Multi-Unit Pay-Your-Bid Auction
with One-Dimensional Multi-Unit Demands

by

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Abstract

An arbitrary number of units of a good is sold to two bidders through a discriminatory auction. The bidders are homogenous ex-ante and their demand functions are two-step functions that depend on a single parameter. In this paper, we give a characterization of the symmetric Bayesian equilibrium and prove its existence and uniqueness. We compare this equilibrium to the equilibrium of the multi-unit Vickrey auction and to the equilibria of the single-unit first price and second price auctions. We examine the consequences of bundling all units for sale into a single package. We study the impact that variations in the demand functions and in the number of units have on the equilbrium, on the bidders'average payoffs per unit, and also on the efficiency of the equilibrium allocation.

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1. Introduction

The discriminatory or "pay-your-bid" auction is a procedure that is used to sell multiple units of a good where each bidder submits a demand curve and where the seller then acts as a standard perfectly discriminating monopolist with zero production cost up to a capacity constraint equal to the number of units to be auctioned. Together with the uniform price auction where the seller acts as a nondiscriminating monopolist, this procedure is one of the most commonly encountered auction procedures. It has been used to sell, for example, bonds, bills, foreign exchange, import quota licenses, airport landing slots, mineral rights, timber rights (see Tenorio 1999, Bikhchandani and Huang 1993, Ausubel and Cramton 1998), SO_2 emission allowances (by the EPA, see Ellerman, Joskow, Schmalensee, Montero, and Bailey 2000), and gold (by the IMF, see Feldman and Reinhart 1995 a and b).

When only one unit is being sold, the discriminatory auction is also called the first price auction. The first price auction and the discriminatory auction with multiple units where every bidder demands only one unit have been studied extensively (see, for the one-unit supply case, the surveys by Mc Affe and McMillan 1987 and Wilson 1992 and see, for the case of one-unit demands, Vickrey 1962 and Weber 1983). Here, we consider the two-bidder case where multiple units are being sold and where both bidders have multiple-unit demands.

In our model, n units of a good are being auctioned and every bidder's inverse demand function is a two-step function, which means that it is constant over $(0, m)$, may exhibit a jump down at m, and is constant over (m, n), with $m \leq n$. A bidder knows only his own demand curve and has subjective beliefs about his opponent's demand curve. This uncertainty is modeled by a random draw of the demand curves at the beginning of the game. After the draw every bidder is informed only of his own curve. We assume that a bidder's inverse demand curve is determined by its first step, or by the bidder's valuation for each one of the first m units. There thus exists a link between a bidder's valuation v_l for each one of his last $m' = n - m$ units and his valuation v_h for each one of his first m units. We make the symmetry assumption that this link is the same for both bidders and we express it by a function g whose value at a high valuation v_h is equal to the bidder's low valuation v_l. The function g is nondecreasing, so we assume that the demand curves do not cross.

Since the demand curves are nonincreasing, we have $g(v) \leq v$, for all v. We assume that $g(v) < v$, for all v in (c, d), or that only the highest and the lowest demand curves may be flat. Because it simplifies the characterization of the equilibrium, we further assume that $g(c) = c$, that is, that the lowest demand curve is flat. As we show in Appendix .. where we study the case $g(c) < c$, by assuming $g(c) = c$ we do not lose any significantly different equilibrium structures. As we will see, the analysis, especially of the case $m > m'$, simplifies considerably when $g(d) = d$, that is, when the highest demand curve is flat. However, throughout we consider the general case where $g(d)$ may be strictly smaller than d because doing otherwise would make us lose interesting equilibrium structures that are particular to the case $g(d) < d$.

Since it determines all the relevant characteristics of a bidder, we also refer to a bidder's valuation for each of his first m units as his type. The initial random draw of the demand curves is thus equivalent to a draw of the types. We assume that the bidders' types are drawn according to identical and independent probability distributions. We denote the common type distribution by F and we assume that, along with the description and rules of the game, it is common knowledge among the two bidders. In a sense, we thus work within an "independent private type" or "independent private value" model. To simplify the analysis, throughout most of our paper we make a further assumption about the rate of increase of the function g which is equivalent to requiring a relation of stochastic dominance between the probability distributions of the high and low valuations.

The seller sets a reserve price that may or may not be binding. Our model is symmetric in that the bidders are ex-ante homogenous and the rules of the games are anonymous.

We study the symmetric equilibria of this game where the bidders use the same "pure" strategy which satisfies some standard regularity assumptions. Although the bidders' real inverse demand curves are two-stepped with a jump down at m, the bidders may submit any nonincreasing demand curve. However, we prove that when they submit "serious" bids, that is, bids above the seller's reserve price, they will submit identical bids for units of identical valuations. They will thus submit two-step demand curves and, in the words of Tenorio (1999), we will observe "lumpy bids". In the special case of our model, where $m = n$, the bidders have flat demand curves and will thus submit flat curves. The equilibrium is then equivalent to the equilibrium of the first price auction where the n units are bundled into a single package. Ausubel and Cramton (1996) fully describe an equilibrium in the case of an arbitrary number of bidders who all have flat demand curves up to some capacity constraints. When these constraints are not binding, this equilibrium comes from the equilibrium of the single-unit first price auction. Here, with two bidders we show that it is the only regular symmetric equilibrium of the discriminatory auction.

From the property of lumpy bidding, any regular symmetric equilibrium is determined by two "bid functions": the serious bid function for high valuation units and the serious bid function for low valuation units. We characterize the inverse bid functions from a symmetric regular equilibrium as solutions of systems of differential equations with boundary conditions. In this characterization, we need to distinguish two cases: $m \leq m'$ and $m > m'$. From this characterization, we prove the existence of a symmetric regular equilibrium. Although results from Jackson and Swinkels (1999) or from Reny (1999) would have given us the existence of an (even pure) equilibrium, these results would not have guaranteed the regularity assumptions we require.

From our characterization, we next prove the uniqueness of the symmetric regular equilibrium, thereby extending our first result about flat demands to our general case of two-step demands. This uniqueness contrasts with the multiplicity of equilibria of the uniform price auction (see Engelbrecht-Wiggans and Kahn 1998b and, although in a common value environment, Back and Zender 1993).

Although our characterization differs according to how the number of high valuation units m compares with the number of low valuation units m', some properties are always displayed by the equilibrium, whatever different values m and m' may take. Among these properties is the strict inequality between the high and low bids for all types in the interior of the type interval. Since in our model the distribution F is atomless there is a probability zero that a bidder will submit a flat demand curve. According to the terminology of Engelbrecht-Wiggans and Kahn (1998a), almost-surely there is no pooling of bids. This property may seem to contradict the results of Engelbrecht-Wiggans and Kahn (1998a). In this paper, the authors consider the discriminatory auction between two bidders with two units where the bidders' types are bi-dimensional. In the framework they use, which was introduced by Noussair (1995) and also studied by Katzmann (1999), a bidder's high and low valuations are the maximum and minimum, respectively, of two independent draws from the same distribution with interval support. The connection between high and low valuations is stochastic and not deterministic as in our model, and the support of the high and low valuation couples is a complete triangle under the 45-degree line. In neighborhoods of the two summits along this line, the support has then nonempty intersections with the "pooling region" where the first order conditions imply the same bid for the first and second units.

However there is no real contradiction with Engelbrecht and Wiggans (1998a). In fact, in our paper the bidders' types are one-dimensional and the support of the high and low valuations is a line, actually the graph of the function g. It then turns out that this graph goes under the endogenous boundary of the pooling region and thus, except for its extremities, lies entirely within the "separating region", as Engelbrecht Wiggans and Kahn (1998a) call it. Notice that this property of almost surely no pooling is not particular to one-dimensional types and that it is robust to a "thickening" of the support of types. It is in fact possible to show[1] that the same boundary between the pooling and separating regions arises for a "thick" support of dimension 2 which approximates the graph of g and which also lies within the separating region. Moreover, even if pooling occurs with probability zero in our model it takes place at the lower extremity of the type interval for all values of m and m' (when $r \leq c$) and at the upper extremity of this interval when $m \leq m'$, and in some cases, when $m > m'$. As in Engelbrecht-Wiggans and Kahn (1998a) we thus also have pooling of bids at the extremities of the valuation supports. The probability of this nonempty event is, however, equal to zero.

We remark that thanks to the strictly smaller bids on low valuation units than on high valuation units, the equilibrium of the multiple-unit discriminatory auction is more efficient than the equilibrium of the single-item discriminatory auction, or the first price auction, when all units for sale are bundled into a single package.

Our choice of one-dimensional types versus Engelbrecht-Wiggans and Kahn (1998a)'s choice of multi-dimensional types is not directly related to the issues of dimensionality addressed by Jehiel and Moldovanu (2000). These authors study the existence of an ex-post incentive compatible mechanism in an environment with multi-dimensional and interdependent types. They show that, generically, no such mechanism exists. However, in our model a bidders' valuations for the units being sold are "totally private" since they depend only on this bidder's type and since the types are independently distributed. In this pure case of private values, it is well known that the Clarke-Groves-Vickrey mechanism is efficient (see Clarke 1971, Groves 1973, and Vickrey 1961). The issue of existence of an efficient mechanism is thus immediate in our model. Choosing one-dimensional types, as Maskin and Riley (1989) do in their study of multi-unit optimal mechanisms, allows us to simplify the analysis, obtain sharp expressions of the properties of the equilibria, and observe similarities between multi-unit auction procedures and single unit auction procedures.

A second property shared by all equilibria and a property which the one-dimensionality of types allows us to express and prove simply is the "more aggressive bidding" on low valuation units than on high valuation units. A similar property was displayed by the example worked out by Engelbrecht-Wiggans and Kahn (1998a). From the first property introduced above, we know that the bid for low valuation units is strictly smaller than the bid for high valuation units. However a bidder of type v, that is, a bidder who values at v each of his first m units, values only at $g(v) < v$ each of his last m' units. Thus if we compare this bidder's low bid $\beta'(v)$ with his high bid $\beta(g(v))$ when his type is equal to $g(v)$, we compare bids for units of same valuation $g(v)$. The property is simply expressed by the inequality $\beta'(v) > \beta(g(v))$. Thus a bidder submits a strictly higher bid for one of his last m' units than for one of his first m units of identical valuation. This property, which tends to "flatten" demand curves, contrasts what happens in the uniform price auction (see, for example, Engelbrecht-Wiggans and Kahn 1998b).

Because of the more aggressive bidding on low valuation units, the equilibrium of the discriminatory auction is not efficient. In fact, with a strictly positive probability a bidder is awarded all n units because his bids were higher than his opponent's, while his valuation for each of his last m' units is lower than his opponent's valuation for each of his first m units. An increase in total welfare could thus be achieved by reallocating $min(m, m')$ units from the highest bidder to the lowest bidder.

As mentioned above, in the case of the one-unit supply case $m = n = 1$ the discriminatory auction is the first price auction. Since the bidders' types are identically distributed in our model, the equilibrium of the discriminatory auction equals the equilibrium of the first price auction in the symmetric case where the bidders are ex-ante homogeneous (see Riley and Samuelson 1981). This is actually a special example of the case $m = n$ of flat demand curves. However, despite the ex-ante homogeneity of the bidders in our model, we observe a link between the equilibria of the discriminatory auction and the equilibria of the single unit first price auction between ex-ante heterogenous bidders, or with different distributions of types. In fact, as Swinkels (1999, pp 509-510) notices: "the presence of multiple unit demands introduces a form of endogenous asymmetry." The link is most apparent in the case of an even number n of units where the jump down in the demand curves occurs exactly at half $n/2$ the total number of units. In this case, a bider wins his first $m = n/2$ units if and only if (neglecting ties) his high bid is larger than his opponent's low bid. The flip side of this situation is that a bidder will receive his second $m' = n/2$ units if and only if his low bid is higher than his opponent's high bid. A bidder's competition for his first $n/2$ units is therefore the other bidder's competition for his last $n/2$ units. A bidder's valuation for each of his last $n/2$ units is $g(v)$, if his valuation for each of his first $n/2$ units is v, and is thus distributed according to $H = F \circ g^{-1}$, which is stochastically dominated by F. Assume that the equilibrium $(\beta, \zeta\ )$ of the single unit first price auction with two bidders where one bidder's valuation distribution is F and where the other bidder's valuation distribution is H is such that the latter bidder's bid $\zeta(g(v))$ at $g(v)$ is not larger than the former bidder's bid $\beta(v)$ at v. Then the equilbirum $(\beta, \zeta\ )$ translates into a symmetric equilibrium of the discriminatory auction where each bidder uses the strategy composed of the bid function β for the high valuation units and $\zeta \circ g$ for the low valuation units. The inequality $\zeta(g(v)) \leq \beta(v)$ actually holds true because of a property of the single unit first price auction with heterogenous bidders according to which the same relation of stochastic dominance passes from the valuation distributions to the bid distributions (see Lebrun 1997, 1999a, or Maskin and Riley 1998).

The link between the discriminatory auction with $m = m' = n/2$ and the first price auction allows us to prove easily statements regarding the discriminatory auction in this case by simply translating known properties of the first price auction with heterogenous bidders. For example, the property of more aggressive bidding for low valuation units that was alluded to above is a direct consequence of the property of more aggressive bidding in the first price auction by the bidder whose valuation distribution is dominated (see Lebrun 1997 and 1999a)[2] . Results of comparative statics are also more easily obtained by using the link with the first price auction. We also use this link to rule out the trade-off, introduced in the next paragraph, between revenue and efficiency in the bundling decision. When m and thus m' are different from $n/2$, there is no such explicit link between equilibria of the discriminatory auction and the first price auction. Still, the methods of proof used in the study of the first price auction with heterogenous bidders are useful in the study of the discriminatory auction.

As the discriminatory auction becomes the first price auction in the one-unit supply case $m = n = 1$, the Clarke-Groves-Vickrey mechanism and its ascending price version with nonincreasing demand curves and independent private values, the Ausubel ascending auction (1995), become the second price auction and the English auction, respectively. Consider the more general model with n units and two bidders with nonincreasing demand curves. Assume that bidder j's valuation for his ith unit is equal the random variable v_i^j, $j = 1, 2$, $i = 1, ..., n$. With probability one we have $v_1^j \geq ... \geq v_n^j$, for $j = 1, 2$. Denote by F_i^j the probability distribution of v_i^j. In this model, with multi-unit supply $n > 1$, there is also an explicit link, albeit different from the link with one-unit supply, between the Vickrey and Ausubel auctions on the one hand and the second price auction on the other hand. In the Vickrey auction, for the ith unit he is awarded a bidder pays the ith highest rejected bid from his opponents. In our case of two bidders, if bidder 1 receives k units he will pay the smallest k bids from bidder 2. We would obtain an equivalent outcome if, for all i, bidder 1 competed for his ith first unit in a second price auction with bidder 2 who, himself actually competed for his ith to the last unit, or for his $(n - i + 1)\, th$ first unit. In this second price auction, the bidders' valuations are in general distributed differently since one bidder's valuation is distributed according to F_i^1 and the other bidder's valuation is distributed according to F_{n-i+1}^2. From this link between the Vickrey auction and the second price auction and a result by Milgrom (2000) and Jehiel and Moldovanu (1999), we prove that for the Vickrey auction in this two-bidder model, bundling all units into a single package decreases efficiency but increases the seller's revenue. Using the connection between the discriminatory auction and the first price auction in our less general model with $m = m' = n/2$, we show no such trade-off in the decision of bundling all units exists for the discriminatory auction.

Although from the Revenue Equivalence Theorem (see, for example, Riley and Samuelson 1981) the first and second price auctions are equivalent when the bidders' valuations are identically distributed, it is also known that with different type distributions there is no general ranking between the seller's revenues at these two auctions. No such general ranking exists between the discriminatory auction on the one hand and the Vickrey and Ausubel auctions, on the other hand, since already none exists in the one-unit supply case. Considering the links introduced above between multi-unit and single-unit auctions, to compare the discriminatory auction with the Vickrey auction with homogeneous bidders when $m = m' = n/2$ is to compare the first price and second price auctions with heterogenous bidders. In our model, the nonexistence of a general ranking between the revenues at these last two one-unit auctions implies the nonexistence of such a ranking between the discriminatory and Vickrey auctions even in the symmetric case of homogenous bidders.

Finally, from our characterizations we derive comparative statics results. We show first that the equilibrium bid functions depend only on the ratio m/m' of the number of high valuation units over the number of low valuation units. We then investigate the effects of changing this ratio. We show that if $m \geq m'$, that is, if there is more high valuation units than low valuation units, an increase in the relative number of high valuation units results in a "deterministic increase" in the bids on high valuation units. Thus, the high bid increases for every type. Furthermore, an "average" of the probability distributions of the low and high bids shifts upwards. This average is the probability distribution a bid on a high valuation unit competes against. In general there is no "deterministic increase" of the low bids. The difference in "bid-shading" between high and low valuation units due to the more aggressive bidding on low valuation units decreases, and the equilibrium allocation is therefore efficient with a higher probability. Each unit contributes a higher average expected surplus to the total welfare. On each unit, every bidder makes a smaller interim and ex-ante expected payoff.

