and to the function defined in (A3.2-9) over $[d_i, d(\eta)]$. Proceeding as in the proof of the previous lemma, we can see, from (A3.2-9) and (A3.2-8),

[24] Here, $\alpha_j(\eta')$ and $\alpha_i(\eta')$ are the limits from the left of α_j and α_i at η'. That is, for example, $\alpha_j(\eta') = \lim_{b \to < \eta'} \alpha_j(b)$.

that the inverses $\alpha_1, ..., \alpha_{k(\eta)}, \alpha_i$, $i \geq k(\eta) + 1$ such that $\eta_i = \eta'$, satisfy the system (A3.2-18) over the interval $(\eta'', \eta]$, where s is the number of index values i such that $\eta_i \geq \eta'$ and where η'' is the maximum of the η_j such that $\eta_j < \eta'$.

We then define α_i, for all $i \geq k(\eta) + 1$ such that $\eta_i < \eta'$, over (η'', η') as follows:

$$\frac{1}{\alpha_i(b) - b} = \frac{1}{s-1} \sum_{\substack{j \\ \eta_j \geq \eta'}} \frac{1}{\alpha_j(b) - b}.$$

For all such i, it is simple to show that the continuous extension of α_i so defined at η' agrees with the continuous extension of α_i as we defined it over $(\eta', \eta]$. Again, from Lemma A5.1-4, α_i is strictly increasing, and, proceeding as in the proof of the previous lemma, we can then show that $(\alpha_1, ..., \alpha_n)$ is a solution of (1) and (3') over $(\eta'', \eta]$. As above, we can also show that $\lim_{b \longrightarrow> \eta''} \alpha_i(b) = d_i$, for all $i \geq k(\eta) + 1$ such that $\eta_i = \eta''$, and $d_i = d_j$, for all i and j such that $\eta_i = \eta_j = \eta''$.

Repeating this construction, we see that the inverses of the bid functions can be extended into solutions of (1) and (3'). Equivalently, the bid functions can be extended such that the inverses of the extensions form a solution of (1) and (3'). Moreover, the extensions of the bid functions for different bidders agree in the complements of their valuation supports in the interval $[\underline{v}, d(\eta)]$. Consequently, for all i, j such that $d_i, d_j \leq d(\eta)$, $\eta_i < \eta_j$ if and only if $d_i < d_j$. From (A3.2-17), we can easily show that the value of the extension of the bid function β_i at v_i in $[d_i, d(\eta)]$ is the best bid from bidder i with valuation v_i. ||

The characterization C.4 (Section 4) follows from Lemmas A3.2-2 and A3.2-6.

Appendix 4

Lemma A4-1: *Assume $d_1 \geq ... \geq d_n$. For all $\eta < d_2$, there exists one* *and only one k such that*

$$\begin{aligned} \eta &< d_k, \frac{1}{d_k-\eta} \leq \frac{1}{k-1}\sum_{i=1}^{k}\frac{1}{d_i-\eta} \\ & \quad \textit{and} \\ \textit{if } \eta &< d_{k+1}, \textit{ then } \frac{1}{k-1}\sum_{i=1}^{k}\frac{1}{d_i-\eta} < \frac{1}{d_{k+1}-\eta}. \textit{(A4-1)} \end{aligned}$$

Proof: Let l be the largest value of the index i such that $\eta < d_i$. From $\eta < d_2$, we have $l \geq 2$. Either $l = n$ and $\eta < d_i$, for all $1 \leq i \leq n$, or $l < n$ and $d_{l+1} \leq \eta < d_l$. Consider j such that $2 \leq j \leq l$. A simple computation shows that the inequality $\frac{1}{d_j-\eta} \leq \frac{1}{j-1}\sum_{i=1}^{j}\frac{1}{d_j-\eta}$ is equivalent to the inequality (A4-2) below

$$\sum_{i=1}^{j-1}\frac{d_i-d_j}{d_i-\eta} \leq 1 \text{ (A4-2)}.$$

Let Λ be the function defined over $\{1,...,l\}$ whose value at j is equal to the L.H.S. of (A4-2). Since d_i is nonincreasing in i and since $\eta < d_l$, Λ is a nondecrasing function. Consequently, $k = \max\{j \text{ such that } 1 \leq j \leq l \text{ and } \Lambda(j) \leq 1\}$ is the only value of the index that satisfies (A4-1). ||

Appendix 5

In this appendix, we consider the system (1) of differential equations over the domain $D' = \{(b, \alpha_1, ..., \alpha_n) \, | c_i < \alpha_i \text{ and } b < \alpha_i, \text{ for all } i\}$. A n-tuple $(\alpha_1, ..., \alpha_n)$ of continuous functions over $(\gamma, \gamma']$, with $\gamma < \gamma'$, is piecewise differentiable if and only if $\alpha_1, ..., \alpha_n$ are differentiable everywhere in $(\gamma, \gamma']$ except, possibly, at a finite number of points, the left-hand derivatives $\frac{d_l}{db}\alpha_1(b), ..., \frac{d_l}{db}\alpha_n(b)$ exist and are finite, for all b in $(\gamma, \gamma']$, and the right-hand derivatives $\frac{d_r}{db}\alpha_1(b), ..., \frac{d_r}{db}\alpha_n(b)$ exist and are finite, for all b in (γ, γ'). A solution $(\alpha_1, ..., \alpha_n)$ of the system (1) over $(\gamma, \gamma']$ is such a continuous and piecewise differentiable n-tuple such that $(b, \alpha_1(b), ..., \alpha_n(b))$ belongs to D', for all b in $(\gamma, \gamma']$, $(\ln F_1\alpha_1, ..., \ln F_n\alpha_n)$ is differentiable[25] over $(\gamma, \gamma']$, and the equations (1) hold true over this interval.

[25] The derivative at γ' is a left-hand derivative.

