Supplementary Material to
"Optimality and the English and Second-Price Auctions with Resale"
by

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Zheng (2002) proves that the following condition has to hold true when one-shot deviations are not profitable:

$$\int_{v_i'}^{v_i} \{q_i(w_i, w_i) - q_i(v_i', w_i)\}\, dw_i \geq 0, \text{ (1)}$$

for all v_i, v_i'; where $q_i(w_i', w_i)$ denotes the probability that bidder i would eventually receive the item if he acted at the first stage as if his value was w_i' and then at the subsequent stages as if it was w_i.

Since no deviation from our PBE is profitable, we know that (1) holds true. Nevertheless, we now prove it directly in the case of Section 3, where a uniform bias exists.

We actually prove the stronger "single-crossing" property below:

$$q_i(w_i', w_i) \geq q_i(w_i, w_i), \text{ (2)}$$

for all $w_i' \geq w_i$, and

$$q_i(w_i', w_i) \leq q_i(w_i, w_i), \text{ (3)}$$

for all $w_i' \leq w_i$.

Since $q_2(d_2, d_2) = 1$, (2, 3) always hold true for $i = 2$ and $w_2 = d_2$. Take w_2 in (c_2', d_2). Then:

$$q_2(w_2', w_2) = F_1\left(\omega_1^{-1}\omega_2(w_2)\right) G_2(w_2|w_2') + \int_{w_2}^{w_2'} F_1\left(\omega_1^{-1}\omega_2(b)\right) dG_2(b|w_2'), \text{ (4)}$$

if $w_2' \geq w_2$, and

$$q_2(w_2', w_2) = F_1\left(\omega_1^{-1}\omega_2(w_2)\right), (5)$$

if $w_2' \leq w_2$.

For example, assume bidder 2 with value w_2 bids at auction as if his value was $w_2' > w_2$. With probability $G_2(w_2|w_2')$, he bids in $[r, w_2]$. Whatever his bid b in this range is, he will eventually own the item if and only if bidder 1's value is smaller than $\omega_1^{-1}\omega_2(w_2)$. If bidder 1 with such a value wins the auction, there is no profitable resale. If such a bidder 1 wins the auction, he then demands b as a resale price, which bidder 2 accepts. If bidder 1's value v_1 is above $\omega_1^{-1}\omega_2(w_2)$, he wins the auction and demands the price $\omega_2^{-1}\omega_1(v_1)$, which bidder 2 refuses. The first term in the RHS of (4) follows.

When bidder 2 bids, according to $G_2(.|w_2')$, in $[w_2, w_2']$, the price he will be charged at resale will exceed his value. Consequently, he can receive the item only after winning the auction. The second term in the RHS. of (4) follows. Proving (5) is similar.

(3) with $i = 2$ follows trivially from (5). Integrating (4) by parts and using $G_2(w_2'|w_2') = 1$ give:

$$q_2(w_2', w_2) = F_1\left(\omega_1^{-1}\omega_2(w_2')\right) - \int_{w_2}^{w_2'} G_2(b|w_2')\, dF_1\left(\omega_1^{-1}\omega_2(b)\right).$$

Consequently, for $w_2' \geq w_2$, the inequality (2) is equivalent to:

$$\begin{aligned} & F_1\left(\omega_1^{-1}\omega_2(w_2')\right) - F_1\left(\omega_1^{-1}\omega_2(w_2)\right) \\ \geq & \int_{w_2}^{w_2'} G_2(b|w_2')\, dF_1\left(\omega_1^{-1}\omega_2(b)\right), \end{aligned}$$

which clearly holds true.

Take next w_2 in $[c_2, c_2']$. Then, it is simple to check that $q_2(w_2', w_2) = 0$, for all $w_2' \leq c_2'$. (2) and (3) then follow immediately from the inequality $q_2(w_2', w_2) \geq 0$, which trivially holds true for all w_2'.

Consider next $i = 1$ and w_1 in $[c_1', d_1]$. Then:

$$q_1(w_1', w_1) = F_2\left(\omega_2^{-1}\omega_1(w_1)\right), \ (6)$$

for all $w_1' \geq w_1$;

$$q_1(w_1', w_1) = G_2(r) + \int_r^{\omega_2^{-1}\omega_1\left(w_1'\right)} F_2\left(\omega_2^{-1}\omega_1(w_1)\,|b\right) dG_2(b), \ (7)$$

for all $c_1' < w_1' \leq w_1$; and

$$q_1(w_1', w_1) = 0,$$

for all $w_1' \leq c_1'$. (6) and (7) hold true because bidder 1 with value w_1 always demands a resale price that bidder 2 accepts when his value is larger than $\omega_2^{-1}\omega_1(v_1)$.

The only part of the proof of the single-crossing property that is not immediate is the proof of the inequality $q_1(w_1', w_1) \leq q_1(w_1, w_1)$, for $c_1' < w_1' \leq w_1$. It can proceed as follows:

$$\begin{aligned} q_1(w_1', w_1) &\leq G_2(r) + \int_r^{\omega_2^{-1}\omega_1(w_1)} F_2\left(\omega_2^{-1}\omega_1(w_1)\,|b\right) dG_2(b) \\ &= F_2\left(\omega_2^{-1}\omega_1(w_1)\right) \\ &= q_1(w_1, w_1). \end{aligned}$$

The inequality above and the second equality follow from (7); and the first equality from the fact that bidder 2 does not bid higher than his value.

Finally, the single-crossing property for $w_1 \leq c_1'$ follows from $q_1(w_1', w_1) = 0$, for all $w_1' \leq c_1'$.