We apply this result to an increase in the number of units supplied when the characteristics of the bidders, and thus of the demand functions, are kept fixed. When the number m of high valuation units is equal to the total number n of units, all units supplied are of identical valuations to each bidder, and the equilibrium of the discriminatory auction therefore equals the equilibrium of the first price auction. When $n = 2m$, the market is split in half between high and low valuation units, and the equilibrium is derived from the equilibrium of the single item first price auction with heterogenous bidders. We show that the average surplus per unit supplied, as a function of the total number of units, decreases monotonically over the interval $[m, 2m]$.

When $m < m'$, that is, when there is a majority of low valuation units, we show that a relative increase in the number of high valuation units will increase the bid function on the low valuation units as well as the "average" bid probability distribution a bid on a low valuation unit competes against. This increase will decrease the bidders' interim and ex-ante expected payoffs on each unit and increase the probability of an inefficient allocation.

We finally briefly mention other possible comparative statics analysis.

Section 2 introduces the model. Then, we characterize the equilibrium and prove its existence and uniqueness in Section 3. In Section 3, we also establish the property of more aggressive bidding on the low valuation units and address the issues of efficiency and bundling. In Section 4, we outline the proofs of the results of Section 3. We study the case with an equal number of high and low valuation units in Section 5. Sections 6 and 7 are devoted to comparative statics. Section 8 concludes. Our proofs can be found in Appendices 1 to 4.

2. The Model

N units of a good are sold to two bidders through a pay-your-bid or discriminatory auction with reserve price r. According to this auction procedure, every bidder submits n bids. All bids are submitted simultaneously. Every bid among the n highest submitted bids not smaller than r entitles the bidder who has submitted it to buy one unit at the price equal to this bid.

We assume that a bidder's demand function is a nondecreasing "two-step" function. That is, there exists an integer m such that the inverse demand function exhibits the following properties: is flat for quantities from 0 to m, may exhibit a jump downward at m, and is flat again for quantities strictly larger than m.

A bidder knows his own demand function with certainty and has some beliefs about the other bidder's demand function. We model this structure of uncertainty by assuming that a bidder's demand function is determined by his "type", which is his private information, and that the bidders' types are distributed according to a probability distribution which is common knowledge to both bidders. We assume that according to this probability distribution, the bidders' types are independent. We thus work within what may be called an "independent private type" model. The bidders are risk-neutral.

We further assume that the "second step" of a bidder's inverse demand function is determined by its "first step". In other words, if one knows a bidder's valuation for his first unit, then one knows his entire demand curve. We can thus assume that a bidder's type is equal to his valuation for his first unit.

We assume that the bidders' types v_1 and v_2 are independently and identically distributed over $[c, d]$, with $c < d$, according to the atomless probability distribution F. We also assume that there exists a strictly increasing[3] continuous function g over $[c, d]$ such that if bidder i's type is v_i then his "high valuation" or valuation for his kth unit is also v_i, for all $1 \leq k \leq m$, and his "low valuation" or valuation for his lth unit is equal to $g(v_i)$, for all $l > m$. When $m = n$, we have the case where both bidders have flat demand curves.

Our main interest lies primarily in the natural case where $g(v) < v$, for all v. It turns out that most of our results can be expressed and derived more simply when $g(c) = c$ or $g(d) = d$, that is, when the lowest or the highest (or both) possible demand curves is flat. As can be seen from Appendix 3 where we examine the case $g(c) < c$, we do not lose any significantly different equilibrium structures by assuming that $g(c) = c$. However, requiring $g(d) = d$ would cause the loss of some original equilibrium structures, especially when $g(d) < d$ and $m > m'$. We thus want a model that allows $g(d) = d$ without imposing it. Consequently, we require only that $g(v) < v$, for all v in (c, d), and $g(c) = c$. We also assume[4] that the reserve price r is strictly smaller than $g(d)$, that is, $r < g(d)$.

We need some technical assumptions. We also denote by F the right continuous cumulative distribution function of the high valuation probability distribution and we assume that this cumulative distribution function is differentiable over $(c, d]$ with a derivative f locally bounded away from zero over this interval.

We assume that the function g is differentiable over $(c, d]$ with a derivative locally bounded away from zero. Furthermore, we require that $\frac{d}{dv}\frac{F \circ g}{F}(v) >: 0$, for all v in $(c, d]$, or, equivalently, that $\frac{d}{dv}\frac{F}{H}(v) > 0$, for all v in $(c, g(d)]$ and thus for all v in $(c, d]$, where H is defined as follows:

$$H(v) = F\left(g^{-1}(v)\right)$$

for all v in $[c, g(d)]$. The function H is the cumulative distribution function of the probability distribution of the low valuations, or the valuations for each one of the m' last units. If the derivative of the ratio F/H is strictly positive over $(c, g(d)]$, then it is strictly increasing over this interval. As it can be easily verified, this property in turn implies that, for any interval $(c, u]$, with $c < u \leq d$, the conditional of F on this interval first order stochastically dominates the conditional of H on the same interval. The assumption is thus an assumption of stochastic dominance between the high and low valuation distributions. An example satisfying this assumption is $F(x) = x$ over $[c, d] = [0, 1]$ and $g(x) = x^2/2$. In this case, $H(x) = \sqrt{2x}$ for all x in $[c, g(d)] = [0, 1/2]$.

Let $b_1 \geq ... \geq b_n \in \mathfrak{R}^n$ be the n bids submitted by bidder 1 and let $b'_1 \geq ... \geq b'_n$ be the n bids submitted by bidder 2. Without loss of generality, we can assume that $b_1 \geq ... \geq b_n$ and $b'_1 \geq ... \geq b'_n$. Let $b_{(n)}$ be the nth-highest bid among all bids submitted by both biders and let D_1 and D_2 be the "quantities demanded" at $b_{(n)}$, such that $D_1 = \#\left\{1 \leq l \leq n \mid b_l \geq b_{(n)}\right\}$ and $D_2 = \#\left\{1 \leq l \leq n \mid b'_l \geq b_{(n)}\right\}$. Any allowable bid b_k, that is, any bid $b_k \geq r$, which is strictly larger than $b_{(n)}$ is a winning bid. When $D_1 + D_2 = n$, there is no "excess demand" so any allowable bid equal to $b_{(n)}$ is also a winning bid. When $D_1 + D_2 > n$, there is excess demand so a tie breaking rule determines which bids among those allowable bids equal to $b_{(n)}$ are winning. Our results hold true for any tie breaking rule. However, to complete the definition of the auction, we assume that the tie is broken according to a fair lottery without replacement. That is, if bidder i has submitted exactly k_i bids strictly larger than $b_{(n)} \geq r$ and k'_i equal to $b_{(n)}$, then the tie breaking rule has to choose $n - k_1 - k_2$ bids among the $k'_1 + k'_2$ bids tying at $b_{(n)}$. One bid is first chosen according to a fair lottery. Then a second unit nit is chosen according to a fair lottery among the remaining tying bids, and so on. The probability of the tie being broken l times in favor of bidder i, or the probability that exactly $k_1 + l$ bids from bidder i be winning, is thus$\binom{l}{k'_1}\binom{n-k_1-k_2-l}{k'_2}/\binom{n-k_1-k_2}{k'_1+k'_2}$.

If b_k is a winning bid from a bidder with type v, then this bid will contribute the amount

$$v - b_k, \text{ if } k \leq m$$

$$g(v) - b_k, \text{ if } k > m$$

to the bidder's payoff. A bidder's payoff is separable in the different units since it is the sum of all the contributions from his winning bids.

A (pure) strategy is an "ordered" n-tuple of bid functions $\beta_1, ..., \beta_n$ from $[c, d]$ to $\mathcal{R}$, that is, $\beta_1(v) \geq ... \geq \beta_n(v)$, for all v in $[c, d]$. The function β_k is thus the kth highest bid function or the bid function for the kth unit.

Let σ be a strategy $(\beta_1, ..., \beta_n)$. A regular strategy σ is a strategy $(\beta_1, ..., \beta_n)$ such that the bid functions $\beta_1, ..., \beta_n$ are strictly increasing continuous functions over $[c, d]$ such that the bids never exceed the valuations. Moreover, $\beta_1, ..., \beta_n$ must be differentiable with strictly positive and possibly infinite derivatives everywhere except possibly when their values belong to the set $E = \{r, \beta_1(c), ..., \beta_n(c), \beta_1(d), ..., \beta_n(d)\}$. When their values belong to the set $M = \{\beta_1(d), ..., \beta_n(d)\}$ they are only required to have strictly positive and possibly infinite left-hand and right-hand derivatives. We thus allow for infinite derivatives. We also allow for nondifferentiability when the bid is equal to the reserve price or to the maximum or minimum bid of a possibly different bid function in the n-tuple. The reason we allow for this nondifferentiability is that the incentives faced by a bidder in a symmetric equilibrium may be different below the maximum, say, of a bid function β_k where he must take into account his opponent's kth bid than these incentives above this maximum, where the opponent's kth to the nth bids play no role. We allow for this difference in incentives to imply a difference in the slopes of the bidder's bid functions. Thus, for all v in $[c, d]$, $\beta_i\,(v) \leq v$, for all $i \leq m$, and $\beta_i\,(v) \leq g\,(v)$, for all $i > m$. When $\beta_i\,(v) \notin E$, $\frac{d}{dv}\beta_i\,(v)$ exists, can be infinite, and is strictly positive. When $\beta_i\,(v) \in M$, $\frac{d_l}{dv}\beta_i\,(v)$ and $\frac{d_r}{dv}\beta_i\,(v)$ exist, can be infinite, and are strictly positive[5].

A symmetric regular equilibrium $(\sigma; \sigma) = (\beta_1, ..., \beta_n; \beta_1, ..., \beta_n)$ is a Bayesian equilibrium where both bidders use the same regular strategy.

3. The Equilibria

Let $\sigma = (\beta_1, ..., \beta_n)$ be a regular strategy. Assume bidder 2 follows σ, and assume bidder 1 has type v and submits $b_1 \geq ... \geq b_n$. If bidder 2 submits $b'_1 \geq ... \geq b'_n$, then bidder 1's bid b_k will certainly be among the n highest submitted bids when $b_k > b'_{n-k+1}$. When $b_k < b'_{n-k+1}$, then b_k will certainly not belong to the n highest submitted bid. When $b_k = b'_{n-k+1}$, then b_k will tie with bids from bidder 2 at the nth highest submitted bid and the tie will be broken in favor of b_k with a certain strictly positive probability. Following the terminology in Reny (1999, footnote 36), the bids b_k and b'_{n-k+1} are "competing". From the definition of regular strategies, the probability of ties are equal to 0. Consequently, the contribution of $b_k \geq r$ to bidder 1's expected payoff is equal to $(v - b_k)\,F\left(\gamma_{n-k+1}\,(b_k)\right)$, if $k \leq m$, and to $(g\,(v) - b_k)\,F\left(\gamma_{n-k+1}\,(b_k)\right)$, if $k > m$, where γ_{n-k+1} is the "extended" inverse of β_{n-k+1}. By extended inverse of β_{n-k+1}, we mean that $\gamma_{n-k+1}\,(b) = \beta^{-1}_{n-k+1}\,(b)$, for all b in $\left[\beta_{n-k+1}\,(c)\,, \beta_{n-k+1}\,(d)\right]$, $\gamma_{n-k+1}\,(b) = c$, for all $b \leq \beta_{n-k+1}\,(c)$, and $\gamma_{n-k+1}\,(b) = d$, for all $b \geq \beta_{n-k+1}\,(d)$. Obviously, the contribution of $b_k < r$ is equal to 0. We have thus proved Lemma 1 below where $I\,\{b_i \geq r\}$ is the indicatrix of $\{b_i \geq r\}$ or is equal to 1 if $b_i \geq r$ and is equal to 0 otherwise.

Lemma 1: *Let $(\beta_1, ..., \beta_n)$ be a regular strategy. Then $(\beta_1, ..., \beta_n; \beta_1, ..., \beta_n)$ is a symmetric regular equilibrium if and only if*

$$(\beta_1\,(v)\,, ..., \beta_n\,(v)) \in \arg \max_{\substack{(b_1, ..., b_n) \\ b_1 \geq ... \geq b_n}} \mathcal{P}\,(v; b_1, ..., b_n) \quad (3\text{-}1)$$

where

$$\mathcal{P}\,(v; b_1, ..., b_n) = \sum_{i=1}^{m} (v - b_i)\,F\left(\gamma_{n-i+1}\,(b_i)\right) I\,\{b_i \geq r\} + \sum_{j=m+1}^{n} (g\,(v) - b_j)\,F\left(\gamma_{n-j+1}\,(b_j)\right) I\,\{b_j \geq r\}$$

for all v in $[c, d]$.

Some of our results vary according to whether the majority of units is of high or low valuations. Let m' be the number of low valuation units, such that $m' = n - m$. Theorem 1 below follows from Lemma 1. We sketch its proof in the next section. A detailed proof can be found Appendix 1.

Theorem 1: *("lumpy bidding" (a), different bids for high and low valuation units (b), and boundary conditions (c), (d)): Let $(\sigma, \sigma) = (\beta_1, ..., \beta_n; \beta_1, ..., \beta_n)$ be a symmetric regular equilibrium and let $\underline{c}$ be the maximum of r and c, that is, $\underline{c} = \max(r, c)$.*

(a) $\beta_1(v) = ... = \beta_m(v)$, $\beta_{m+1}(w) = ... = \beta_n(w)$, for all v in $[\underline{c}, d]$ and all w in $\left[g^{-1}(\underline{c}), d\right]$

(b) $\beta_1(v) = ... = \beta_m(v) > \beta_{m+1}(v), ..., \beta_n(v)$, for all v in $(\underline{c}, d)$

(c) $\beta_1(\underline{c}) = ... = \beta_m(\underline{c}) = \beta_{m+1}\left(g^{-1}(\underline{c})\right) = ... = \beta_n\left(g^{-1}(\underline{c})\right) = \underline{c}$.

(d) If $g(d) = d$ or if $m \leq m'$, then $\beta_1(d) = ... = \beta_n(d)$

To prove Theorem 1 (see the next section and Appendix 1) we first establish (c) and (d) and the equalities $\beta_1(d) = ... = \beta_m(d)$ and $\beta_{m+1}(d) = ... = \beta_n(d)$. We then prove (a) by proving the equivalent equalities for the inverse bid functions. For example, the first equalities in (a) are equivalent to $\gamma_1(b) = ... = \gamma_m(b)$, for all b in $[\underline{c}, \beta_m(d)]$. We know that these last equalities hold true at the extremities of this interval. We show that if they did not hold true everywhere in this interval then there would exist two consecutive groups which "separate" at a bid b^*, that is, there would exist $1 \leq i \leq j \leq m-1$, $(j+1) \leq k \leq m$, and $\varepsilon > 0$ such that $\gamma_{i-1}(b) < \gamma_i(b) = ... = \gamma_j(b) < \gamma_{j+1}(b) = ... = \gamma_k(b) < \gamma_{k+1}(b)$, for all b in $(b^* - \varepsilon, b^*)$, and $\gamma_i(b^*) = ... = \gamma_j(b^*) = \gamma_{j+1}(b^*) = ... = \gamma_k(b^*)$. We finally prove (a) by ruling out such "separations".

According to (c) in Theorem 1, bidders submit their valuation on all units of smallest possible valuation $\underline{c}$ at least equal to the reserve price. In fact, a type $g^{-1}(\underline{c})$ bidder has a valuation of $\underline{c}$ for any ith unit, with $i \geq m+1$. According to (d), if the highest possible demand curve is flat or if there are at least as many low valuation units than high valuation units, then all bid functions have the same maximum. This is to say that the highest possible submitted demand curve is flat. Below, we will show equilibria when $g(d) < d$ and $m > m'$where the bid functions for the high valuation units have a strictly larger maximum than the bid functions for the low valuation units.

From Theorem 1 (a) the equilibrium bid functions for units of identical valuations are identical over the range of types where bids are not smaller than the reserve price. When a bidder submits "serious" bids, that is, bids at least equal to the reserve price, he will submit "lumps" of bids: m identical bids for his high valuation units and m' identical bids for his low valuation units. The serious bidding behaviors at any regular symmetric equilibrium $(\beta_1, ..., \beta_n; \beta_1, ..., \beta_n)$ are thus completely determined by two functions: the serious bid function for the first m units, which is defined over $[\underline{c}, d]$, and the serious bid function for the remaining units, which is defined over $\left[g^{-1}(\underline{c}), d\right]$. We denote the former function by β and the latter by β', such that $\beta_1 = ... = \beta_m = \beta$ over $[\underline{c}, d]$ and $\beta_{m+1} = ... = \beta_n = \beta'$ over $\left[g^{-1}(\underline{c}), d\right]$. The equilibrium bid functions $\beta_1, ..., \beta_n$ outside these respective intervals cannot take values larger than the valuations but are otherwise arbitrary and uninteresting. We focus on the serious parts β and β' of the equilibrium bid functions and neglect their uninteresting components below the reserve price. We denote then the equilibrium simply by (β, β').