Note that if $(\alpha_1, ..., \alpha_n)$ is a solution of (1) over $(\gamma, \gamma']$, since F_i is differentiable over (c_i, d_i) with a strictly positive derivative over this interval, α_i is differentiable at b, for all b such that $\alpha_i(b) \in (c_i, d_i)$ and all $1 \leq i \leq n$.

Appendix 5.1

Lemma A5.1-1: *Let Assumptions A.1 and A.2 be satisfied. Let $(\alpha_1, ..., \alpha_n)$ be a solution over $(\eta - \varepsilon, \eta]$, with $\varepsilon > 0$, of the system (1). Then, the following equations hold true:*

$$\frac{d}{db}\sum_{k\neq i}\ln F_k(\alpha_k(b)) = \frac{1}{\alpha_i(b) - b} \text{(A5.1-1)}$$

$$\frac{d}{db}\ln F_j(\alpha_j(b)) - \frac{d}{db}\ln F_i(\alpha_i(b)) = \frac{1}{\alpha_i(b) - b} - \frac{1}{\alpha_j(b) - b} \text{(A5.1-2),}$$

for all $1 \leq i, j \leq n$ and $b \in (\eta - \varepsilon, \eta]$.

Proof: By summing all equations in (1) except the ith equation, we find (A5.1-1). It suffices to subtract the ith equation in (1) from the jth equation in order to prove (A5.1-2). ||

Lemma A5.1-2: *Let Assumptions A.1 and A.2 be satisfied. Let $(\alpha_1, ..., \alpha_n)$ be a solution over $(\zeta - \varepsilon, \zeta]$, with $\varepsilon > 0$, of the system (1') and the initial condition (A5.1-3) below*

$$\alpha_i(\zeta) = f_i,\ 1 \leq i \leq n,\ \text{(A5.1-3)}$$

where $d_i < f_i$ and $\zeta < f_i$, for all $i > k$, and $\zeta < f_i \leq d_i$, for all $1 \leq i \leq k$. Assume that $k \geq 2$ and that $\alpha_i(b) > d_i$, for all $i > k$ and all b in $(\zeta - \varepsilon, \zeta]$. Then, $(\alpha_1, ..., \alpha_n)$ is a solution of (1) over $(\zeta - \varepsilon, \zeta]$ if an only if:

$$\frac{d}{db}\ln F_i(\alpha_i(b)) = \frac{1}{k-1}\left\{-\frac{k-2}{\alpha_i(b) - b} + \sum_{k\neq i}\frac{1}{\alpha_j(b) - b}\right\},\ \textit{for all } 1 \leq i \leq k, \text{(A5.1-4)}$$

and $a_{k+1}(b) = ... = \alpha_n(b)$ and $\alpha_i(b)$ is the unique solution of the equation (A5.1-5) below, for all b in $(\zeta - \varepsilon, \zeta]$ and $k + 1 \leq i \leq n$:

$$\frac{1}{\alpha_i(b) - b} = \frac{1}{k-1}\sum_{1\leq j\leq k}\frac{1}{\alpha_j(b) - b} \text{(A5.1-5).}$$

:

Proof: Assume $(\alpha_1, ..., \alpha_n)$ is a solution of (1). Since $\frac{d}{db} \ln F_i (\alpha_i (b)) = 0$, for all $k+1 \leq i \leq n$ and b in $(\zeta', \zeta]$, $(\alpha_{k+1}, ..., \alpha_n)$ is a solution of the following system:

$$A \left(\frac{1}{\alpha_i (b) - b} \right)_{k+1 \leq i \leq n} = E_{n-k.k} \left(\frac{1}{\alpha_j (b) - b} \right)_{1 \leq j \leq k} ,\text{(A5.1-6)}$$

where $A = ((n-k) I_{n-k} - K_{n-k})$, with I_{n-k}, K_{n-k}, and $E_{n-k,k}$ as defined in Definition A3.2-1. Thus, we have:

$$\left(\frac{1}{\alpha_i (b) - b} \right)_{k+1 \leq i \leq n} = A^{-1} E_{n-k.k} \left(\frac{1}{\alpha_j (b) - b} \right)_{1 \leq j \leq k} .\text{(A5.1-7)}$$

(A5.1-5) then follows from (A5.1-7) and the equalities $A^{-1} = \frac{1}{(k-1)(n-1)} (kI_{n-k} + K_{n-k})$ (from Lemma A3.2-4) and $\frac{1}{(k-1)(n-1)} (kI_{n-k} + K_{n-k}) E_{n-k,k} = \frac{1}{k-1} E_{n-k,k}$.

From the system (1), we have:

$$\left(\frac{d}{db} \ln F_i (\alpha_i (b)) \right)_{1 \leq i \leq k} = \frac{1}{n-1} \left\{ E_{k,n-k} \left(\frac{1}{\alpha_i (b) - b} \right)_{k+1 \leq i \leq n} + B \left(\frac{1}{\alpha_j (b) - b} \right)_{1 \leq j \leq k} \right\} ,\text{(A5.1-8)}$$

where $B = (-(n-2) I_k + K_k)$, with $E_{k,n-k}, I_k$, and K_k as defined in Definition A3.2-1.

Substituting to $\left(\frac{1}{\alpha_i (b) - b} \right)_{k+1 \leq i \leq n}$ in (A5.1-8) its value from (A5.1-7), we find

$$\left(\frac{d}{db} \ln F_i (\alpha_i (b)) \right)_{1 \leq i \leq k} = \frac{1}{n-1} E_{k,n-k} A^{-1} E_{n-k,k} + B \left(\frac{1}{\alpha_j (b) - b} \right)_{1 \leq j \leq k} .\text{(A5.1-9)}$$

The product $E_{k,n-k} A^{-1} E_{n-k,k}$ is equal to $\frac{n-k}{n-1} E_k$ and the sum $E_{k,n-k} A^{-1} E_{n-k,k} + B$ is equal to $\frac{n-1}{k-1} (-(k-2) I_k + K_k)$. Substituting this value to the matrix between braces in (A5.1-9) immediately gives (A5.1-4).