In the particular case of flat demand curves, that is, when there is no low valuation unit for sale and when thus $m = n$, Theorem 1 (a) implies that both bidders use the same bid function on all their units. At an equilibrium, we can thus assume that the constraints in the maximization problem of Lemma 1 are $b_1 = ... = b_n$. The objective function or the expected payoff then reduces to $n(v-b)F(\gamma(b))I\{b \geq r\}$, where $\gamma = \beta^{-1}$ and where b is the common bid on all units. Up to the constant factor n, this is the same expected payoff that a bidder maximizes at the equilibrium of the first-price auction where the bidders' valuations for the single item for sale are distributed according to F. From Riley and Samuelson (1981), the only symmetric equilibrium of this auction is given by the formula in Corollary 1 below.

Corollary 1: *(the case of flat demand curves) If $m = n$, there exists one and only one regular symmetric equilibrium of the discriminatory auction. This equilibrium is one where both bidders use the same bid function β on all their units, such that*

$$\beta(v) = \frac{\underline{c}F(\underline{c}) + \int_{\underline{c}}^{v} w dF(w)}{F(v)}$$

for all v in $[\underline{c}, d]$.

To end the proof of Corollary 1 above, we still have to show that the bid function β defines an equilibrium of the discriminatory auction. Since β defines an equilibrium of the first price auction, $\beta(v)$ is a solution of the unconstrained problem $\max_b (v-b)F(\gamma(b))I\{b \geq r\}$, for all $v \geq \underline{c}$. It is then immediate that $(\beta(v), ..., \beta(v))$ is a solution of the constrained problem $\max_{\substack{(b_1,...,b_n) \\ b_1 \geq ... \geq b_n}} \sum_{i=1}^{m} (v-b_i)F(\gamma(b_i))I\{b_i \geq r\}$ and, from Lemma 1, β thus also defines an equilibrium of the discriminatory auction when the demand curves are flat. Corollary 1 above is proved. Since the single-item first price auction in the symmetric model where the bidders are homogeneous ex-ante has been extensively studied in the literature, we focus in the rest of the paper on the case where the demand curves are not flat, that is, where $m' \neq 0$ and thus $m < n$.

From Lemma 1 and Theorem 1, we prove the existence of equilibria and characterize them. Theorem 2 below is our existence and uniqueness result and

Theorems 3 and 4 give our characterizations.

Theorem 2: *(existence and uniqueness of the equilibrium) There exists one and only one symmetric regular equilibrium of the pay-your-bid-auction.*

Theorem 3: *(not larger number of high valuation units than of low valuation units) Assume that $0 < m \leq m'$. Then, there is one and only one $\underline{c} < \eta < g(d)$, such that there exists a solution (γ, σ') over $(\underline{c}, \eta]$ to the following system of differential equations with boundary conditions:*

$$\frac{d}{db}\ln F(\gamma(b)) = \frac{1}{\sigma'(b) - b}\left\{1 + \left(\frac{m'}{m} - 1\right)\frac{H(\sigma'(b))}{F(\gamma(b))}\left(\frac{\gamma(b) - \sigma'(b)}{\gamma(b) - b}\right)\right\}$$

$$\frac{d}{db}\ln H(\sigma'(b)) = \frac{1}{\gamma(b) - b}$$

$$\gamma(\underline{c}) = \sigma'(\underline{c}) = \underline{c}, \gamma(\eta) = d, \sigma'(\eta) = g(d).$$

Furthermore, the unique symmetric regular equilibrium (β, β') can be obtained from (γ, σ') and vice-versa through the equations $\gamma = \beta^{-1}, \sigma' = g \circ \beta'^{-1}$, and thus $\beta = \gamma^{-1}, \beta' = \sigma'^{-1} \circ g$.

Theorem 4: *(strictly larger number of high valuation units than of low valuation units, see Figure 1) Assume that $m > m' > 0$. Then there are one and only one $g(d) \leq d' \leq d$ and one and only one $\underline{c} < \eta' < d$, with $g(d) < d'$ if $g(d) < d$, such that there exists a solution (γ, σ')* over *$(\underline{c}, \eta']$ to the following system of differential equations with boundary conditions:*

$$\frac{d}{db}\ln F(\gamma(b)) = \frac{1}{\sigma'(b) - b} \text{(3-2)}$$

$$\frac{d}{db}\ln H(\sigma'(b)) = \frac{1}{\gamma(b) - b}\left\{1 + \left(\frac{m}{m'} - 1\right)\frac{F(\gamma(b))}{H(\sigma'(b))}\left(\frac{\sigma'(b) - \gamma(b)}{\sigma'(b) - b}\right)\right\} \text{(3-3)}$$

$$\gamma(\underline{c}) = \sigma'(\underline{c}) = \underline{c}, \gamma(\eta') = d', \sigma'(\eta') = g(d) \text{(3-4)}$$

with

$$\eta' = g(d) + \left(\frac{m}{m'} - 1\right)(g(d) - d')F(d') \text{ if } d' < d \text{(3-5).}$$

Moreover, the unique symmetric regular equilibrium (β, β') can be obtained over $[\underline{c}, d']$ from (γ, σ') through the equations $\gamma = \beta^{-1}, \sigma' = g \circ \beta'^{-1}$,or, equivalently, $\beta = \gamma^{-1}, \beta' = \sigma'^{-1} \circ g$. The bid function β over $[d', d]$ is given by the following equation:

$$\beta(v) = v - \frac{(d' - \eta')\left(1 + \left(\frac{m}{m'} - 1\right)F(d')\right) + \int_{d'}^{v}\left(1 + \left(\frac{m}{m'} - 1\right)F(u)\right)du}{1 + \left(\frac{m}{m'} - 1\right)F(v)} \text{(3-6).}$$

Since $\sigma'(b) = g \circ \beta'^{-1}(b)$ is equal to the valuation of any of the last m' units for which the bidder bids b on these units, and since a bidder's type is equal to his valuation for any of the first m units, we can say that the systems and boundary conditions in Theorems 3 and 4 are written in the "valuation" space. Equivalent expressions in the "type" space can be easily obtained. For example, the system in Theorem 4 is equivalent to the system below where $\gamma' = \beta'^{-1}$:

$$\frac{d}{db}\ln F(\gamma(b)) = \frac{1}{g(\gamma'(b)) - b} \text{(3-7)}$$

$$\frac{d}{db}\ln F(\gamma'(b)) = \frac{1}{\gamma(b) - b}\left\{1 + \left(\frac{m}{m'} - 1\right)\frac{F(\gamma(b))}{F(\gamma'(b))}\left(\frac{g(\gamma'(b)) - \gamma(b)}{g(\gamma'(b)) - b}\right)\right\} \text{(3-8)}$$

$$\gamma(\underline{c}) = \underline{c}, \gamma'(\underline{c}) = g^{-1}(\underline{c}), \gamma(\eta') = d', \gamma'(\eta') = d \text{(3-9)}$$

for all b in $(\underline{c}, \eta']$.

The necessity of the differential systems in Theorems 3 and 4 above follows easily from Lemma 1 and Theorem 1. For example, consider the case $m > m'$. From Theorem 1, the bid functions for identical valuation units are identical and so are thus the inverse bid functions. Consequently, the objective function in Theorem 1 when a type v bidder has to submit the same bid b for his first m units and the same bid b' for his last m' units is equal to[6], up to the factor m', $(v-b)F(\gamma'(b)) + \left(\frac{m}{m'} - 1\right)(v-b)F(\gamma(b)) + (g(v) - b')F(\gamma(b'))$. The derivative with respect to b' of this expression is

$$-F(\gamma(b')) + (g(v) - b')\frac{d}{db}F(\gamma(b')) \text{ (3-10).}$$

The first term is the marginal cost, due to an increase of payment in case of winning, of an increase of the lower bid. The second term is the marginal benefit, due to an increase in the probability of winning low valuation units, of such a change. At the equilibrium, marginal cost and benefit are equal at the lower bid the bidder submits. Since the bidder submits $\beta'(v)$ when his type is v in $\left(g^{-1}(\underline{c}), d\right)$ or b in $(\underline{c}, \eta')$ when his type is $\gamma'(v)$, we find the first equation of the characterization in Theorem 4 of the inverse bid functions over $(\underline{c}, \eta')$.

Taking the derivative of the objective function with respect to b gives, up to the factor m',

$$-F(\gamma'(b)) - \left(\frac{m}{m'} - 1\right)F(\gamma(b)) + (v-b)\frac{d}{db}F(\gamma'(b)) + \left(\frac{m}{m'} - 1\right)(v-b)\frac{d}{db}F(\gamma(b)) \text{ (3-11).}$$

The first two terms make up the marginal cost of an increase of the higher bid and the two last terms are its marginal benefit. When b lies in $(\underline{c}, \eta')$, we can substitute to $\frac{d}{db}F(\gamma(b))$ in this derivative its value from the first differential equation. Solving for $\frac{d}{db}F(\gamma'(b))$ gives the second differential equation of the characterization over $(\underline{c}, \eta')$.

When b is strictly higher than η', b is strictly larger than the opponent's low bid and $F(\gamma'(b))$ is identically equal to 1 and its derivative therefore vanishes.

The marginal effect of a change of high bid is only

$$-1-\left(\frac{m}{m'}-1\right)F\left(\gamma\left(b\right)\right)+\left(\frac{m}{m'}-1\right)\left(v-b\right)\frac{d}{db}F\left(\gamma\left(b\right)\right).$$

Since a bidder submits such a bid b for $v=\gamma\left(b\right)>d'$, this marginal change must be equal to zero at such a type. We thus find a differential equation with only one unknown function γ. An initial condition is given by $\gamma\left(\eta'\right)=d'$. Proceeding as in Riley and Samuelson (1981), solving this equation gives the expression for $\beta\left(v\right)$, with $v\geq d'$, in Theorem 4.

The necessity of the link (3-5) between η' and d' is also easily established. Since the function σ' is nondecreasing, its derivative at η' is nonnegative. From the initial condition at η' and from the second differential equation in Theorem 4, we find $\frac{d}{db}\ln H\left(\sigma'\left(\eta'\right)\right)=\left\{1+\left(\frac{m}{m'}-1\right)F\left(d'\right)\left(\frac{g(d)-d'}{g(d)-\eta'}\right)\right\}/\left(d'-\eta'\right)\geq 0$. The factor between braces must be nonnegative. Rearranging this factor, we find the inequality $\eta'\leq g\left(d\right)+\left(\frac{m}{m'}-1\right)\left(g\left(d\right)-d'\right)F\left(d'\right)$. The reverse inequality is obtained by ruling out profitable increases of the type d bidder's low bid. Assume $d'<d$ and thus $\eta'<\eta$. A type d bidder can thus increase slightly his bid on his last m' units while still satisfying the constraints in the maximization problem of Lemma 1. Such an increase will change only the term $m'\left(g\left(d\right)-b'\right)F\left(\gamma\left(b'\right)\right)$. The logarithmic derivative of this term is $-1/\left(g\left(d\right)-b'\right)+\frac{d}{db}\ln F\left(\gamma\left(b'\right)\right)$ and it must be nonpositive at $b'=\eta'$, otherwise it will be in the bidder's interest to raise his bid above η'. Substituting to $\frac{d}{db}\ln F\left(\gamma\left(b'\right)\right)$ its value from the differential equation used to derive the expression for β above η', substituting η' to b', and using the initial condition at η', we find the reverse inequality, and the link (3-5) between η' and d' in Theorem 4 must therefore hold true. Notice that this link implies that the function γ is differentiable at the border η' of the two "regimes" and thus everywhere over $(\underline{c},\eta]$. In fact, the value of $\frac{d_l}{db}\ln F\left(\gamma\left(\eta'\right)\right)$ obtained from the differential equation (3-2) is identical to the value of $\frac{d_r}{db}\ln F\left(\gamma\left(\eta'\right)\right)$ obtained from the differential equation we solved to find the expression (3-6).

The proofs of the sufficiency of the characterizations in Theorems 3 and 4, as well as the proof of the existence and uniqueness of the equilibrium (Theorem 2), are similar to the proof of the sufficiency of the characterization of the first price auction equilibria in Lebrun (1997,1999a). We actually first prove the characterization in Theorems 3 and 4 of the equilibrium bid functions as inverses of the solutions of differential systems with boundary conditions. From these characterizations, we prove the existence and uniqueness of the equilibrium (Theorem 2). In these proofs, the case $m>m'$ of a larger number of high valuation units is the least straightforward. The complete proofs can be found in Appendix 2. We sketch these proofs in the next section.

The differential systems in Theorems 3 and 4 above are singular at the first initial condition $\gamma\left(\underline{c}\right)=\sigma\left(\underline{c}\right)=\underline{c}$ in their respective boundary conditions. Consequently, in the existence and uniqueness proofs we cannot apply the standard theorems of the theory of ordinary differential equations to these systems with this initial condition. As already stated in Theorem 1 (d), the second part

$\gamma(\eta) = d, \sigma(\eta) = g(d)$ of the boundary condition in the characterization in Theorem 3 implies that when $m \leq m'$, the functions β and β' have the same maximum η. We can consider the solution of the differential system with only the second part of the boundary condition as an initial condition. In this way, η is a parameter which determines the corresponding solution of the differential system. Finding an equilibrium will then be equivalent to finding a value of this parameter such that the corresponding solution also satisfies the first part $\gamma(\underline{c}) = \sigma(\underline{c}) = \underline{c}$. This last initial condition uniquely determines the value of the unknown parameter η.

The situation is different when $m > m'$ and $g(d) < d$. In this case, the maximum $\eta = \beta(d)$ of β can be strictly larger than the maximum of $\eta' = \beta'(d)$ of β' and thus $d' = \beta^{-1}\left(\beta'(d)\right)$ can be strictly smaller than d. Over the common range $[\underline{c}, \eta']$ of β and β', where the bids on the first units of a bidder compete with the bids on the last units of the other bidder, the differential system in Theorem 4 must be satisfied. Now the initial condition $\gamma(\eta') = d', \sigma(\eta') = g(d)$, which is the second part of the boundary condition includes the *two* parameters η' and d'. However, the relationship (3-5) in Theorem 4 that links η' and d' decreases by one the number of "degrees of freedom", so the other initial condition $\gamma(\underline{c}) = \sigma(\underline{c}) = \underline{c}$ in the boundary condition will again uniquely determine η', d', and the equilibrium.

Although the characterization of the equilibria depends on the number of high valuation units relative to the number of low valuation units, all equilibria display the property stated in Theorem 5 below.

Theorem 5: *("more aggressive" bidding on the low valuation units) Let* $\left(\beta, \beta'\right)$ *be a symmetric regular equilibrium. Then we have*

$$\beta(g(v)) < \beta'(v)$$

for all v *in* $(\underline{c}, d]$ *.*

The value $\beta'(v)$ is the bid a type v bidder submits for any one of his last m' units. Such a bidder has a valuation $g(v)$ for each one of these units. The value $\beta(g(v))$ is the bid a type $g(v)$ bidder would submit for anyone of his first m units. His valuation for each such unit is $g(v)$. In the inequality in Theorem 5, we thus compare bids on units of equal valuations. We find that a bidder would bid higher for a unit of fixed valuation if this unit was one of his last m' units than if it was one of his first m units. Thus in a sense, bidders use more aggressive bidding for their low valuation units than for their high valuation units.

An intuition for Theorem 5 is as follows. A bidder's bids on his last m' units compete with his opponent's bids on his first m' units. Among those units, will be $\min(m, m')$ units that his opponent highly values. On the other hand, a bidder's bids on his first m units compete with his opponent's bids on his last m units, among which $\min(m, m')$ have low valuations. Thus $m - \min(m, m')$ have high valuations. Per low valuation unit, a bidder will thus face the competition coming from his opponent's $\frac{\min\left(m,m'\right)}{m'}$ high valuation units and per high valuation unit he will face the competition coming from his opponent's $\frac{m-\min\left(m,m'\right)}{m} = 1-\frac{\min\left(m,m'\right)}{m}$ high valuation units. From the obvious inequality $\min\left(m,m'\right)\left(\frac{1}{m}+\frac{1}{m'}\right) > 1$, we see that because it comes from a higher number of his opponent's high valuation units, the competition that a bidder faces on a low valuation unit is likely to be fiercer that the competition he faces on a high valuation unit. It is thus not surprising that his bidding will be more aggressive on a low valuation unit.

The formal proof of Theorem 5 is different according to whether $m \leq m'$ or $m > m'$. In the case where there is a majority of high valuation units, so that $m > m'$, Theorem 5 is easily proved by ruling out profitable increases of the bidder's high bid when his type lies in the interval $(\underline{c}, d')$. In the maximization problem of Lemma 1, the constraints will obviously still be satisfied if the highest bid is increased. Such an increase will only change the first term $(v-b)\,F\left(\gamma'\left(b\right)\right)$ of the objective function. The logarithmic derivative of this term is $-1/\left(v-b\right)+\frac{d}{db}\ln F\left(\gamma'\left(b\right)\right)$ and it must be nonpositive at $\gamma\left(b\right)$; otherwise, it will be in the type $\gamma\left(b\right)$ bidder's interest to increase his first bid above $b=\beta\left(\gamma\left(b\right)\right)$. For $b \leq \eta'$, substituting $\gamma\left(b\right)$ to v and its value from the second differential equation in Theorem 4 to $\frac{d}{db}\ln H\left(\sigma'\left(b\right)\right) = \frac{d}{db}\ln F\left(\gamma'\left(b\right)\right)$, we see that this derivative is equal, up to the strictly positive factor $\frac{1}{\left(\gamma(b)-b\right)}$, to the second term $\left(\frac{m}{m'}-1\right)\frac{F(\gamma(b))}{F(\gamma'(b))}\left(\frac{g\left(\gamma'(b)\right)-\gamma(b)}{g(\gamma'(b))-b}\right)$ between braces in the R.H.S. of the differential equation (3-3). Consequently, $g\left(\gamma'\left(b\right)\right) \leq \gamma\left(b\right)$ for all b in $(\underline{c}, \eta')$ or, equivalently by taking the inverses, $\beta'\left(v\right) \geq \beta\left(g\left(v\right)\right)$ for all v in $(\underline{c}, d')$.

From the characterization of the equilibrium, we see that the only kind of inefficient allocation or "misallocation" that can occur at the equilibrium happens when a bidder is awarded all units because his low bid is higher than his opponent's high bid and when the bidder's valuation for each one of his last m' units is smaller than his opponent's valuation for each one of his first m units. That is, a misallocation to the benefit of bidder 2 occurs if and only if, neglecting ties, bidder 2's low bid $\beta'\left(v_2\right)$ is larger than bidder 1's high bid $\beta\left(v_1\right)$ and when bidder 2's low valuation $g\left(v_2\right)$ is smaller than bidder 1's high valuation v_1. The allocation could be improved in the sense of Pareto if $\min\left(m,m'\right)$ units were transferred from bidder 2 to bidder 1 (with an appropriate transfer of money). Notice that $\beta'\left(v_2\right) \geq \beta\left(v_1\right)$ if and only if $g\left(v_2\right) \geq \sigma' \circ \beta\left(v_1\right)$. We thus have the following lemma.

Lemma 2: *(the set of realizations of types for which the equilibrium allocation is inefficient, see Figure 2) Let* $\left(\beta,\beta'\right)$ *be the unique symmetric regular equilibrium. Then almost surely, a misallocation to the benefit of bidder j occurs if and only if bidder i's type* v_i *and bidder j's type* v_j *are such that* $\varphi\left(v_i\right) \leq g\left(v_j\right) \leq v_i$, *where* $\varphi = g \circ \beta'^{-1} \circ \beta = \sigma' \circ \beta$ *and* $i \neq j$.

From Theorem 5, $\beta'\left(v\right) > \beta\left(g\left(v\right)\right)$ or, equivalently, $\varphi\left(v\right) < v$, for all v in (c, d), and with a strictly positive probability an inefficient allocation takes place at the equilibrium. Bidders bid more aggressively on their last m' units. Consequently, a bidder's bids on his last units can be larger than his opponent's bids on his first units while having lower valuations than his opponent's for those units. If β' was equal to $\beta \circ g$ and thus if φ was equal to the identity function, there would be no difference in bid shading on high and low valuation units, and the equilibrium would therefore be efficient. In our framework of "pure private values", it is well known that the Clarke-Groves-Vickrey mechanism gives efficient results. The Vickrey auction and the Ausubel auction, which implements the Vickrey auction when the demand functions are nonincreasing, would then be efficient auctions in our model. We compare these two auctions to the discriminatory auction in the next section.

If the n units were bundled into one single package, the discriminatory auction would reduce to the single-item first price auction with homogenous bidders. The only equilibrium in this case is the symmetric equilibrium where both bidders use the same bid function β_b such that[7] $\beta_b(v) = \frac{n\underline{c}F(\underline{v})+\int_{\underline{v}}^{v}\left(mw+m'g(w)\right)dF(w)}{F(v)}$ (see Riley and Samuelson 1981, Lebrun 1997, 1999a, Maskin and Riley 1998), where $\underline{v}$ is the solution of the equation $m\underline{v} + m'g(\underline{v}) = n\underline{c}$. The equilibrium is equivalent to the equilibrium of a discriminatory auction where bidders are limited to bidding the same amount on all units and thus to using the same bid function β_b for all units. In Lemma 2 above, the function φ would thus be equal to $g \circ \beta_b^{-1} \circ \beta_b = g$. Since at the equilibrium of the discriminatory auction, $\varphi(v) = g \circ \beta'^{-1} \circ \beta(v) > g(v)$, for all v in (c, d), which is to say that bids on low and high valuation units are different, a better allocation of units can take place. The equilibrium of the discriminatory auction without bundling is thus strictly more efficient than with bundling. We therefore have the corollary that follows below.

Corollary 2: *(bundling is more inefficient) If a realization of types results in an inefficient allocation of units in the equilibrium of the discriminatory auction without bundling, then it also results in an inefficient allocation of units with bundling. Moreover, the probability that there will be an inefficient allocation of units in the equilibrium with bundling is strictly higher than in the equilibrium without bundling.*

When k heterogenous goods are being sold through k second price auctions to two bidders with separable and independent utility functions, Milgrom (2000) and Jehiel and Moldovanu (1999) show that there is a trade-off in the decision of bundling between increasing the seller's revenue and increasing efficiency. They show that a coarser partition of the goods or more bundling leads to lower efficiency and higher expected revenue. We now show how this result implies that when n units of a homogenous good are put for sale through a Vickrey auction to two bidders with independent nonincreasing demand curves, bundling all units into a single package also decreases efficiency and increases revenues.

As in the introduction, consider the model where bidder j's valuation for his ith unit is equal the random variable v_i^j, $j = 1, 2$, $i = 1, ..., n$. With probability one we have $v_1^j \geq ... \geq v_n^j$, for $j = 1, 2$, and we take the n-tuple $(v_1^j, ..., v_n^j)$ to be bidder j's type. Denote by F_i^j the marginal probability distribution of v_i^j. This model is more general than the model we have studied so far since the bidders' demand curves are not required to be two-stepped and the bidder's types may be multi-dimensional. In the Vickrey auction, if a bidder receives k units he will pay the smallest k bids from the other bidder. Since in the unique equilibrium in weakly dominating strategies of the Vickrey auction bidders submit their true valuations, the seller would obtain the same revenue if, for all i, bidder 1 competed for his ith first unit in a second price auction with bidder 2 who in turn actually competed for his ith to the last unit or his $(n-i+1)\,th$ first unit. In this second price auction, bidder 1's valuation v_i^1 is distributed according to F_i^1 and bidder 2's valuation v_{n-i+1}^2 is distributed according to F_{n-i+1}^2. Consequently, the seller would obtain the same expected revenue in the sale of n different objects to two bidders through n second price auctions, where bidder 1's valuation $w_i^1 = v_i^1$ for object i is distributed according to F_i^1 and bidder 2's valuation $w_i^2 = v_{n-i+1}^2$ for the same object is distributed according to F_{n-i+1}^2, for $1 \leq i \leq n$. From the result by Milgrom (2000) and Jehiel and Moldovanu (1999), this expected revenue is smaller than the expected revenue the seller would obtain if he bundled all units in the same package. If the seller does so, his expected revenue will then be equal to $E\left(\min\left(w_1^1 + ... + w_n^1, w_1^2 + ... + w_n^2\right)\right)$. Since $w_1^2 + ... + w_n^2 = v_n^2 + ... + v_1^2$, this expectation is also equal to $E\left(\min\left(v_1^1 + ... + v_n^1, v_1^2 + ... + v_n^2\right)\right)$, which is the expected revenue the seller would obtain in the initial setting with n units of a homogenous good when these units are all bundled into a single package[8].

Proposition 1 *(bundling increases revenue in the Vickrey auction) Let* n *units of a good be sold to bidder 1 and bidder 2. Let* $\left(v_1^1, ..., v_n^1\right)$ *and* $\left(v_1^2, ..., v_n^2\right)$ *be two independent random vectors. Assume that if bidder* j *receives* l *units and pays an amount* p*, his utility is equal to* $\sum_{h=1}^{l} v_h^j - p$*, for* $j = 1, 2$ *and* $l = 1, ..., n$*. Then the seller's expected revenue at the Vickrey auction where all units are bundled into a single package is higher than his expected revenue at the Vickrey auction without bundling.*

Since the Vickrey auction without bundling is ex-post efficient, it is thus apparent that bundling will decrease efficiency. As we have seen in Corollary 2 above, that the bundling of all units into a single package also decreases efficiency in the discriminatory auction. However, we show in Section 5 that it does not necessarily increase the seller's revenues. Thus, contrary to the Vickrey auction, bundling in discriminatory auctions with two bidders does not always entail a trade-off between efficiency and revenue.

4. Outline of the Proofs of the Results of Section 3

Outline of the Proof of Theorem 1

We first prove (*Lemma A1-1* in Appendix 1) Theorem 1 (c). Assume that $r < c$, that is, that the reserve price is not binding. From the definition of regular strategies, we immediately obtain $\beta_1(c), ..., \beta_n(c) \leq c$. Suppose that not all equalities in $\beta_1(c) = ... = \beta_n(c)$ hold true. Then, there will exist $j \leq (n+1)/2$ such that $\beta_{j-1}(c) > \beta_j(c) = ... = \beta_{n-j+1}(c)$ or $\beta_j(c) = ... = \beta_{n-j+1}(c) > \beta_{n-j+2}(c)$ (or both). In the former case, it will be more advantageous for a type c bidder to change his jth, ..., $(n-j+1)$th bids from $\beta_j(c) = ... = \beta_{n-j+1}(c)$, which wins a probability zero, to any strictly higher bid $b \leq \beta_{j-1}(c)$. This former case is thus impossible and we must have $\beta_{j-1}(c) = \beta_j(c) = ... = \beta_{n-j+1}(c)$. Then in the latter case, it will be strictly profitable to a type v bidder, with v close to c, to increase his $(n-j+2)$th bid from $\beta_{n-j+2}(v) < \beta_{n-j+1}(c)$ to, for example, $\left(\beta_{n-j+2}(v) + \beta_{n-j+1}(c)\right)/2$ since $\left(\beta_{n-j+2}(v) + \beta_{n-j+1}(c)\right)/2$ is strictly larger than $\beta_{j-1}(c)$ and thus wins with a strictly positive probability.

The proof in the case $r \geq c$, or when the reserve price is binding, is simpler. Since bids do not exceed valuations, a bid on a unit of valuation smaller than r cannot be larger than r. Furthermore, a bid on a unit of valuation strictly larger than r must at least be equal to r. Consider a type v bidder and let i be the smallest index such that the bidder's valuation for the ith unit is strictly larger than r and his bid on this unit is strictly smaller than r. Since the valuation of a unit is nondecreasing in the number of this unit, the bidder's valuation on his $(i-1)$th unit is also strictly larger than r. From the definition of i, the bidder's bid on this unit $\beta_{i-1}(v)$ is strictly larger than r and he can thus increase his ith bid to, for example, $\left(\beta_{i-1}(v) + r\right)/2$, which is strictly larger than r and thus also strictly larger than c. Because the opponent does not submit bids higher than his valuations, with a strictly positive probability this opponent's $(n-i+1)$th bid will be smaller than $\left(\beta_{i-1}(v) + r\right)/2$, and this new ith bid will thus be among the winning bids with a strictly positive probability. With this new ith bid, the bidder obtains a strictly positive expected payoff on his ith unit. Such a deviation is thus profitable, which is impossible at an equilibrium. Since the bid is strictly smaller than r on a unit of valuation strictly smaller than r, and since the bid is strictly larger than r on a unit of valuation strictly larger than r, we find by continuity that the bid must be equal to r on a unit of valuation equal to r. (c) is therefore proved.

We next prove (*Lemma A1-2*) that bid functions for units of identical valuations have the same maximum, or $\beta_1(d) = ... = \beta_m(d)$ and $\beta_{m+1}(d) = ... = \beta_n(d)$. Assume, for example, that there exists $2 \leq k \leq m$ such that $\beta_k(d) < \beta_1(d)$. Let k be the smallest such index. Then $\beta_k(d) < \beta_{k-1}(d) = \beta_1(d)$. If the bid function β_{n-k+2} with which β_{k-1} "competes" (see Lemma 1 in Section 3) did not go as high as $\beta_1(d)$, it would be strictly more advantageous for a type d bidder to lower his $(k-1)$th bid. Thus, we also have $\beta_{n-k+2}(d) = \beta_{n-k+1}(d) = \beta_1(d)$. Moreover, since β_{n-k+1} competes with β_k, a bidder has no interest in submitting a $(n-k+1)$th bid strictly larger than his $(n-k+2)$th bid when this latter bid is already larger than $\beta_k(d)$. Consequently, β_{n-k+2} and β_{n-k+1} are identical over the interval $\left[\gamma_{n-k+2}(\beta_k(d)), d\right]$ where the $(n-k+2)$th bid is larger than $\beta_k(d)$, and their inverses γ_{n-k+2} and γ_{n-k+1} coincide over the gap $[\beta_k(d), \beta_1(d)]$.

Any bid in the gap $[\beta_k(d), \beta_1(d)]$ thus contributes the same amount to the bidder's expected payoff whether it is submitted as the $(k-1)$th bid or the kth bid. Since $\beta_{k-1}(d) = \beta_1(d)$ is chosen as a best $(k-1)$th bid and $\beta_k(d)$ is chosen as a best kth bid when the bidder's type is d, both these bids must contribute equally to the expected payoff . However, this is impossible since this contribution is a strictly increasing function of the bid b in the gap $[\beta_k(d), \beta_1(d)]$. In fact, every b in this gap is chosen as the best $(k-1)$th bid among the interval $[\beta_k(v), b]$, which is the bid that maximizes $(w-b)\,F\left(\gamma_{n-k+2}(b)\right)$ or, equivalently, its logarithm, when the type w is equal to $\gamma_{k-1}(b)$. A bidder with this type has no interest in locally lowering his $(k-1)$th bid, so his marginal cost $\frac{d_l}{db}\ln F\left(\gamma_{n-k+2}(b)\right)$ of lowering his bid must be at least as large as his marginal revenue $\frac{1}{\gamma_{n-k+2}(b)-b}$. If the bidder's type is d, the left-hand logarithmic derivative of the contribution $(d-b)\,F\left(\gamma_{n-k+1}(b)\right) = (d-b)\,F\left(\gamma_{n-k+2}(b)\right)$ from his kth bid is equal to $\frac{d_l}{db}\ln F\left(\gamma_{n-k+2}(b)\right) - \frac{1}{d-b}$, which is thus not smaller than $\frac{1}{\gamma_{n-k+2}(b)-b} - \frac{1}{d-b}$, which in turn is strictly positive since $\gamma_{n-k+2}(b) < d$. This contribution is thus a strictly increasing function of b and it cannot reach its maximum at both extremities of the gap $[\beta_k(d), \beta_1(d)]$.

Similar arguments (*Lemma A1-3)* allow to show that all bid functions have the same maximum when $g(d) = d$. When there is a majority of low valuation units, that is, when $m \leq m'$, the same results can be obtained very easily. In fact, if $\beta_1\ (d) = ... = \beta_m(d) > \beta_{m+1}\ (d) = ... = \beta_n(d)$ a type d bidder would increase his payoff strictly if he lowered his mth bid to $\beta_{m+1}(d)$. Statement (a) in Theorem 1 is thus proved.

We then prove (in *Lemma* A1-6) that the bid functions $\beta_1, ..., \beta_m$ on high valuation units coincide when their values are larger than the common maximum $\beta_{m+1}(d)$ of the bid functions on low valuation units. That is, we prove $\gamma_1(b) = ... = \gamma_m(b)$, for all b in $\left[\beta_{m+1}(d), \beta_m(d)\right]$. We already know that these equalities hold true at $b = \beta_m(d)$. Assume that some of these equalities do not hold true over the interval $\left[\beta_{m+1}(d), \beta_m(d)\right]$. Then, as we show in *Lemma* A1-4, there would exist a bid b^* in $(\beta_{m+1}(d), \beta_m(d)]$, $\delta > 0$ and two consecutive groups (possibly counting only one element) of inverse bid functions which "separate" at b^* . There would thus exist i, k, j such that $1 \leq i \leq k < k+1 \leq j \leq m$,

$$\gamma_{i-1}(b) < \gamma_i(b) = ... = \gamma_k(b) < \gamma_{k+1}(b) = ... = \gamma_j(b) < \gamma_{j+1}(b)\ (4\text{-}1)$$

for all b in $(b^* - \delta, b^*)$, and

$$\gamma_i(b^*) = ... = \gamma_k(b^*) = \gamma_{k+1}(b^*) = ... = \gamma_j(b^*)\ (4\text{-}2).$$

If $n-k+1 \geq m+1$, (4-1) is clearly impossible. In fact, othwerwise the type $\gamma_k(b)$ bidder would strictly increase his expected payoff if he decreased his kth bid b to, for example, $\max\left(\beta_{k+1}(\gamma_k(b)), \beta_{n-k+1}(\gamma_k(b)) = \beta_{m+1}(\gamma_k(b))\right) < b$.