The proof that (A5.1-4) and (A5.1-5) imply (1) can proceed as in the proof of Lemma A3.2-5. ||

Lemma A5.1-3: *Let Assumptions A.1 and A.2 be satisfied. Let $(\alpha_1, ..., \alpha_n)$ be a solution over $(\zeta - \varepsilon, \zeta]$, with $\varepsilon > 0$, of the system (1) and the* *initial condition (A5.1-3) in Lemma A5.1-2 above where* $d_i < f_i$ *and* $\zeta < f_i$, *for all* $i > k$, *and* $\zeta < f_i \leq d_i$, *for all* $1 \leq i \leq k$. *Assume that* $k \geq 2$ *and that* $\alpha_i(b) > d_i$, *for all* $i > k$ *and all* b *in* $(\zeta - \varepsilon, \zeta]$, *and that* $\frac{d}{db}\alpha_j(\zeta) \geq 0$, *for all* $1 \leq j \leq k$. *Then,* $\frac{d}{db}\alpha_i(b)$ *exists and*

$$\frac{d}{db}\alpha_i(b) > 0, \text{ (A5.1-10)}$$

for all $1 \leq i \leq n$ *and all* b *in* $(\zeta - \varepsilon, \zeta)$,

$$\alpha_{k+1}(b) = ... = \alpha_n(b) < \alpha_j(b), \text{ (A5.1-11)}$$

for all $1 \leq j \leq k$ *and* b *in* $(\zeta - \varepsilon, \zeta)$. *Moreover,*

$$f_1 = ... = f_k \leq f_j, \text{(A5.1-12)}$$

and *if* $\frac{d}{db}\alpha_j(\zeta) > 0$, *then*

$$f_1 = ... = f_k < f_j, \text{ (A5.1-13)}$$

for all $1 \leq j \leq k$ *and* $k + 1 \leq i \leq n$.

Proof: From (A5.1-3) and (A5.1-4) in the previous lemma and from Lemma A1-1, $\frac{d}{db}\alpha_i(b)$ exists and $\frac{d}{db}\alpha_i(b) > 0$, for all $1 \leq i \leq k$ and all b in $(\zeta - \varepsilon, \zeta)$. The function α_i is thus strictly increasing over $(\zeta - \varepsilon, \zeta]$, for all $1 \leq i \leq k$. Since α_i is strictly increasing over $(\zeta - \varepsilon, \zeta]$ and $\alpha_i(\zeta) = f_i \leq d_i$, we have $\alpha_i(b) < d_i$, for all $1 \leq i \leq k$.

Since $\alpha_j(b) < d_j$ and $\frac{d}{db}\alpha_j(b) > 0$, we have $\frac{d}{db}\ln F_j(\alpha_j(b)) > 0$, for all $1 \leq j \leq k$ and b in $(\zeta - \varepsilon, \zeta)$. Moreover, $\frac{d}{db}\ln F_i(\alpha_i(b)) = 0$, for all $k + 1 \leq i \leq n$ and b in $(\zeta - \varepsilon, \zeta]$. The equality (A5.1-2) in Lemma A5.1-1 thus implies

$$\alpha_i(b) < \alpha_j(b), \text{(A5.1-14)}$$

for all $1 \leq j \leq k$, $k + 1 \leq i \leq n$, and b in $(\zeta - \varepsilon, \zeta)$. (A5.1-11) follows from the previous inequality and from (A5.1-5) in the previous lemma. Making b in (A5.1-11) tend towards ζ and using (A5.1-3), we find (A5.1-12). If $\frac{d}{db}\alpha_j(\zeta) > 0$, $\alpha_j(\zeta) = f_j < d_j$ implies $\frac{d_l}{db}\ln F_j(\alpha_j(\zeta)) > 0$, for all $1 \leq j \leq k$. From $\frac{d}{db}\ln F_i(\alpha_i(\zeta)) = 0$ and (A5.1-2) in Lemma A5.1-1, (A5.1-14) holds true at $b = \zeta$, for all $k + 1 \leq i \leq n$. Substituting ζ to b in (A5.1-14) and using (A5.1-3), we find (A5.1-13).

From (A5.1-5) in the previous lemma, $\frac{d}{db}\alpha_i(b)$ exists, for all b in $(\zeta-\varepsilon,\zeta)$ and all $i \geq k+1$. From (A5.1-5), we also have $1 = \frac{1}{k-1}\sum_{1\leq j\leq k}\frac{\alpha_i(b)-b}{\alpha_j(b)-b}$, for all $k+1 \leq i \leq n$ and all b in $(\zeta-\varepsilon,\zeta)$. Taking the derivative with respect to b, we find:

$$0 = \sum_{1\leq j\leq k}\frac{1}{(\alpha_j(b)-b)^2}\left[\left(\frac{d}{db}\alpha_i(b)-1\right)(\alpha_j(b)-b)-(\alpha_i(b)-b)\left(\frac{d}{db}\alpha_j(b)-1\right)\right].\text{(A5.1-15)}$$

Suppose $\frac{d}{db}\alpha_i(b) \leq 0$. The expression between brackets in the R.H.S. of (A5.1-15) would not be larger than $(\alpha_i(b)-\alpha_j(b))-(\alpha_i(b)-b)\frac{d}{db}\alpha_j(b)$. Since $\alpha_i(b) \leq \alpha_j(b)$, $\alpha_i(b) > b$, and $\frac{d}{db}\alpha_j(b) > 0$, every term in the sum in (A5.1-15) would be strictly negative and the equality (A5.1-15) could not hold true. Consequently, $\frac{d}{db}\alpha_i(b) > 0$, for all b in $(\zeta-\varepsilon,\zeta)$ and all $k+1 \leq i \leq n$, and (A5.1-10) is proved. ||