Assume next that $n-k+1<m+1$ or, equivalently, that $n-k<m$. We prove (in *Lemma* A1-5) the following useful result

$$\gamma_r(b)<\gamma_{r+1}(b) \text{ if and only if } \gamma_{n-r}(b)<\gamma_{n-r+1}(b) \text{ (4-3)}$$

for all b in $(\underline{c},\beta_m(d)]$ such that $b\neq\beta_{m+1}(d)$ and for all r such that $r+1\leq m$ and $n-r+1\leq m$. Assume first that $\gamma_r(b)<\gamma_{r+1}(b)$, with $\beta_m(d)>b>\underline{c}$, $b\neq\beta_{m+1}(d)$, $r+1\leq m$, and $n-r+1\leq m$. From the definition of regular strategies, we know that all inverse bid functions are differentiable at b. Moreover, b is chosen as the best rth bid in the nonempty interval $\left[\beta_{r+1}(\gamma_r(b)),b\right]$ when the bidder's type is $\gamma_r(b)$. Such a bidder's marginal cost from lowering his rth bid must outweigh his marginal benefit and thus $\frac{d}{db}\ln F\left(\gamma_{n-r+1}(b)\right)\geq\frac{1}{\gamma_r(b)-b}$. Similarly, b is chosen as the best $(r+1)$th bid in the nonempty interval $\left[b,\beta_r\left(\gamma_{r+1}(b)\right)\right]$ when the bidder's type is $\gamma_{r+1}(b)$. The bidder's marginal cost from increasing his $(r+1)$th bid must outweigh the marginal benefit. We thus have $\frac{d}{db}\ln F\left(\gamma_{n-r}(b)\right)\leq\frac{1}{\gamma_{r+1}(b)-b}$. Since $\gamma_r(b)<\gamma_{r+1}(b)$, we then have $\frac{d}{db}\ln F\left(\gamma_{n-r+1}(b)\right)>\frac{d}{db}\ln F\left(\gamma_{n-r}(b)\right)$. Therefore γ_{n-r+1} has a strictly higher slope at b than γ_{n-r}. If these last two inverse bid functions coincided at b, γ_{n-r+1} would go under γ_{n-r} to the left of b. This is clearly impossible, so $\gamma_{n-r}(b)<\gamma_{n-r+1}(b)$. (4-1) is thus proved for b in $(\underline{c},\beta_m(d))\setminus\left\{\beta_{m+1}(d)\right\}$. Since $\gamma_r(\beta_m(d))=d$, for all r such that $1\leq r\leq m$, it immediately holds true for all b in $(\underline{c},\beta_m(d)]\setminus\left\{\beta_{m+1}(d)\right\}$.

(4-3), (4-1) and (4-2) imply

$$\begin{aligned}\gamma_{n-j}(b) &< \gamma_{n-j+1}(b)=...=\gamma_{n-k}(b)<\gamma_{n-k+1}(b)=...=\gamma_{\min(n-i+1,m)}(b)\\ &< \gamma_{\min(n-i+1,m)+1}(b) \text{ (4-4)}\end{aligned}$$

for all b in $(b^*-\delta,b^*)$ and

$$\gamma_{n-j+1}(b^*)=...=\gamma_{n-k}(b^*)=\gamma_{n-k+1}(b^*) \text{ (4-5)}.$$

From (4-1), small equal and simultaneous changes of the $(k+1)$th bid to the jth bid are feasible and therefore unprofitable. We thus find the first order condition $\frac{d}{db}\ln F\left(\gamma_{n-k}(b)\right)=\frac{1}{\gamma_{k+1}(b)-b}$, for all b in $(b^*-\delta,b^*)$. Similarly, from (4-4), simultaneous changes of the $(n-j+1)$th bid to the $(n-k)$th bid are feasible and hence unprofitable. We thus have also $\frac{d}{db}\ln F\left(\gamma_{k+1}(b)\right)=\frac{1}{\gamma_{n-k}(b)-b}$, for all b in $(b^*-\delta,b^*)$. From (4-4), simultaneous changes of the $(n-k+1)$th bid to the $\min(n-i+1,m)$th bid are feasible, and we thus find that $\frac{d}{db}\ln F(\gamma_k(b))=\frac{1}{\gamma_{n-k+1}(b)-b}$, for all b in $(b^*-\delta,b^*)$. From (4-1), small decreases of the kth bid are possible and are thus unprofitable. Consequently $\frac{d}{db}\ln F\left(\gamma_{n-k+1}(b)\right)\geq\frac{1}{\gamma_k(b)-b}$, for all b in $(b^*-\delta,b^*)$. Summing up, over the interval $(b^*-\delta,b^*)$ $\left(\gamma_{k+1},\gamma_{n-k}\right)$ satisfies the system of differential equations (4-6) and $\left(\gamma_k,\gamma_{n-k+1}\right)$ satisfies the system of differential inequations (4-7) below:

$$\begin{aligned}\frac{d}{db}\ln F\left(\gamma_{k+1}(b)\right) &= \frac{1}{\gamma_{n-k}(b)-b},\frac{d}{db}\ln F\left(\gamma_{n-k}(b)\right)=\frac{1}{\gamma_{k+1}(b)-b}\text{(4-6)}\\ \frac{d}{db}\ln F\left(\gamma_k(b)\right) &= \frac{1}{\gamma_{n-k+1}(b)-b},\frac{d}{db}\ln F\left(\gamma_{n-k+1}(b)\right)\geq\frac{1}{\gamma_k(b)-b}\text{(4-7)}.\end{aligned}$$

Moreover, from (4-2) and (4-5), $\left(\gamma_{k+1}, \gamma_{n-k}\right)$ and $\left(\gamma_k, \gamma_{n-k+1}\right)$ satisfy the same initial condition at b^*:

$$\gamma_{k+1}\left(b^*\right) = \gamma_k\left(b^*\right), \gamma_{n-k}\left(b^*\right) = \gamma_{n-k+1}\left(b^*\right).$$

As it can then be easily seen, the equalities above and (4-6, 4-7) imply that the inequalities $\gamma_k(b) \leq \gamma_{k+1}(b)$ and $\gamma_{n-k+1}(b) \leq \gamma_{n-k}(b)$ hold true over $[b^* - \delta, b^*]$. The latter inequality contradicts the second inequality in (4-4) and we have thus proved $\gamma_1(b) = ... = \gamma_m(b)$, for all b in $\left[\beta_{m+1}(d), \beta_m(d)\right]$.

We then go on to the range of bids $\left[\underline{c}, \beta_{m+1}(d)\right]$. We first extend (*Lemma* A1-7) the result (4-3) as follows:

$$\begin{aligned} \gamma_i(b) &< \gamma_{i+1}(b) \text{ implies } \gamma_{n-i}(b) < \gamma_{n-i+1}(b) \quad (4\text{-}8) \\ \gamma_{n-i}(b) &= \gamma_{n-i+1}(b) \text{ implies } \gamma_i(b) = \gamma_{i+1}(b) \quad (4\text{-}9) \end{aligned}$$

for all b in $\left(\underline{c}, \beta_{m+1}(d)\right]$ and for all i such that $i \neq m$. For b in $(\underline{c}, \beta_{m+1}(d))$, the proof of (4-8) is similar to the proof of (4-3). From the results we already obtained, we know that $\gamma_1\left(\beta_{m+1}(d)\right) = ... = \gamma_m\left(\beta_{m+1}(d)\right)$ and $\gamma_{m+1}\left(\beta_{m+1}(d)\right) = ... = \gamma_n\left(\beta_{m+1}(d)\right)$. Consequently, (4-8) holds true for all b in $\left(\underline{c}, \beta_{m+1}(d)\right]$. (4-9) is the contraposition of (4-8).

We then establish (in *Lemma* A1-8) that β_m and β_{m+1} are never equal over $(\underline{c}, d)$, that is, that $\beta_m(v) > \beta_{m+1}(v)$, for all v in $(\underline{c}, d)$. It there existed v in this interval such that $\beta_m(v) = \beta_{m+1}(v)$, then we would have $\gamma_m(b^*) = \gamma_{m+1}(b^*)$ with $b^* = \beta_m(v) = \beta_{m+1}(v)$ and thus $\underline{c} < b^* < \beta_{m+1}(d)$. Let i be the smallest index (not larger than m) such that $\beta_i(v) = b^*$ and let j be the largest index (not smaller than $(m+1)$) such that $\beta_j(v) = b^*$. From the definitions of i and j, we have

$$\gamma_{i-1}\left(b^*\right) < v = \gamma_i\left(b^*\right) = ... = \gamma_m\left(b^*\right) = \gamma_{m+1}\left(b^*\right) = ... = \gamma_j\left(b^*\right) < \gamma_{j+1}\left(b^*\right).$$

From the definition of regular strategies, all inverse bid functions are differentiable at b^*. Since an increase of the ith bid alone is feasible, it must not be profitable. We thus obtain the inequality below

$$\frac{d}{db}\ln F\left(\gamma_{n-i+1}\left(b^*\right)\right) \leq \frac{1}{v - b^*}.$$

A decrease of the jth bid is also feasible and thus cannot be profitable. We find

$$\frac{d}{db}\ln F\left(\gamma_{n-j+1}\left(b^*\right)\right) \geq \frac{1}{g(v) - b^*}$$

and consequently $\frac{d}{db}\ln F\left(\gamma_{n-j+1}\left(b^*\right)\right) > \frac{d}{db}\ln F\left(\gamma_{n-i+1}\left(b^*\right)\right)$. Since their derivatives are different at b^*, γ_{n-j+1} and γ_{n-i+1} also differ at this bid since otherwise the two functions would cross. We thus have

$$\gamma_{n-j+1}\left(b^*\right) < \gamma_{n-i+1}\left(b^*\right) \quad (4\text{-}10).$$

From (4-8) and $\gamma_i(b^*) = \gamma_j(b^*)$, we must have $n-j+1 \le m$ and $m+1 \le n-i+1$. The first inequality implies $(n+1)/2 < j$ and the second inequality implies $i < (n+1)/2$. Since $(n+1)/2$ is strictly between i and j (4-8), (4-9), and the definitions of i and j imply that $n-j+1 = i$ or, equivalently, $n-i+1 = j$. The inequality (4-10) then contradicts $\gamma_i(b^*) = \gamma_j(b^*)$.

We can now prove the identities $\gamma_1 = ... = \gamma_m$ and $\gamma_{m+1} = ... = \gamma_n$ over $(\underline{c}, \beta_{m+1}(d)]$. We know that these equalities hold at $b = \beta_{m+1}(d)$. If they did not hold true everywhere over the interval $(\underline{c}, \beta_{m+1}(d)]$, there would exist (again from *Lemma* A1-4) b^* in $(\underline{c}, \beta_{m+1}(d)]$ at which some of the first m inverse bid functions or some of the last $m' = n-m$ inverse bid functions "separate". That is, there would exist an interval $(b^* - \delta, b^*)$, with $\delta > 0$, to the left of b^* where the inverse bid functions can be grouped consecutively in several bundles (possibly counting only one element) as follows:

$$\gamma_1(b) = ... = \gamma_{i_1}(b) < \gamma_{i_1+1}(b) = ... = \gamma_{i_2}(b) < ... < \gamma_{i_t+1}(b) = ... = \gamma_m(b) \text{ (4-11)}$$

$$\gamma_{m+1}(b) = ... = \gamma_{k_1}(b) < \gamma_{k_1+1}(b) = ... = \gamma_{k_2}(b) < ... < \gamma_{k_l+1}(b) = ... = \gamma_n(b) \text{ (4-12)}$$

for all b in $(b^* - \delta, b^*)$. Furthermore, two groups both in the same line among the line of inequalities above coincide at b^*. Assume for example, that these two groups belong to the first line. There thus exists $1 \le t^* \le t-1$ such that

$$\gamma_{i_{t^*-1}}(b^*) = ... = \gamma_{i_{t^*}}(b^*) = \gamma_{i_{t^*}+1}(b^*) = ... = \gamma_{i_{t^*+1}}(b^*) \text{ (4-13).}$$

From the previous paragraph, we know that $\gamma_m(b) < \gamma_{m+1}(b)$, for all b in $(b^* - \delta, b^*)$.

To simplify the notations, denote i_{t^*-1} by i, i_{t^*} by k, and i_{t^*+1} by j. Summing up, we have $\underline{c} < b^* \le \beta_{m+1}(d)$, $i < k < k+1 < j \le m$, and

$$\gamma_{i-1}(b) < \gamma_i(b) = ... = \gamma_k(b) < \gamma_{k+1}(b) = ... = \gamma_j(b) < \gamma_{j+1}(b) \text{ (4-14)}$$

$$\gamma_m(b) < \gamma_{m+1}(b) \text{ (4-15),}$$

for all b in $(b^* - \delta, b^*)$, and

$$\gamma_i(b^*) = ... = \gamma_k(b^*) = \gamma_{k+1}(b^*) = ... = \gamma_j(b^*) \text{ (4-16).}$$

Assume first that $n-k \ge m+1$. Then, from (4-8) and (4-9) we see that (4-14) and (4-16) imply

$$\begin{aligned}\gamma_{\max(m+1,n-j+1)-1}(b) &< \gamma_{\max(m+1,n-j+1)}(b) = ... = \gamma_{n-k}(b) \\ &< \gamma_{n-k+1}(b) = ... = \gamma_{n-i+1}(b) < \gamma_{n-i+2}(b) \text{ (4-17),}\end{aligned}$$

for all b in $(b^* - \delta, b^*)$, and

$$\gamma_{\max(m+1,n-j+1)}(b^*) = ... = \gamma_{n-k}(b^*) = \gamma_{n-k+1}(b^*) = ... = \gamma_{n-i+1}(b^*).$$

Equal deviations of the $(n-k+1)$th bid to the $(n-i+1)$th bid are feasible and thus unprofitable. We thus find $\frac{d}{db}\ln F(\gamma_k(b)) = \frac{1}{g(\gamma_{n-k+1}(b)) - b}$, for all b in

$(b^* - \delta, b^*)$. Similarly, equal deviations of the ih bid to the kth bid are feasible and must thus be unprofitable. We find $\frac{d}{db} \ln F\left(\gamma_{n-k+1}(b)\right) = \frac{1}{\gamma_k(b)-b}$, for all b in $(b^* - \delta, b^*)$. Simultaneous deviations of the $\max(m+1, n-j+1)$th bid to the $(n-k)$th bid are also possible and cannot be profitable. Therefore, $\frac{d}{db} \ln F\left(\gamma_{k+1}(b)\right) = \frac{1}{g\left(\gamma_{n-k}(b)\right)-b}$, for all b in $(b^* - \delta, b^*)$. Small increases of the $(k+1)$th bid are also possible, and we thus obtain the condition $\frac{d}{db} \ln F\left(\gamma_{n-k}(b)\right) \leq \frac{1}{\gamma_{k+1}(b)-b}$, for all b in $(b^* - \delta, b^*)$. Over the interval $(b^* - \delta, b^*)$, $\left(\gamma_k, \gamma_{n-k+1}\right)$ is thus a solution of the system (4-18) of differential equations and $\left(\gamma_{k+1}, \gamma_{n-k}\right)$ is a solution of the system (4-19) of differential inequations below:

$$\frac{d}{db} \ln F\left(\gamma_k(b)\right) = \frac{1}{g\left(\gamma_{n-k+1}(b)\right) - b}, \frac{d}{db} \ln F\left(\gamma_{n-k+1}(b)\right) = \frac{1}{\gamma_k(b) - b} \text{(4-18)}$$

$$\frac{d}{db} \ln F\left(\gamma_{k+1}(b)\right) = \frac{1}{g\left(\gamma_{n-k}(b)\right) - b}, \frac{d}{db} \ln F\left(\gamma_{n-k}(b)\right) \leq \frac{1}{\gamma_{k+1}(b) - b} \text{(4-19)}.$$

Moreover, at $b = b^*$ they satisfy the same initial condition, such that

$$\gamma_k(b^*) = \gamma_{k+1}(b^*), \gamma_{n-k+1}(b^*) = \gamma_{n-k}(b^*).$$

As it can be easily seen, the equalities above and (4-18, 4-19) imply $\gamma_{k+1}(b) \geq \gamma_k(b)$ and $\gamma_{n-k}(b) \geq \gamma_{n-k+1}(b)$, for all b in $(b^* - \delta, b^*]$. The latter inequality contradicts the second inequality in (4-17) and we have thus ruled out the case $n - k \geq m + 1$. Ruling out the case $n - k + 1 \leq m$ is similar.