Lemma A5.1-4: *Let Assumptions A.1 and A.2 be satisfied. Let $\alpha_1,...,\alpha_n$ be continuous and piecewise differentiable functions over $(\gamma,\eta]$, with $\gamma<\eta$, such that $(\alpha_1,...,\alpha_n)$ is a solution over this interval of the system (1) and the initial condition (A5.1-16) below*

$$\alpha_i(\eta) = f_i, \text{ (A5.1-16)}$$

where $d_i < f_i$ and $\eta < f_i$, for all $i > k$, and $\eta < d_i = f_i$ and $d(\eta) \leq d_i = f_i$, for all $1 \leq i \leq k$. Assume that $k \geq 2$ and that $f_i < d_j$, for all $1 \leq j \leq k$ and $k+1 \leq i \leq n$. Then, α_i is strictly increasing and $\frac{d}{db}\alpha_i(b)$ exists and

$$\frac{d}{db}\alpha_i(b) > 0,$$

for all $1 \leq i \leq n$ and all b in (γ,η) such that $\alpha_i(b) \neq d_i$.

Proof: We can easily show that $d(\eta) \leq d_i$, for all $1 \leq i \leq k$, implies $\frac{d}{db}\alpha_i(\eta) \geq 0$, for all $1 \leq i \leq k$ (see the proof of Lemma A3.2-3) Lemma A5.1-4 then follows from repeated applications of Lemma A5.1-3. ||

Appendix 5.2

Lemma A5.2-1: *Let Assumptions A.1 and A.2 be satisfied. Let $\alpha_1, ..., \alpha_n$ be continuous and piecewise differentiable functions over $(\gamma, \eta]$, with $\gamma < \eta$, such that $(\alpha_1, ..., \alpha_n)$ is a solution over this interval of (1) and (3') for a value $\eta < d_2$ of the parameter. Let $\alpha_1^*, ..., \alpha_n^*$ be continuous and piecewise differentiable functions over $(\gamma, \eta']$, with $\gamma < \eta'$, such that $(\alpha_1^*, ..., \alpha_n^*)$ is a solution over this interval of (1) and (3') for a value $\eta' < \eta$ of the parameter. Then, $\alpha_i^*(\eta') > \alpha_i(\eta')$, for all $1 \leq i \leq n$.*

Proof: Let L be the set of index values i defined as follows:

$$L = \left\{ i \text{ such that } 1 \leq i \leq n \text{ and } \frac{d}{db} \ln F_i(\alpha_i(\eta')) = 0 \right\}.$$

Let k be the minimum of L. From our assumption that d_i does not increase with i, from (3'), and from Lemma A5.1-4, we have $L = \{k, ..., n\}$. For all $1 \leq i \leq k(\eta)$, since $\alpha_i(\eta) = d_i$ we have $\frac{d}{db} \ln F_i(\alpha_i(\eta')) > 0$ and thus $i \notin L$. Consequently, $k > k(\eta)$. In particular, $k > 2$. For all $i < k$, since $\frac{d}{db} \ln F_i(\alpha_i(\eta')) > 0$ we have $\alpha_i(\eta') < \alpha_i(\eta) = d_i = \alpha_i^*(\eta')$. For all $i \geq k$, from $\frac{d}{db} \ln F_i(\alpha_i(\eta')) = 0$ and (1), we have $-\frac{n-2}{\alpha_i(\eta')-\eta'} + \sum_{j \neq i} \frac{1}{\alpha_j(\eta')-b} = 0$. Consequently, $(\alpha_k(\eta'), ..., \alpha_n(\eta'))$ is a solution of the system below:

$$((n-2) I_{n-k+1} - K_{n-k+1}) \left(\frac{1}{\alpha_i(\eta') - \eta'} \right)_{k \leq i \leq n} = E_{n-k+1,k-1} \left(\frac{1}{\alpha_j(\eta') - \eta'} \right)_{1 \leq j \leq k-1}, \text{(A5.2-1)}$$

where I_{n-k+1}, K_{n-k+1}, and $E_{n-k+1,k-1}$ are as in Definition A3.2-1. From Lemma A3.2-4, the matrix $A = ((n-2) I_{n-k+1} - K_{n-k+1})$ is regular and its inverse is equal to $A^{-1} = \frac{1}{(k-2)(n-1)} ((k-1) I_{n-k+1} + K_{n-k+1})$. From (A5.2-1), we have:

$$\left(\frac{1}{\alpha_i(\eta') - \eta'} \right)_{k \leq i \leq n} = A^{-1} E_{n-k+1,k-1} \left(\frac{1}{\alpha_j(\eta') - \eta'} \right)_{1 \leq j \leq k-1}. \text{(A5.2-2)}$$

From $\frac{d_l}{db} \ln F_i(\alpha_i^*(\eta')) \geq 0$ and (1), we have:

$$A \left(\frac{1}{\alpha_i^*(\eta') - \eta'} \right)_{k \leq i \leq n} \leq E_{n-k+1,k-1} \left(\frac{1}{\alpha_j^*(\eta') - \eta'} \right)_{1 \leq j \leq k-1}.$$

Since all the elements in the inverse $A^{-1} = \frac{1}{(k-2)(n-1)} ((k-1) I_{n-k+1} + K_{n-k+1})$ are nonnegative, we have:

$$\left(\frac{1}{\alpha_i^*(\eta') - \eta'} \right)_{k \leq i \leq n} \leq A^{-1} E_{n-k+1.k-1} \left(\frac{1}{\alpha_j^*(\eta') - \eta'} \right)_{1 \leq j \leq k-1}. \text{(A5.2-3)}$$

All the elements in $A^{-1} = \frac{1}{(k-2)(n-1)} \left((k-1) I_{n-k+1} + K_{n-k+1} \right)$ and $E_{n-k+1,n=k}$, and thus in their product are strictly positive. Moreover, we have already proved that $\alpha_j^* (\eta') > \alpha_j (\eta')$, for all $j \leq k-1$. (A5.2-2) and (A5.2-3) thus imply $\frac{1}{\alpha_i^*(\eta')-\eta'} < \frac{1}{\alpha_i(\eta')-\eta'}$ or, equivalently, $\alpha_i^* (\eta') > \alpha_i (\eta')$, for all $i \geq k$, and the lemma is proved. ||

Lemma A5.2-2: *Let Assumptions A.1 and A.2 be satisfied. Let $\alpha_1, ..., \alpha_n$ be continuous and piecewise differentiable functions over $(\gamma, \eta]$, with $\gamma < \eta$, such that $(\alpha_1, ..., \alpha_n)$ is a solution over this interval of (1) and (3') for a value $\eta < d_2$ of the parameter. Let $\alpha_1^*, ..., \alpha_n^*$ be continuous and piecewise differentiable functions over $(\gamma, \eta']$, with $\gamma < \eta'$, such that $(\alpha_1^*, ..., \alpha_n^*)$ is a solution over this interval of (1) and (3') for a value $\eta' < \eta$ of the parameter. Then, $\alpha_i^* (b) > \alpha_i (b)$, for all $1 \leq i \leq n$ and all b in $(\gamma, \eta']$.*

Proof: From the previous lemma, we have $\alpha_i^* (\eta') > \alpha_i (\eta')$, for all $1 \leq i \leq n$. We define h in $(\gamma, \eta']$ as follows:

$$h = \inf \left\{ b \in [\gamma, \eta'] \left| \alpha_i^* (b') > \alpha_i (b') \right., \text{ for all } 1 \leq i \leq n \text{ and all } b' \text{ in } (b, \eta'] \right\}.$$

We want to prove that $h = \gamma$. By continuity and $\alpha_i^* (\eta') > \alpha_i (\eta')$, for all $1 \leq i \leq n$, we have $h < \eta'$. Suppose that $h > \gamma$. There must exist i such that $\alpha_i^* (h) = \alpha_i (h)$. By continuity, we also have $\alpha_j^* (h) \geq \alpha_j (h)$, for all $1 \leq j \leq n$. Moreover, there exists $l \neq i$ such that $\alpha_l^* (h) > \alpha_l (h)$. In fact, if it was not the case the continuous and piecewise differentiable solutions $(\alpha_1, ..., \alpha_n)$ and $(\alpha_1^*, ..., \alpha_n^*)$ would be equal at h and would thus be equal everywhere over their common definition domain, which is impossible since $\alpha_i^* (\eta') > \alpha_i (\eta')$, for all $1 \leq i \leq n$.

Assume that $\frac{d}{db} \ln F_i (\alpha_i (h)) = 0$. From (1), we have

$$\frac{1}{\alpha_i (h) - h} = \frac{1}{n-2} \sum_{j \neq i} \frac{1}{\alpha_j (h) - h}. \text{(A5.2-4)}$$

From $\frac{d}{db} \ln F_i (\alpha_i^* (h)) \geq 0$ and (1), we have

$$\frac{1}{\alpha_i^* (h) - h} \leq \frac{1}{n-2} \sum_{j \neq i} \frac{1}{\alpha_j^* (h) - h}. \text{(A5.2-5)}$$

Since $\alpha_j^* (h) \geq \alpha_j (h)$, for all $j \neq i$, and $\alpha_l^* (h) > \alpha_l (h)$, (A5.2-4) and (A5.2-5) imply $\frac{1}{\alpha_i^*(h)-h} < \frac{1}{\alpha_i(h)-h}$ and thus $\alpha_i^* (h) > \alpha_i (h)$. This contradicts $\alpha_i^* (h) = \alpha_i (h)$ and $\frac{d}{db} \ln F_i (\alpha_i (h)) = 0$ is impossible.

From the previous paragraph, we must have $\frac{d}{db}\ln F_i(\alpha_i(h)) > 0$ and, consequently, $\alpha_i(h) < d_i$. Since $\alpha_i^*(h) = \alpha_i(h)$, we obviously have $\alpha_i^*(h) < d_i$ and $\frac{d}{db}\ln F_i(\alpha_i^*(h)) > 0$.

From (1), we have:

$$\frac{d}{db}\alpha_i(h) = \frac{f_i(\alpha_i(h))}{F_i(\alpha_i(h))}\frac{1}{n-1}\left\{-\frac{n-2}{\alpha_i(h)-h} + \sum_{k\neq i}\frac{1}{\alpha_j(h)-h}\right\} > 0 \text{ (A5.2-6)}$$

$$\frac{d}{db}\alpha_i^*(h) = \frac{f_i(\alpha_i^*(h))}{F_i(\alpha_i^*(h))}\frac{1}{n-1}\left\{-\frac{n-2}{\alpha_i^*(h)-h} + \sum_{k\neq i}\frac{1}{\alpha_j^*(h)-h}\right\} > 0 \text{ (A5.2-7).}$$

From $\alpha_i(h) = \alpha_i^*(h)$, $\alpha_j(h) \leq \alpha_j^*(h)$, for all j, and $\alpha_l(h) < \alpha_l^*(h)$, the inequalities (A5.2-6) and (A5.2-7) imply

$$\frac{d}{db}\alpha_i^*(h) < \frac{d}{db}\alpha_i(h).$$

There thus exists $\delta > 0$ such that $\alpha_i(b) > \alpha_i^*(b)$, for all b in $(h, h+\delta)$. However, this contradicts the definition of h. We have thus proved that $h > \gamma$ is impossible and thus that $h = \gamma$. The lemma is proved. ||

Appendix 6

Existing Results III (Lebrun 1999a)*: Let Assumptions A.1 be satisfied. Assume also $c_i = c$, $d_i = d$, for all i, and $r \leq c$. If $F_1 = ... = F_m = G_1$ and $F_{m+1} = ... = F_n = G_2$, where $1 \leq m \leq n$, $G_1(c) = G_2(c) = 0$, and $\frac{d}{dv}\frac{G_1}{G_2}(v) > 0$, for all v in $(c, d]$, then there exists one and only one equilibrium.*

The assumption $\frac{d}{dv}\frac{G_1}{G_2}(v) > 0$, over $(c, d]$, is the assumption of (strict) reverse hazard rate stochastic dominance by G_1 over G_2 (see Krishna 2002). Conditionally on any interval $[c, e]$, with $c \leq e \leq d$, the distribution G_1 first order dominates the distribution G_2. A close examination of the proof in Lebrun (1999a) reveals that this relation of stochastic dominance is only needed in a nondegenerate interval with c as its lower extremity. We have the following extension.