The only case we still have to rule out is the case $m-1 < n-k < m+1$, that is, the case $n-k = m$. Suppose thus that $n-k = m$ (and thus $m \geq (n+1)/2$). The definitions of i and j, (4-11), (4-12), and the properties (4-8) and (4-9) then imply $n - i + 1 = k_1$ and (4-20) below

$$\begin{aligned}\gamma_{i-1}(b) &< \gamma_i(b) = ... = \gamma_{n-m}(b) < \gamma_{n-m+1}(b) = ... = \gamma_m(b) \\ &< \gamma_{m+1}(b) = ... = \gamma_{n-i+1}(b) < \gamma_{n-i+2}(b) \text{(4-20)},\end{aligned}$$

for all b in $(b^* - \delta, b^*)$. We can rewrite (4-16) as (4-21) below:

$$\gamma_i(b^*) = ... = \gamma_{n-m}(b^*) = \gamma_{n-m+1}(b^*) = ... = \gamma_m(b^*) \text{(4-21)}.$$

Since we have ruled out any other possible "separation" point, the inequality (4-20) must hold over $(\underline{b}, b^*)$ and $\gamma_1(\underline{b}) = ... = \gamma_m(\underline{b})$ and $\gamma_{m+1}(\underline{b}) = ... = \gamma_n(\underline{b})$, with $\underline{b} \leq b^* - \delta$. From Theorem 1 (c), which we already proved, we have $\gamma_1(\underline{c}) = ... = \gamma_m(\underline{c}) = \underline{c}$ and $\gamma_{m+1}(\underline{c}) = ... = \gamma_n(\underline{c}) = g^{-1}(\underline{c})$. The existence of such a $\underline{b}$ thus follows from the properties (4-8) and (4-9). Since equal changes of all bids from the $(n-m+1)$th bid to the mth bid are feasible and thus unprofitable, we obatin the first order condition $\frac{d}{db} \ln F\left(\gamma_m(b)\right) = \frac{1}{\gamma_m(b)-b}$, for all b in $(\underline{b}, b^*)$. Similarly, by considering equal changes of all bids from the $(m+1)$th bid to the $(n-i+1)$th bid we find $\frac{d}{db} \ln F\left(\gamma_{n-m}(b)\right) = \frac{1}{g\left(\gamma_{m+1}(b)\right)-b}$

. By considering equal changes of all bids from the ith bid to the $(n-m)$th bid, we find $\frac{d}{db}\ln F\left(\gamma_{m+1}(b)\right) = \frac{1}{\gamma_{n-m}(b)-b}$, for all b in $(\underline{b}, b^*)$. Moreover, since $\gamma_{n-m}(b^*) = \gamma_m(b^*)$, and since γ_{n-m} can never be strictly larger than γ_m, the left-hand derivative of $\ln F\left(\gamma_{n-m}\right)$ at b^* is at least as large as the left-hand derivative of $\ln F(\gamma_m)$ at b^*. Since these left-hand derivatives are the limits of the corresponding two-sided derivatives at b for b tending from below to b^*, the previous differential equations imply $g\left(\gamma_{m+1}(b^*)\right) \leq \gamma_m(b^*)$. Denote $\underline{v}$ the common value of γ_{n-m} and γ_m at $\underline{b}$. Summing up our conclusions, we know that γ_m is the solution over $(\underline{b}, b^*]$ of the differential equation (4-22) with initial condition (4-23) below:

$$\frac{d}{db}\ln F(\gamma_m(b)) = \frac{1}{\gamma_m(b)-b}(4\text{-}22)$$

$$\gamma_m(\underline{b}) = \underline{v}(4\text{-}23).$$

We also know that $\left(\gamma_{n-m}, \gamma_{m+1}\right)$ is the solution over $(\underline{b}, b^*]$ of the system of differential equations (4-24) with (partial) initial condition (4-25) below:

$$\frac{d}{db}\ln F\left(\gamma_{n-m}(b)\right) = \frac{1}{g\left(\gamma_{m+1}(b)\right)-b}, \frac{d}{db}\ln F\left(\gamma_{m+1}(b)\right) = \frac{1}{\gamma_{n-m}(b)-b}(4\text{-}24)$$

$$\gamma_{n-m}(\underline{b}) = \underline{v}(4\text{-}25).$$

Moreover, we know that (4-26) below holds true

$$\gamma_{n-m}(b^*) \leq \gamma_m(b^*), \, g\left(\gamma_{m+1}(b^*)\right) \leq \gamma_m(b^*)\,(4\text{-}26).$$

As we explain in the next paragraphs, it is a property of the equation (4-22) and the system (4-24) that no such solutions can exist. We have thus ruled out the last possible case of "separation", and Theorem 1 (a) is proved.

Let $\widehat{\gamma}$ be a solution of the differential equation (4-22), and let $(\widehat{\gamma}_1, \widehat{\gamma}_2)$ be a solution of the differential system (4-24)[9] over the same interval $\left(\rho, \overline{b}\right]$, with $\overline{b} > \rho$, such that

$$\widehat{\gamma}(\rho) = \widehat{\gamma}_1(\rho)\,(4\text{-}26).$$

The property alluded to above is that the inequalities

$$\widehat{\gamma}_1\left(\overline{b}\right) \leq \widehat{\gamma}\left(\overline{b}\right) \text{ and } g\left(\widehat{\gamma}_2\left(\overline{b}\right)\right) \leq \widehat{\gamma}\left(\overline{b}\right)(4\text{-}27)$$

cannot both simultaneously hold. Along lines developed in Lebrun (1997 and 1999), we prove in Appendix 6 this property. We give here the main ideas of the proof. The couple $(\widehat{\gamma}_1, \widehat{\sigma}_2) = (\widehat{\gamma}_1, g \circ \widehat{\gamma}_2)$ is a solution over $\left(\rho, \overline{b}\right]$ of the system (4-28) below:

$$\frac{d}{db}\ln F(\widehat{\gamma}_1(b)) = \frac{1}{\widehat{\sigma}_2(b)-b}, \frac{d}{db}\ln H(\widehat{\sigma}_2(b)) = \frac{1}{\widehat{\gamma}_1(b)-b}(4\text{-}28).$$

A first property of the system (4-28) is that

$$\widehat{\sigma}_2(\rho) = \rho \text{ if only if } \widehat{\gamma}_1(\rho) = \rho(4\text{-}29),$$

for every solution of (4-28) over $\left(\rho,\overline{b}\right]$. It then turns out (*Lemma A6-1*) that under our assumption of stochastic dominance, if $\widehat{\gamma}$ and $(\widehat{\gamma}_1,\widehat{\sigma}_2)$ are solutions of, respectively, the equation (4-22) and the system (4-28) over the interval $\left(\rho,\overline{b}\right]$, then

$$\begin{aligned}\widehat{\gamma}_1\left(\overline{b}\right) &\leq \widehat{\gamma}\left(\overline{b}\right) \text{ and } \widehat{\sigma}_2\left(\overline{b}\right)\leq \widehat{\gamma}\left(\overline{b}\right) \text{ imply}\\ \widehat{\gamma}_1\left(b\right) &< \widehat{\gamma}\left(b\right) \text{ and } \widehat{\sigma}_2\left(b\right)<\widehat{\gamma}\left(b\right), \text{ for all b in } \left(\rho,\overline{b}\right) \text{ (4-30)}.\end{aligned}$$

Let $\widehat{\gamma}$ and $(\widehat{\gamma}_1,\widehat{\sigma}_2)$ be solutions of, respectively, (4-22) and (4-28) over $\left(\rho,\overline{b}\right]$, with $\overline{b}>\rho\geq\underline{c}$, such that (4-26) holds true. Suppose that $\widehat{\gamma}_1\left(\overline{b}\right)\leq\widehat{\gamma}\left(\overline{b}\right)$ and $\widehat{\sigma}_2\left(\overline{b}\right)\leq\widehat{\gamma}\left(\overline{b}\right)$. Then, from (4-30) above, we have $\widehat{\gamma}_1\left(b\right)<\widehat{\gamma}\left(b\right)$ and $\widehat{\sigma}_2\left(b\right)<\widehat{\gamma}\left(b\right)$, for all b in $\left(\rho,\overline{b}\right)$. Assume first that $\widehat{\gamma}_1\left(\rho\right)>\rho$. Then $(\rho,\widehat{\gamma}\left(\rho\right))$ and $(\rho,\widehat{\gamma}_1\left(\rho\right),\widehat{\sigma}_2\left(\rho\right))$ belong to the (interiors of the) domains where their respective differential equation and system satisfy the standard assumptions of the theory of ordinary differential equations. The solutions can thus be continued to the left of these points, so $\widehat{\gamma}$ and $(\widehat{\gamma}_1,\widehat{\sigma}_2)$ are solutions of (4-22) and (4-28), respectively, over an interval $\left(\rho',\overline{b}\right]$ with $\rho'<\rho$. From (4-30), we have $\widehat{\gamma}_1\left(\rho\right)<\widehat{\gamma}\left(\rho\right)$, which contradicts (4-26).

Suppose now that $\widehat{\gamma}_1\left(\rho\right)=\rho$ and thus, from (4-29), $\widehat{\sigma}_2\left(\rho\right)=\rho$. From (4-26), we have $\widehat{\gamma}\left(\rho\right)=\rho$. However, it is well known that the solution (4-22) with this last initial condition is the inverse of the equilibrium bid function $\widehat{\beta}$ of the first price auction with reserve price ρ and two homogenous bidders with valuation distribution F (see, for example, Riley and Samuelson 1981). More generally, the solution of the differential equation (4-22) with initial condition (4-31) below, where ρ' belongs to $[\underline{c},d]$,

$$\widehat{\gamma}\left(\rho'\right)=\rho'\text{(4-31)}$$

is the inverse of the function $\widehat{\beta}$ such that $\widehat{\beta}\left(v\right)=\frac{\rho' F\left(\rho'\right)+\int_{\rho'}^{v} wdF(w)}{F(v)}$, for all v in $\left[\rho',\widehat{\gamma}\left(\overline{b}\right)\right]$. This inverse is thus a continuous and strictly decreasing function of ρ', and we denote it by $\widehat{\gamma}\left(.;\rho'\right)$. Take b' in $\left(\rho,\overline{b}\right)$. From (4-30), we have $\widehat{\gamma}_1\left(b'\right)<\widehat{\gamma}\left(b'\right)=\widehat{\gamma}\left(b';\rho\right)$ and $\widehat{\sigma}_2\left(b'\right)<\widehat{\gamma}\left(b'\right)=\widehat{\gamma}\left(b';\rho\right)$. There thus exists $\rho'>\rho$ such that $\widehat{\gamma}\left(;\rho'\right)$ is defined at b' and is such that $\widehat{\gamma}_1\left(b'\right)<\widehat{\gamma}\left(b';\rho'\right)<\widehat{\gamma}\left(b'\right)$ and $\widehat{\sigma}_2\left(b'\right)<\widehat{\gamma}\left(b';\rho'\right)<\widehat{\gamma}\left(b'\right)$. From the property (4-30), we then obtain $\widehat{\gamma}_1\left(b\right)<\widehat{\gamma}\left(b;\rho'\right)$ and $\widehat{\sigma}_2\left(b'\right)<\widehat{\gamma}\left(b;\rho'\right)$, for all b in $\left(\rho',b'\right]$, which is clearly impossible since, for example, $\widehat{\gamma}_1\left(\rho'\right)>\rho'$ while $\widehat{\gamma}\left(\rho';\rho'\right)=\rho'$. We have thus proved that for all solutions $\widehat{\gamma}$ of (4-22) and $(\widehat{\gamma}_1,\widehat{\gamma}_2)$ of (4-24) such that (4-26) holds true, both inequalities in (4-27) cannot simultaneously hold (*Lemma A6-2*).

Outline of the Proofs of Theorems 2, 3, and 4

In this subsection, we consider only the case $m>m'$. The proofs in the case $m\leq m'$ are similar and simpler. We first prove in *Lemma A2-1* (Appendix 2) that a regular strategy $\left(\beta, \beta'\right)$ gives a symmetric regular equilibrium if and only if the conditions in Theorem 4 and the inequality

$$\sigma'(b) \leq \gamma(b) \text{ (4-32)}$$

or, equivalently, $g\left(\gamma'(b)\right) \leq \gamma(b)$, for all b in $[c, d']$, are satisfied. We have already shown in the previous section (after the statements of the theorems) that these conditions are necessary. In order to prove their sufficiency, we distinguish the following three components in the objective function of Lemma 1:

$$\begin{aligned}
\kappa\left(v; b_1, ..., b_{m'}\right) &= \sum_{i=1}^{m'} \left(v - b_i\right) F\left(\gamma'\left(b_i\right)\right), \\
\lambda\left(v; b_{m'+1}, ..., b_m\right) &= \sum_{i=m'+1}^{m} \left(v - b_i\right) F\left(\gamma\left(b_i\right)\right), \\
\mu\left(v; b_{m+1}, ..., b_n\right) &= \sum_{i=m+1}^{n} \left(g(v) - b_i\right) F\left(\gamma\left(b_i\right)\right).
\end{aligned}$$

Denote the objective function or expected payoff by $\pi\left(v; b_1, ..., b_n\right)$. We then have

$$\pi\left(v; b_1, ..., b_n\right) = \kappa\left(v; b_1, ..., b_{m'}\right) + \lambda\left(v; b_{m'+1}, ..., b_m\right) + \mu\left(v; b_{m+1}, ..., b_n\right).$$

The proof that $\beta'(v)$ maximizes $(g(v) - b) F(\gamma(b))$ over $[c, \eta']$ is actually routine. In order to prove that $\beta'(v)$ is a solution of this maximization problem when bids higher than η' are allowed, we use the inequality $g(d) \leq d'$ and the differential equation we solved to find the expression (3-6). Thus, $\beta'(v) \in \arg\max_{b \geq c} (g(v) - b) F(\gamma(b))$. From the definition of the third component μ, we immediately find

$$\max_{c \leq b_{m+1}, ..., b_n} \mu\left(v; b_{m+1}, ..., b_n\right) = \mu\left(v; \beta'(v), ..., \beta'(v)\right) \text{ (4-33).}$$

Let $\left(\widetilde{b}_1, ..., \widetilde{b}_m\right)$ be a solution of the maximization problem $\max_{c \leq b_m \leq ... \leq b_1}$ $\left(\kappa\left(v; b_1, ..., b_{m'}\right) + \lambda\left(v; b_{m'+1}, ..., b_n\right)\right)$ involving the two other components of π. Obviously $\kappa\left(v; \widetilde{b}_1, ..., \widetilde{b}_{m'}\right) + \lambda\left(v; \widetilde{b}_{m'+1}, ..., \widetilde{b}_n\right)$ will not decrease if we replace all $\widetilde{b}_1, ..., \widetilde{b}_{m'}$ by $\overline{b}$ and all $\widetilde{b}_{m'+1}, ..., \widetilde{b}_n$ by $\widehat{b}$ where $\overline{b}$ is a solution of $\max_{b \in \left[\widetilde{b}_{m'}, \widetilde{b}_1\right]}$ $(v - b) F\left(\gamma'(b)\right)$ and $\widehat{b}$ is a solution of $\max_{b \in \left[\widetilde{b}_n, \widetilde{b}_{m'+1}\right]} (v - b) F(\gamma(b))$. In maximizing the sum of κ and λ, we can thus assume that bids are identical if they are arguments of the same function. We therefore find the following equivalent maximization problem:

$$\max_{c \leq b'' \leq b} m' (v - b) F\left(\gamma'(b)\right) + (m - m') (v - b'') F\left(\gamma(b'')\right).$$

Using the differential equation (3-3) and the condition (4-32), we show that $(v-b)\,F\left(\gamma'(b)\right)$ is nonincreasing in b to the right of $\beta(v)$. The differential equation (3-2) and (4-32) imply that $(v-b'')\,F\left(\gamma(b'')\right)$ is nondecreasing in b" to the left of $\beta(v)$. Let $b'' \leq b$ be a solution of the last maximization problem. If $b'' \leq b \leq \beta(v)$, (b,b) is thus a solution of the maximization problem. If $\beta(v) \leq b'' \leq b$, (b'',b'') is such a solution. If $b'' \leq \beta(v) \leq b$, $(\beta(v),\beta(v))$ is a solution. We can thus further reduce the problem by assuming that all (bid) arguments are equal and we therefore find the equivalent problem $\max_{c\leq b}$ $m'(v-b)\,F\left(\gamma'(b)\right) + (m-m')(v-b)\,F\left(\gamma(b)\right)$. Using the assumption that γ and γ' come from regular strategies and are thus strictly increasing (and their derivatives are nonnegative) and appealing once again to the differential equations (3-2) and (3-3), it is easy to show that $\beta(v)$ is a solution of this problem. Consequently,

$$\begin{aligned}
&\max_{c\leq b_m\leq\ldots\leq b_1}\left(\kappa\left(v;b_1,...,b_{m'}\right)+\lambda\left(v;b_{m'+1},...,b_n\right)\right)\\
=\ &\kappa\left(v;\beta(v),...,\beta(v)\right)+\lambda\left(v;\beta(v),...,\beta(v)\right) \text{(4-34)}.
\end{aligned}$$

(4-33) and (4-34) then immediately imply $\left(\beta(v),...,\beta(v),\beta'(v),...,\beta'(v)\right) \in \arg\max_{\underline{c}\leq b_n\leq\ldots\leq b_1}\pi\left(v;b_1,...,b_n\right)$. Therefore, from Proposition 1, $\left(\beta,\beta'\right)$ defines a symmetric regular equilibrium.