Extension of Results III: *Let Assumptions A.1 be satisfied. Assume also* $c_i = c$, $d_i = d$, *for all* i, $r \leq c$, $F_1 = ... = F_m = G_1$, $F_{m+1} = ... = F_n = G_2$, *with* $1 \leq m \leq n$, *and* $G_1(c) = G_2(c) = 0$. *If there exists* $\varepsilon > 0$ *such that* $\frac{d}{dv}\frac{G_1}{G_2}(v) > 0$, *for all* v *in* $(c, c+\varepsilon]$, *then there exists one and only one equilibrium.*

In the proof of this extension above, we will use Lemma A6-1 below.

Lemma A6-1: *Let Assumptions A.1 be satisfied. Assume* $c_i = c$, $d_i = d$, for all i, and $r \leq c$. *Let* F_1,...,F_n *be differentiable over* $(c, d]$ *with derivatives* $f_1, ..., f_n$ *locally bounded away from zero over this interval. Assume that there exists* $\delta > 0$, m, G_1, *and* G_2 *such that* $G_1(c) = G_2(c) = 0$, $F_i = G_1$, *for all* $1 \leq i \leq m$, $F_j = G_2$, *for all* $m < j \leq n$, *and* $\frac{d}{dv}\frac{G_1}{G_2}(v) > 0$, *for all* v *in* $(c, c+\delta]$. *Let* $(\beta_1, ..., \beta_n)$ *and* $\left(\widetilde{\beta}_1, ..., \widetilde{\beta}_n\right)$ *be two different equilibria of the first-price auction. Then there exist* $\beta_1^*, \beta_2^*, \widetilde{\beta}_1^*$, *and* $\widetilde{\beta}_2^*$ *such that* $\beta_i = \beta_1^*$ *and* $\widetilde{\beta}_i = \widetilde{\beta}_1^*$, *for all* $1 \leq i \leq m$, $\beta_j = \beta_2^*$ *and* $\widetilde{\beta}_j = \widetilde{\beta}_2^*$, *for all* $m < j \leq n$. *There also exists* $\gamma > 0$ *such that either* $\varphi_{21}^*(v) < \widetilde{\varphi}_{21}^*(v)$, *for all* v *in* $(c, c+\gamma)$, *or* $\varphi_{21}^*(v) > \widetilde{\varphi}_{21}^*(v)$, *for all* v *in* $(c, c+\gamma)$, *where* $\varphi_{21}^* = \beta_2^{*-1} \circ \beta_1^*$ *and* $\widetilde{\varphi}_{21}^* = \widetilde{\beta}_2^{*-1} \circ \widetilde{\beta}_1^*$.

Proof: The first part of Lemma A6-1 is from Corollary 3 (iv) in Lebrun (1999a). From the property of monotonicity with respect to η (see Lemma 2, Section 3, and its proof in Appendix 1), we know that, over $(c, d]$, either $\widetilde{\beta}_1^* > \beta_1^*$ and $\widetilde{\beta}_2^* > \beta_2^*$ or $\widetilde{\beta}_1^* < \beta_1^*$ and $\widetilde{\beta}_2^* < \beta_2^*$. Without loss of generality, we can assume that $\widetilde{\beta}_1^* > \beta_1^*$ and $\widetilde{\beta}_2^* > \beta_2^*$.

Substituting in (1) α_1^* to α_i and G_1 to F_i, for all $1 \leq i \leq m$, and α_2^* to α_j and G_2 to F_j, for all $m+1 \leq j \leq n$, and rearranging, the system (1) reduces to the two following equations:

$$\frac{d}{db}\alpha_1^*(b) = \frac{G_1(\alpha_1^*(b))}{g_1(\alpha_1^*(b))}\frac{1}{n-1}\frac{(n-m)\alpha_1^*(b) - (n-m-1)\alpha_2^*(b) - b}{(\alpha_1^*(b) - b)(\alpha_2^*(b) - b)} \text{(A6-1)}$$

$$\frac{d}{db}\alpha_2^*(b) = \frac{G_2(\alpha_2^*(b))}{g_2(\alpha_2^*(b))}\frac{1}{n-1}\frac{m\alpha_2^*(b) - (m-1)\alpha_1^*(b) - b}{(\alpha_1^*(b) - b)(\alpha_2^*(b) - b)} \text{(A6-2).}$$

Since the bid functions are differentiable over $(c, d]$ or since the derivatives of the inverse bid functions are strictly positive over $(c, \eta]$ (see Lemma A1-2), the functions φ_{21}^* and β_1^* are differentiable at v, $\frac{d}{dv}\varphi_{21}^*(v) = \left[\frac{d}{db}\alpha_2^*(b)\right]_{b=\alpha_1^{*-1}(v)} \frac{1}{\left[\frac{d}{db}\alpha_1^*(b)\right]_{b=\alpha_1^{*-1}(v)}}$, and $\frac{d}{dv}\beta_1^*(v) = \frac{1}{\left[\frac{d}{db}\alpha_1^*(b)\right]_{b=\alpha_1^{*-1}(v)}}$, for all v in $(c, d]$. Dividing (A6-2) by (A6-1) and simplifying, we find that $(\varphi_{21}^*, \beta_1^*)$ is a solution over $(c, d]$ of the system (A6-3) and (A6-4) below –considered on the domain:

$$\{(v, \varphi_{21}, \beta_1) \,|c < v, \varphi_{21} \leq d,\ (n-m)\,v - (n-m-1)\,\varphi_{21} - \beta_1 > 0\}.$$

$$\frac{d}{dv}\varphi_{21}^*(v) = \frac{g_1(v)}{G_1(v)} \frac{G_2(\varphi_{21}^*(v))}{g_2(\varphi_{21}^*(v))} \frac{m\varphi_{21}^*(v) - (m-1)\,v - \beta_1^*(v)}{(n-m)\,v - (n-m-1)\,\varphi_{21}^*(v) - \beta_1^*(v)} \text{(A6-3)}$$

$$\frac{d}{dv}\beta_1^*(v) = \frac{g_1(v)}{G_1(v)} \frac{(n-1)\,(v - \beta_1^*(v))\,(\varphi_{21}^*(v) - \beta_1^*(v))}{(n-m)\,v - (n-m-1)\,\varphi_{21}^*(v) - \beta_1^*(v)} \text{(A6-4).}$$

From our assumption of stochastic dominance between G_1 and G_2 over $(c, c+\gamma)$, we have $\frac{d}{dv}\varphi_{21}^*(v) < 1$, for all solution $(\varphi_{21}^*, \beta_1^*)$ of (A6-3) and (A6-4) and all v in $(c, c+\gamma)$ such that $\varphi_{21}^*(v) = v$. From (A6-3) and (A6-4), this property implies, through a standard proof (see, for example, Lemma 2 in Milgrom and Weber 1982, Lemma A7 in Lebrun 1998, or the proof of Lemma A1-1 in Appendix 1 of the present paper), that there exists $\delta > 0$ such that either $\varphi_{21}^*(v) > v$, for all v in $(c, c+\delta)$, or $\varphi_{21}^*(v) < v$, for all v in $(c, c+\delta)$. Similarly, there exists $\widetilde{\delta} > 0$ such that either $\widetilde{\varphi}_{21}^*(v) > v$, for all v in $(c, c+\delta)$, or $\widetilde{\varphi}_{21}^*(v) < v$, for all v in $(c, c+\delta)$. We can assume that[27] $\delta = \widetilde{\delta}$.

If $\varphi_{21}^*(v) > v$ and $\widetilde{\varphi}_{21}^*(v) < v$, for all v in $(c, c+\delta)$, or if $\varphi_{21}^*(v) < v$ and $\widetilde{\varphi}_{21}^*(v) > v$, for all v in $(c, c+\delta)$, Lemma A6-1 is proved. Assume that $\varphi_{21}^*(v) < v$ and $\widetilde{\varphi}_{21}^*(v) < v$, for all v in $(c, c+\delta)$. Since (A6-3) can be rewritten as $\frac{d}{dv}\varphi_{21}^*(v) = \frac{g_1(v)}{G_1(v)} \frac{G_2\left(\varphi_{21}^*(v)\right)}{g_2\left(\varphi_{21}^*(v)\right)} \left\{1 + \frac{(n-m)(\varphi_{21}^*(v) - v)}{(n-m)v - (n-m-1)\varphi_{21}^*(v) - \beta_1^*(v)}\right\}$ and since $\widetilde{\beta}_1^* > \beta_1^*$, over $(c, d]$, we see that if $\varphi_{21}^*(v) = \widetilde{\varphi}_{21}^*(v)$ then $\frac{d}{dv}\varphi_{21}^*(v) > \frac{d}{dv}\widetilde{\varphi}_{21}^*(v)$, for all v in $(c, c+\delta)$. Again through a standard proof, this implies the existence of $0 < \mu < \delta$ such that either $\varphi_{21}^*(v) > \widetilde{\varphi}_{21}^*(v)$, for all v in $(c, c+\mu)$, or $\varphi_{21}^*(v) < \widetilde{\varphi}_{21}^*(v)$, for all v in $(c, c+\mu)$, and Lemma A6-1 is proved in this case. The proof in the case $\varphi_{21}^*(v) > v$ and $\widetilde{\varphi}_{21}^*(v) > v$, for all v in $(c, c+\delta)$, is similar. ||

Proof of Extension of Results III: Suppose that there are two different equilibria $(\beta_1, ..., \beta_n)$ and $\left(\widetilde{\beta}_1, ..., \widetilde{\beta}_n\right)$. Let (β_1^*, β_2^*), $\left(\widetilde{\beta}_1^*, \widetilde{\beta}_2^*\right)$, and $\gamma > 0$ be as in Lemma A6-1. As in the proof of Lemma A6-1, we can assume, without loss of generality, that $\widetilde{\beta}_1^* > \beta_1^*$ and $\widetilde{\beta}_2^* > \beta_2^*$, over $(c, d]$. Define w as follows:

$$w = \min\left\{v \in (c, d] \,|\varphi_{21}^* (v) = \widetilde{\varphi}_{21}^* (v)\right\}.$$

Since $\varphi_{21}^* (d) = \widetilde{\varphi}_{21}^* (d)$, the set in the definition of w is not empty. Moreover, from Lemma A6-1, we have either $\varphi_{21}^* (v) < \widetilde{\varphi}_{21}^* (v)$, for all v in $(c, c + \gamma)$, or $\varphi_{21}^* (v) > \widetilde{\varphi}_{21}^* (v)$, for all v in $(c, c + \gamma)$. Consequently, w exists and $w \geq c+\gamma$. Assume that $\varphi_{21}^* (v) < \widetilde{\varphi}_{21}^* (v)$, for all v in $(c, c + \gamma)$. Then, $\varphi_{21}^* (v) < \widetilde{\varphi}_{21}^* (v)$, for all v in (c, w). It is a standard result from the theory of incentive compatible mechanisms that the rate of increase of the interim expected payoff is equal to the probability of winning. We thus have $\beta_1^* (w) = w - \int_c^w G_1 (v)^m G_2 (\varphi_{21}^* (v))^{(n-m-1)} dv / G_1 (w)^m G_2 (\varphi_{21}^* (w))^{(n-m-1)}$ and $\widetilde{\beta}_1^* (w) = w - \int_c^w G_1 (v)^m G_2 (\widetilde{\varphi}_{21}^* (v))^{(n-m-1)} dv / G_1 (w)^m G_2 (\widetilde{\varphi}_{21}^* (w))^{(n-m-1)}$. From the inequality between φ_{21}^* and $\widetilde{\varphi}_{21}^*$ over (c, w) and the equality between them at w, it then follows that $\beta_1^* (w) > \widetilde{\beta}_1^* (w)$. It contradicts $\widetilde{\beta}_1^* > \beta_1^*$ over $(c, d]$ and Extension of Results III is proved in the case $\varphi_{21}^* (v) < \widetilde{\varphi}_{21}^* (v)$, for all v in $(c, c + \delta)$.