We then show that we can drop from the necessary and sufficient conditions for an equilibrium the requirement that $\left(\beta,\beta'\right)$ be a regular strategy and that (4-32) be satisfied. We first notice in *Lemma A2-4* that any solution (γ,γ') of (3-2, 3-3) that satisfies (4-33) is such that $\gamma' \geq \gamma$. We then show in *Lemma A2-5* that any solution of the differential system with boundary conditions (3-2, 3-3, 3-4) (with $g(d) \leq d'$) satisfies condition (4-32) everywhere over $[\underline{c},\eta']$ with a strict inequality over $(\underline{c},\eta')$. Moreover, in *Lemma A2-6,* we show that if the link (3-5) between η' and d' holds true, then all solutions of (3-2, 3-3, 3-4) have strictly positive derivatives over the interior (c,η') of the definition domain. Consequently, any solution (γ,γ') of (3-2, 3-3, 3-4) when the link (3-5) holds true satisfies (4-32) and is formed of strictly increasing functions such that $\gamma' \geq \gamma$ and thus such that their inverses satisfy the inequality $\beta' = \gamma'^{-1} \leq \beta = \gamma^{-1}$. Thus these inverses and the expression (3-6), which has a strictly positive derivative, define a regular strategy. Consequently, we can drop the requirements that (γ,γ') come from a regular strategy and that (4-32) be satisfied and the characterization of Theorem 4 is proved.

The existence and uniqueness of the equilibrium (Theorem 2) is proved roughly along the same lines as the corresponding proofs for the first price auction in Lebrun (1997,1999). Contrary to these papers, here both η' and d' are unknown. However, the known link (3-5) has to hold between them. This link can be expressed by the equality below:

$$d' = \delta'(\eta')$$

where δ' is a known nonincreasing function from $[\underline{c},g(d)]$ to $[g(d),d]$ such that $\delta'(\underline{c}) = d$ and $\delta'(g(d)) = g(d)$. Figure 1 display a possible graph for this function. We thus consider that d' is a function of η' and we are left with η' as the only parameter. We study the solutions of the differential system (3-2, 3-3) and the initial condition below:

$$\gamma(\eta') = \delta'(\eta'), \sigma'(\eta') = g(d) \text{ (4-35)}$$

We obtain (4-35) from the last initial condition in the boundary condition (3-4) by setting $d' = \delta'(\eta')$. The standard theorems of the theory of ordinary differential equations can be applied to the initial condition (4-35).

We show in *Lemmas A2-9* and *A2-10* that if (γ, σ') is a solution of (3-2, 3-3) with initial condition (4-35) and if $(\rho, \eta']$ is its "maximal" (to the left) definition interval, which means that there does not exist any strictly larger interval $(\rho', \eta']$ where (γ, σ') is still a solution of (3-2, 3-3), then either $\rho < \underline{c}$ and $\gamma(\underline{c}), \sigma'(\underline{c}) > \underline{c}$ or $\gamma(\rho) = \sigma'(\rho) = \rho$. Thanks to a property of monotonicity which the solutions of the differential system (3-2, 3-3) display (*Lemma A2-8)*, we show in *Lemma A2-11* that the lower extremity ρ of the maximal definition interval is a nondecreasing function of η'. In *Lemma A2-12,* we prove the continuity from the right of the function $\max(\rho, \underline{c})$, that is, of the lower extremity of the maximal definition interval over $[\underline{c}, \eta']$ of the solution of (3-2,3-3) and (4-35). In *Lemma A2-16,* we establish the continuity of this function by proving its continuity from the left. In *Lemma A2-17,* we show that for all $\eta' < g(d)$ close enough to $g(d)$ the left-hand extremity ρ of the maximal definition interval is not smaller than $\underline{c}$[10] . Obviously there exist values of η', for example $\eta' = \underline{c}$, such that this left-hand extremity is strictly smaller than $\underline{c}$. We then consider the infimum $\eta'^* > \underline{c}$ of the nonempty set of values η' such that the left-hand extremity ρ of the maximal definition interval is not smaller than $\underline{c}$. The continuity of the function $\max(\underline{c}, \rho)$ implies that the left-hand extremity ρ^* of the maximal definition interval of the solution corresponding to η'^* is equal to $\underline{c}$ (*Lemma A2-18*) (see Figure 1). This solution satisfies all the necessary and sufficient requirements to generate an equilibrium, so the existence of an equilibrium is proved.

In order to prove the uniqueness of the equilibrium, we first suppose that there exist two equilibria or, equivalently, two values η' and $\widetilde{\eta}'$ for which there exist solutions of (3-2, 3-3, 3-4) and (3-5) and thus (4-35). Without loss of generality, we can assume that $\widetilde{\eta}' < \eta'$. By studying the differential system the functions $\psi = \gamma' \circ \beta$, which "connects" the bid functions on high and low valuation units, and the bid function β (we actually consider rather $\chi = H \circ \sigma' \circ \gamma^{-1} \circ F^{-1} = F \circ \gamma' \circ \gamma^{-1} \circ F^{-1}$ and $\rho = \gamma^{-1} \circ F^{-1}$), we show in *Lemma A2-19* that $\widetilde{\psi} > \psi$ over $(\underline{c}, d')$ and thus $\widetilde{\psi}^{-1} < \psi^{-1}$. However, the differential equation (3-2) implies the value $(g(d) - \eta') F(d')$ for the integral $\int_c^d F(\gamma \circ \beta'(u))\, dg(u) = \int_c^d F(\psi^{-1}(u))\, dg(u)$ (*Lemma A2-13*). This integral must thus be a strictly decreasing function of η',we obtain a contradiction with $\widetilde{\psi}^{-1} < \psi^{-1}$. We have therefore proved the uniqueness of the equilibrium (*Lemma A2-20*).

5. The Single-Item First Price Auction

and the Discriminatory Auction With an Equal Number of High and Low Valuation Units .

In this section, we consider the special case when there is the same number of high valuation units as low valuation units, or when n is even and $m = m' = n/2$. In this case, the differential system in the characterization of the equilibrium in Theorem 3 (Section 3) reduces to the simpler system below:

$$\frac{d}{db} \ln F(\gamma(b)) = \frac{1}{\sigma'(b) - b}$$

$$\frac{d}{db} \ln H(\sigma'(b)) = \frac{1}{\gamma(b) - b}.$$

The characterization in Theorem 3 (Section 3) is almost identical to the characterization (found in Lebrun 1997, 1999a) of the equilibrium of the single-item first price auction with two heterogeneous bidders whose valuations are distributed according to F and H. This result comes from the separability of the objective function in Lemma 1 (Section 3). Since this objective function is, up to the factor $m = m'$, equal to $(v - b) F(\gamma'(b)) + (g(v) - b') F(\gamma(b')) = (v - b) H(\sigma'(b)) + (g(v) - b') F(\gamma(b'))$, if the constraint $b \leq b'$ is not binding, which from Theorem 1 is the case under our assumptions for all v in $\left(g^{-1}(\underline{c}), d\right)$, then the first order conditions of the constrained problem in Lemma 1 will be identical to the first order conditions of the two separated maximization problems $\max (v - b) H(\sigma'(b))$ and $\max (g(v) - b') F(\gamma(b'))$. The first one of these last two problems arises in the maximization of the expected payoff of a bidder in a first price auction when the bidder's valuation for the single item for sale is v and when his opponent has his valuation distributed according to H and bids according to $\zeta = \sigma'^{-1} = \beta' \circ g^{-1}$. The second one of these problems is the expected payoff maximization problem of a bidder in a first price auction with valuation $g(v)$ when his opponent bids according to $\beta = \gamma^{-1}$ and has his valuation distributed according to F. Obviously, if v is distributed according to F, then $g(v)$ is distributed according to H.

The similarity between the discriminatory and first price auctions is intuitive. In the discriminatory auction, a bidder's bid for his ith unit competes with the other bidder's bid for this bidder's $(n - i + 1)$th unit, for i ranging from 1 to $n/2$. For each one of his high valuation units, a bidder's bid thus competes with the other bidder's bid for one of this bidder's low valuation units. All high valuations or valuations for the first $n/2$ units are distributed according to F and all low valuations or valuations for the last $n/2$ units are distributed according to H. Since in the discriminatory auction, as in the first price auction, if a bidder wins a unit he pays the bid he submitted for this unit, we see that one bidder's trade-offs in the discriminatory auction are related to the trade-offs in the first price auction with two bidders where one bidder's valuation is distributed according to F and the other bidder's valuation is distributed according to H.

The only difference between the characterization in Theorem 3 and the characterization of the equilibrium of the first price auction with heterogenous bidders in Lebrun (1997,1999) is that, in Theorem 3, we added the assumption that (β, β') is a regular strategy, that is, in particular, that $\beta' \leq \beta$. In the discriminatory auction (β, β') is the strategy of every bidder where β is the bid function on the first $n/2$ units and β' is the bid function on the last n/2 units. In the first price auction with heterogenous bidders, β is the bid function of one bidder and $\zeta = \beta' \circ g^{-1}$ is the bid function of the other bidder. Still, because we have assumed nonincreasing demand curves in our model, or because $g(v) \leq v$, for all v, it turns out that any equilibrium (β, ζ) of the first price auction with heterogeneous bidders is such that the inequality $\beta' = \zeta \circ g \leq \beta$ holds true. According to the theory of the first price auction with heterogenous bidders, the same relation of stochastic dominance passes from the valuation distributions to the bid distributions (see Lebrun 1997, 1999a, or Maskin and Riley 1998). Since $g(v) \leq v$, for all v, we have $H = F \circ g^{-1} \geq F$ or the distribution F first order stochastically dominates the distribution H. Consequently, at the equilibrium of the first price auction the bid probability distribution $F \circ \gamma = F \circ \beta^{-1}$ of the bidder whose valuation distribution is F stochastically dominates the bid distribution $H \circ \zeta^{-1} = F \circ g^{-1} \circ g \circ \beta'^{-1} = F \circ \beta'^{-1}$ of the bidder with valuation distribution H. We therefore have[11] $F \circ \beta^{-1} \leq F \circ \beta^{'-1}$ and thus $\beta \geq \beta'$. We formally express the link between equilibria of the discriminatory and first-price auctions in the corollary below.

Corollary 3[12]: *(link between the equilibrium of the discriminatory auction with homogenous bidders and the equilibrium of a first price auction with heterogenous bidders) Let F and g be as in Section 2. Let β and β' be continuous and strictly increasing functions over $[\underline{c}, d]$ which are differentiable over $(\underline{c}, d]$ for β and $\left(g^{-1}(\underline{c}), d\right]$ for β' with strictly positive derivatives (and possibly infinite). Then, for all even integer n, (β, β') is a symmetric regular equilibrium of the discriminatory auction with homogeneous bidders, with reserve price r, and with $m = m' = n/2$ if and only if $(\beta, \beta' \circ g^{-1})$ is an equilibrium of the first price auction with the same reserve price r and with two heterogenous bidders where the bidders' valuations are distributed according to F and $H = F \circ g^{-1}$.*

The equivalence stated in Corollary 2 above between equilibria of the discriminatory and first price auctions when $m = m' = n/2$ allows us to translate results from the literature on first price auctions with heterogenous bidders into results pertaining to the discriminatory auction in this case. For example, the result, expressed in Theorem 5 (Section 3), according to which bidders bid more aggressively on their low valuation units than on their high valuation units is an immediate consequence of a result about first price auctions when $m = m' = n/2$. According to this result about first price auctions, if bidder i's valuation distribution F_i stochastically dominates bidder j's valuation distribution F_j in the sense[13] that F_i/F_j is nondecreasing, then bidder i's equilibrium bid function is not larger than bidder j's bid function. The intuition for this result is that bidder j, whose valuation distribution is dominated, faces a more serious competition than bidder i does. In fact, (in addition to all the other bidders when there are more than two bidders) this bidder faces the competition from bidder i, whose valuation is likely to be high, while bidder i faces only the competition from bidder j, whose valuation is likely to be low. Obviously, the assumption $\frac{d}{dv}\ \frac{F}{H} > 0$ implies that $\frac{F}{H}$ is nondecreasing. Consequently, in any equilibrium of the first price auction and thus in any equilibrium of the discriminatory auction, $\beta \leq \beta' \circ g^{-1}$ or, equivalently, $\beta \circ g \leq \beta'$, as in Theorem 5.

When m is different from m' there does not exit an equivalence between the equilibria of the discriminatory auction and equilibria of some first price auctions. However, as is apparent from the proofs of our Theorems 2, 3, and 4 in Section 3 and Theorems 6 and 7 in Section 5 the methods used in the study of first price auctions with heterogenous bidders (as found in Lebrun 19997, 1999) reveal useful in the study of the discriminatory auction.

Corollary 3 also allows us to compare the discriminatory auction with the Vickrey and Ausubel auctions. The Ausubel auction is an ascending price version (with private values and nonincreasing demand curves) of the Vickrey auction. Before the statement of Proposition 1 in the previous section, we showed a connection between the Vickrey auction and the second price auction. According to this connection, the seller's revenue at the unique equilibrium in weakly dominant strategies of the Vickrey auction, where n units of a good are sold to bidder 1 and bidder 2, is equal to the seller's total revenue at n one-unit second price auctions. In the second price auction i, $i = 1, ..., n$, one bidder's valuation is equal to bidder1's valuation for his ith unit and the other bidder's valuation is equal to bidder 2's valuation for his $(n - i + 1)$th unit. Consequently, the seller's expected revenue R_V in the Vickrey auction for the particular setting of this section is equal to n times the seller's expected revenue R_S in the second price auction with two heterogenous bidders where the valuation distributions are F and H. From Corollary 3, we know that the seller's expected revenue in the discriminatory auction is n times the seller's revenue in the first price auction with the two "same" heterogenous bidders. Corollary 4 follows below.

Corollary 4: *Let n be even and m and m' be equal to n/2 . Let R_V be the seller's expected revenue at the equilibrium in weakly dominant strategies of the Vickrey auction and of the Ausubel auction and let R_D be the seller's expected revenue at the regular symmetric equilibrium of the discriminatory auction with homogenous bidders. We have*

$$R_V \gtreqless R_D \text{ if and only if } R_S \gtreqless R_F$$

where R_S and R_F denote the seller's expected revenue at the equilibrium in weakly dominating strategies of the second price auction and at the equilibrium of the first price auction, respectively, with two heterogenous bidders whose valuation distributions are F and H.

When the bidders' valuation distributions are identical, according to the Revenue Equivalence Theorem (see Riley and Samuelson 1981, Myerson 1981) the first and second price auctions bring the same expected revenue to the seller. However, when the bidders' valuations are distributed differently it is well known that no general ranking between the first and second price auctions holds true. For example, in the case of two bidders as in Corollary 4 above, if the valuation distributions approximate different discrete distributions with the same two values as support[14], the second price auction gives a strictly higher expected revenue than the first price auction (see Maskin and Riley 1985). On the other hand, if one bidder's valuation distribution H is the uniform distribution over $[0, 1]$, and if the other bidder's valuation distribution F is the distribution of the maximum of two independent uniforms over the same interval, that is, $F(v) = v^2$, for all v in $[0, 1]$, then the first price auction gives strictly higher expected revenues than the second price auction when the reserve price r is equal to 0[15] . Consequently, from the corollary above, there is no general ranking even with homogenous bidders between the sellers' revenues in a multi-unit discriminatory auction and the Vickrey and Ausubel auctions. The actual ranking between these two revenues depends on the particular combination of probability distributions of high and low valuations. The reason for this ambiguity, despite the symmetry of our model where we assume the bidders to be homogenous ex-ante, is the link stated in Corollary 3 between the equilibrium of the discriminatory auction in our symmetric model and the equilibrium of a first price auction in an asymmetric model where the bidders are heterogenous ex-ante.

Finally, we consider the issue of bundling. We saw in Proposition 1 in the previous section that, in the Vickrey auction, bundling all units for sale into a single package decreases efficiency and increases the seller's revenue. Here, we see that, contrary to the Vickrey auction, the seller's revenue may or may not increase after bundling. To show this, we consider the case of equal numbers of high and low valuation units.