Assume now that $\varphi_{21}^* (v) > \widetilde{\varphi}_{21}^* (v)$, for all v in $(c, c + \delta)$ and thus for all v in (c, w). By inverting the (strictly increasing) functions φ_{21}^* and $\widetilde{\varphi}_{21}^*$, we find $\varphi_{12}^* (v) < \widetilde{\varphi}_{12}^* (v)$, for all v in (c, w), where $\varphi_{12}^* = \beta_1^{*-1} \circ \beta_2^*$ and $\widetilde{\varphi}_{12}^* = \widetilde{\beta}_1^{*-1} \circ \widetilde{\beta}_2^*$. By applying to the functions β_2^* and $\widetilde{\beta}_2^*$ the same arguments we used in the previous paragraph, we also find a contradiction and Extension of Results III is proved. ||

References

An, M.Y. (1998): "Logconcavity versus Logconvexity: A Complete Characterization," *Journal of Economic Theory*, 80, 350-69.

Athey, S. (2001): "Single Crossing Properties and the Existence of Pure Strategy Equilibria in Games of Incomplete Information," *Econometrica*, 69, 4, 861-89.

Bagnoli, M. and T. Bergstrom (1989): “Log-Concave Probability and Its Applications,” Mimeo.

Bajari, P. (2001): “Comparing Competition and Collusion: a Numerical Approach,” *Economic Theory*, 18, 187-205.

Baldwin, L.H., Marshall, R.C., and J.F. Richard (1997): “Bidder Collusion at Forest Service Timber Sales,” *Journal of Political Economy*, 105, 4, 657-99

Dalkir, S., Logan, J.W., and R.T. Masson, (2000): “Mergers in Symmetric and Asymmetric Noncooperative Auction Markets: The Effects on Prices and Efficiency,” *International Journal of Industrial Organization*, 18, 383-413

Graham, D.A. and R.C. Marshall (1985): “Collusive Behavior At a Single Object English Auction,” Working Paper n0. 85-01. Durham, N.C.: Duke University, Department of Economics.

Hendricks, K. and R.H. Porter (1989): “Collusion in Auctions,” *Annales d’Economie et de Statistique*, 15/16, 217-230.

Jackson, M.O., Simon, L.K., Swinkels, J.M., and W.R. Zame (2002): “Communication and Equilibrium in Discontinuous Games of Incomplete Information,” *Econometrica*, 70, 5, 1711-1740.

Krishna, V. (2002): *Auction Theory*, Academic Press

Lebrun, B. (1996): “Existence of an Equilibrium in First Price Auctions,” *Economic Theory*, 7, 421-43

Lebrun, B. (1997): “First Price Auction in the Asymmetric N Bidder Case,” Les Cahiers de Recherches du GREEN (working paper series) #97-03. Requests should be sent to the author at blebrun@econ.yorku.ca

Lebrun, B. (1998): “Comparative Statics in First Price Auctions,” *Games and Economic Behavior*, 25, 97-110

Lebrun, B. (1999a): “First Price Auctions in the Asymmetric N Bidder Case,” *International Economic Review*, 40 No1, 125-142.

Lebrun, B. (1999b): "The First Price Auction in the IPV Model," Les Cahiers de Recherches du GREEN (working paper series) #99-15.

Lebrun, B. (2002): "Continuity of the First Price Auction Nash Equilibrium Correspondence," *Economic Theory*, 20, 3, 435-453.

Li, H. and J. Riley (1997): "Auction Choice; a Numerical Analysis," Mimeo.

Marshall, R.C., Meurer, M.J., Richard, J.F., and W. Stromquist (1994): "Numerical Analysis of Asymmetric Auctions," *Games and economic behavior,"* 7, 193-220

Maskin, E. and J. Riley (1996): "Uniqueness in Sealed High Bid Auctions," Mimeo (revised version of UCLA Working Paper #407)

Maskin, E. and J. Riley (2000a): "Asymmetric Auctions," *Review of Economic Studies*, 67, 413-38

Maskin, E. and J. Riley (2000b): "Equilibrium in Sealed High Bid Auctions," *Review of Economic Studies*, 67, 439-454.

Milgrom, P.R. and C. Shannon (1994): "Monotone Comparative Statics," *Econometrica*, 62, 157-80.

Milgrom, P.R. and R.J. Weber (1982): "A Theory of Auctions and Competitive Bidding," *Econometrica*, 50, 1089-122

Pesendorfer, M. (2000): "A Study of Collusion in First-Price Auctions," *Review of Economic Studies*, 67, 381-411.

Porter, R.H. and J.D. Zona (1999): "Ohio School Milk Markets: an Analysis of Bidding," *RAND Journal of Economics*, 30, 2, 263-88.

Riley, J.G. and W.F. Samuelson (1981): "Optimal Auctions," *The American Economic Review*, 71, 3, 381-92

Thomas, C.J. (1997): "Disincentives for Cost-Reducing Investment," *Economics Letters*, 57, 359-63.

Topkis, D.M. (1998): *Supermodularity and Complementarity*, Princeton University Press

Waehrer, K. (1999): "Asymmetric Private Value Auctions with Applications to Joint Bidding and Mergers," *International Journal of Industrial Organization*, 17, 437-52