For the sake of simplicity, assume that the reserve price is nonbinding, or that $r \leq c$, and that there are one high valuation unit and one low valuation unit, such that $n = 2$ and $m = m' = 1$. Then, from Corollary 2 above, the seller's revenue at the regular symmetric equilibrium of the discriminatory auction is equal to twice the seller's revenue at the equilibrium of the first price auction where the bidders' valuations are distributed according to F and $H = F \circ g^{-1}$. If there is bundling of the two units into a single package, the seller's revenue at the unique equilibrium of the discriminatory, here, the first price auction will be equal to the expectation of the second highest or, equivalently, the lower valuation for the package. Since a type v bidder's valuation for this package is equal to $(v + g(v))$ and since v is distributed according to F, the seller's revenue will thus be equal to the sum of the expectation of the lower valuation for the first unit and the expectation of the lower valuation for the second unit. The first expectation is equal to the seller's revenue in the first price auction where the bidders' valuations are identically distributed according to F, and the second expectation is equal to the seller's revenue when the bidders' valuations are distributed according to H.

Comparing the revenue at the equilibrium of the discriminatory auction without bundling to the revenue with bundling is thus equivalent to comparing the revenue at the first price auction with heterogenous bidders and a couple of distributions (F, H) to the arithmetic mean of the revenues at the two first price auctions with homogeneous bidders and couples of distributions (F, F) and (H, H). In our first example, H is the uniform distribution over the interval $[0, 1]$ and F is the distribution of the maximum of four random variables independently and uniformly distributed over $[0, 1]$, that is, $F(v) = v^4$ and $H(v) = v$, for all v in $[0, 1]$. In this example, the function g is simply defined as follows: $g(v) = v^4$, for all v in $[0, 1]$. From Marshall, Meurer, Richard, and Stromquist (1994), the seller's revenue at the first price auction with a couple of distributions (F, H) is 0.5057. Because of the simplicity of the distributions involved, it is simple to find[16] that the average of the revenues at the first price auctions with couples of distributions (F, F) and (H, H) is 47/90=0.5222. Thus we find that bundling decreases efficiency and increases revenue in this example.

In our second example, H is the distribution of the maximum of two random variables independently and uniformly distributed over $[0, 1]$, and F is the distribution of the maximum of three independent random variables uniformly distributed over $[0, 1]$, that is, $F(v) = v^3$ and $H(v) = v^2$, for all v in $[0, 1]$. In this example, $g(v) = v^{3/2}$, for all v in $[0, 1]$. Again from Marshall, Meurer, Richard, and Stromquist (1994), the seller's revenue at the first price auction with a couple of distributions (F, H) is 0.5875. A simple computation gives 43/84=0.5119 for the average of the revenues at the first price auctions with couples of distributions (F, F) and (H, H). Thus in this second example, bundling decreases both efficiency *and* revenue. Contrary to the Vickrey auction, no general trade-off thus exists in the discriminatory auction with two bidders.

6. Comparative Statics

We now derive some comparative statics from our characterizations of the equilibria. One observation that follows from the characterizations in Section 3 is that the equilibrium depends only on the ratio m/m' of the number of high valuation units to the number of low valuation units or, equivalently, on the proportion $\alpha = \frac{m/m'}{1+m/m'}$ of high valuation units. As is apparent from the expressions in Section 3, a change of m or m' that keeps this ratio constant would of course only scale the marginal cost and benefit of changes of bids by the same factor, so the trade-offs faced by bidders would stay unchanged. It is interesting, however, to look at how the equilibrium is affected by a change in proportions of high and low valuation units. This issue is addressed in Theorems 6 and 7 in this section.

Assume $m \geq m'$. From Theorem 1 in Section 3 and from the proof (see Appendix.., a sketch of the proof can be found in the next section) of Theorem 4 (Section 3), we know that we can break down the problem (1) in Lemma 1

(Section 3) into the two maximization problems:

$$\max_{\underline{c}\leq b}(v-b)\left(m'F\left(\gamma'(b)\right)+(m-m')F\left(\gamma(b)\right)\right), \text{ for all } v \text{ in } \left[\underline{c},d\right]$$

$$\max_{\underline{c}\leq b}\left(g(v)-b\right)m'F\left(\gamma(b)\right), \text{ for all } v \text{ in } \left[g^{-1}\left(\underline{c}\right),d\right].$$

The first problem consists in the maximization of the total expected payoff on the first m units, and the second problem is the maximization of the total expected payoff on the last m' units. It is thus meaningful to consider the equilibrium expected payoffs on the first m units and on the last m' units. The average (per unit) interim expected payoff $P_H(v)$ on one of the first m units or high valuation units, and the average interim expected payoff $P_L(v)$ on one of the last m' units or low valuation units, are, respectively,

$$P_H(v)=\max_{\underline{c}\leq b}(v-b)\left(\frac{m'}{m}F\left(\gamma'(b)\right)+\left(1-\frac{m'}{m}\right)F\left(\gamma(b)\right)\right), \text{ for all } v \text{ in } \left[\underline{c},d\right]$$

$$P_L(v)=\max_{\underline{c}\leq b}\left(g(v)-b\right)F\left(\gamma(b)\right), \text{ for all } v \text{ in } \left[g^{-1}\left(\underline{c}\right),d\right].$$

In a sense, the factor $\left(\frac{m'}{m}F\left(\gamma'(b)\right)+\left(1-\frac{m'}{m}\right)F\left(\gamma(b)\right)\right)$, which multiplies $(v-b)$ in the first maximization problem, is the "average" bid probability distribution the bidder's high bid b competes against.

Obviously, $P_H(v)=0$, for all v in $\left[c,\underline{c}\right]$, and $P_L(v)=0$, for all v in $\left[c,g^{-1}\left(\underline{c}\right)\right]$. We denote by EP_H and EP_L, a bidder's average ex-ante payoff on one high valuation unit and on one low valuation unit, respectively, such that $EP_H=\int P_H(v)\,dF(v)$ and $EP_L=\int P_L(v)\,dF(v)$. We assume that the seller's valuation for any unit is equal to zero and we denote by TS be the total surplus or welfare and by AS the average surplus per unit, such that $AS=TS/n=TS/(m+m')$. We have Theorem 6 below. The detailed proof can be found in Appendices 3 and 4. We provide a sketch of the proof in the next section.

Theorem 6: *(the percentage of high valuation units grows away from 50%, see Figures 1 and 2) Let $m,m',\widetilde{m}$, and $\widetilde{m}'$ be such that $\frac{\widetilde{m}}{\widetilde{m}'}>\frac{m}{m'}>1$ and*

let $\left(\beta,\beta'\right)$ and $\left(\widetilde{\beta},\widetilde{\beta}'\right)$ the corresponding equilibria. The expected payoffs and surpluses in the equilibrium are EP_H, EP_L,P_H,P_L,TS,AS in the equilibrium $\left(\beta,\beta'\right)$ and $\widetilde{EP}_H$, $\widetilde{EP}_L,\widetilde{P}_H$, $\widetilde{P}_L,\widetilde{TS},\widetilde{AS}$ in the equilibrium $\left(\widetilde{\beta},\widetilde{\beta}'\right)$. The properties below hold true.

(a) (higher bids on the high valuation units)

$$\widetilde{\beta}(v)>\beta(v)$$

, for all v in $\left(\underline{c},d\right]$

(b) (increase of the "average" bid distribution the bid on a high valuation unit competes against)

$$\frac{\widetilde{m}'}{\widetilde{m}}F\left(\widetilde{\gamma}'(b)\right)+\left(1-\frac{\widetilde{m}'}{\widetilde{m}}\right)F\left(\widetilde{\gamma}(b)\right)<\frac{m'}{m}F\left(\gamma'(b)\right)+\left(1-\frac{m'}{m}\right)F\left(\gamma(b)\right)$$

for all b in $(\underline{c}, \eta = \beta(d)]$

(c) (decrease of the inefficiency) If a realization of types (v_1, v_2) *results in an inefficient allocation of units in the equilibrium* $\left(\widetilde{\beta}, \widetilde{\beta}'\right)$, *then it also results in an inefficient allocation of units in the equilibrium* (β, β'). *Moreover, the probability that there will be an inefficient allocation of units in the equilibrium* $\left(\widetilde{\beta}, \widetilde{\beta}'\right)$ *is strictly smaller than in the equilibrium* (β, β').

(d) (decrease of the bidders' interim and ex-ante average expected payoffs on every high valuation unit and on every low valuation unit)

$$\widetilde{P}_H(v) < P_H(v),\ \widetilde{P}_L(w) < P_L(w)$$

, for all v in $(\underline{c}, d]$ *and all w in* $\left(g^{-1}(\underline{c}), d\right]$ *,and*

$$\widetilde{EP}_H < EP_H,\ \widetilde{EP}_L < EP_L$$

(e) (higher average surplus per unit)

$$\widetilde{AS} > AS$$

If we substitute its value $\frac{F(\gamma(b))}{\sigma'(b)-b}$ from the differential equation (2) in Theorem 4 (Section 3) to $\frac{d}{db}F(\gamma(b))$ in the expression (3-11) (Section 3), we find that the marginal profit or net marginal benefit, that is, the difference between the marginal benefit and the marginal cost of an increase of the higher bid b when $m > m'$ is equal to

$$-F\left(\gamma'(b)\right)+(v-b)\frac{d}{db}F\left(\gamma'(b)\right)+\left(\frac{m}{m'}-1\right)\left(\frac{v-\sigma'(b)}{\sigma'(b)-b}\right)\frac{d}{db}F\left(\gamma(b)\right) \text{(6-1)}.$$

At the equilibrium, a bidder will submit a high bid equal to b when his type v is equal to $\gamma(b)$. The factor $\left(\frac{v-\sigma'(b)}{\sigma'(b)-b}\right)=\left(\frac{\gamma(b)-\sigma'(b)}{\sigma'(b)-b}\right)$ above will be strictly positive for all b in $(\underline{c}, d')$, since, from Theorem 5 (Section 3), $\gamma > \sigma'$ over the same interval. Consequently, the expression (6-1) is a strictly increasing function of the ratio m/m'. Starting from an equilibrium where this expression (6-1) vanishes, other things being equal an increase of the ratio m/m' will thus increase the marginal net benefit of an increase of the higher bid. Such an increase will thus be strictly profitable. From Theorem 6 (a), we see that this effect will dominate and the new equilibrium bid function on high valuation units will increase. Intuitively, when the ratio m/m' increases, a bidder's high bid will compete against a larger proportion of bids on high valuation units. Since this type of competition is likely to result in fiercer bidding, the bidder will in turn increase his high bid.

From (3-10) (Section 3), we see that when $m > m'$ the marginal benefit of an increase of the lower bid does not depend directly on the ratio m/m'. It is thus not surprising that, in general, the lower bid function will not increase everywhere. For example, reasoning as in Lebrun (1998), it is straightforward to show that when the reserve price r is strictly larger than c, there exist types v in any neighborhood of $\underline{c} = r$ where the lower bid function will actually decrease. However, there exists a bound on the possible decreases of the lower bid function. In fact, since the marginal net benefit (3-11) (Section 3) vanishes at the equilibrium, we see that $\frac{d}{db} \ln \left\{ \frac{m'}{m} F\left(\gamma'\left(b\right)\right) + \left(1 - \frac{m'}{m}\right) F\left(\gamma\left(b\right)\right) \right\}$ is equal to $1/\left(\gamma\left(b\right) - b\right)$. From Theorem 6 (a) above, $\widetilde{\gamma}\left(b\right) < \gamma\left(b\right)$ and thus $1/\left(\widetilde{\gamma}\left(b\right) - b\right) > 1/\left(\gamma\left(b\right) - b\right)$, for all b in $\left(\underline{c}, \beta\left(d\right)\right]$. Moreover, Theorem 6 (a) again implies that $\frac{m'}{m} F\left(\gamma'\left(\beta\left(d\right)\right)\right) + \left(1 - \frac{m'}{m}\right) F\left(\gamma\left(\beta\left(d\right)\right)\right) = 1 > \frac{\widetilde{m}'}{\widetilde{m}} F\left(\widetilde{\gamma}'\left(\beta\left(d\right)\right)\right) + \left(1 - \frac{\widetilde{m}'}{\widetilde{m}}\right) F\left(\widetilde{\gamma}\left(\beta\left(d\right)\right)\right)$. Since the value of $\frac{\widetilde{m}'}{\widetilde{m}} F\left(\widetilde{\gamma}'\left(b\right)\right) + \left(1 - \frac{\widetilde{m}'}{\widetilde{m}}\right) F\left(\widetilde{\gamma}\left(b\right)\right)$ is lower than the value of $\frac{m'}{m} F\left(\gamma'\left(b\right)\right) + \left(1 - \frac{m'}{m}\right) F\left(\gamma\left(b\right)\right)$ at $b = \beta\left(d\right)$ and the rate of increase of the former function is higher than the rate of increase of the latter, the former function is always smaller and we find Theorem 6 (b). The lower bid function β' may thus decrease for some types and its inverse γ' may this increase for some bids, but never enough to increase the "average" cumulative distribution function a high bid competes against.

The new average probability distribution a high bid competes against thus first order stochastically dominates the old one and the inequality $\widetilde{P}_H\left(v\right) < P_H\left(v\right)$ in Theorem 6 (d) follows. Since the high bid function β increases (and thus its inverse γ decreases), the new high bid probability distribution dominates the old one. Since a bidder's low bid always competes against the other bidder's high bids, the second inequality $\widetilde{P}_L\left(v\right) < P_L\left(v\right)$ in Theorem 6 (d) follows. Since the ex-ante expected payoffs are the expectations of the interim payoffs, the inequalities between ex-ante expected payoffs in Theorem 6 (d) are immediate consequences of the inequalities between interim payoffs.

The statement Theorem 6 (c) provides an upper bound on the possible increases of the lower bid function β'. According to Theorem 6 (c), when the ratio m/m' grows away from 1 the equilibrium will become more efficient. From Theorem 5 and Lemma 2 in Section 3, we know that the more aggressive bidding on the low valuation units than on the high valuation units implies a strictly positive probability of inefficient allocation at equilibrium. When the ratio $m/m' \geq 1$ increases, according to Theorem 6 (a) bidders will be higher on their high valuation units. Even if the low bid function also increases for some types, it will not increase enough to enlarge the set of types for which the equilibrium allocation is inefficient, that is, it will not decrease the function $\varphi = g \circ \gamma' \circ \beta = g \circ \beta'^{-1} \circ \beta$, whose graph is the lower boundary of this set. According to

Theorem 6 (c), there will thus be less difference in bid shading on high and low valuation units.

The inequality in Theorem 6 (e) is the consequence of two effects which go in the same direction. As the ratio $m/m' \geq 1$ grows, the proportion of high valuation units in the total number of units grows. Since the total surplus on every high valuation unit is larger than on a low valuation unit, this effect increases the average surplus. Moreover, from Theorem 6 (c), the equilibrium allocation is more efficient. This second effect also goes in the direction of a higher average surplus.

Theorem 6 above can be applied to an increase of the number of units being auctioned while keeping constant the bidders' preferences. Let $m > 0$ be constant. The following corollary is an immediate consequence of Theorem 6 (c) above. In this corollary, we consider the average surplus AS as a function of n.

Corollary 5:*(decreasing average surplus curve when $n \leq 2m$) The function AS is strictly decreasing over $[m, 2m]$.*

When the ratio m/m' is smaller than 1 and increases, different results from those of Theorem 6 hold true. However, the intuition for these results as well as their proofs are similar. We gather these new results in Theorem 7 below.

Theorem 7: *(the percentage of high valuation units grows closer to 50%)*

Let $m, m', \widetilde{m}$, and $\widetilde{m}'$ be such that $1 > \frac{\widetilde{m}}{\widetilde{m}'} > \frac{m}{m'}$ and let (β, β') and $\left(\widetilde{\beta}, \widetilde{\beta}'\right)$ the corresponding equilibria. The expected payoffs and surpluses in the equilibrium are EP_H, EP_L, P_H, P_L, TS, AS in the equilibrium (β, β') and $\widetilde{EP}_H$, $\widetilde{EP}_L, \widetilde{P}_H$, $\widetilde{P}_L, \widetilde{TS}, \widetilde{AS}$ in the equilibrium $\left(\widetilde{\beta}, \widetilde{\beta}'\right)$. The properties below hold true.

(a) (higher bids on the low valuation units)

$$\widetilde{\beta}'(v) > \beta'(v)$$

, for all v in $\left(g^{-1}(\underline{c}), d\right]$

(b) (increase of the "average" bid distribution the bid on a low valuation unit competes against)

$$\frac{\widetilde{m}}{\widetilde{m}'} F(\widetilde{\gamma}(b)) + \left(1 - \frac{\widetilde{m}}{\widetilde{m}'}\right) F\left(\widetilde{\gamma}'(b)\right) < \frac{m}{m'} F(\gamma(b)) + \left(1 - \frac{m}{m'}\right) F(\gamma'(b))$$

for all b in $(\underline{c}, \eta = \beta(d)]$

(c) (increase of the inefficiency) If a realization of types (v_1, v_2) results in an inefficient allocation of units in the equilibrium (β, β') then it also results in an inefficient allocation of units in the equilibrium $\left(\widetilde{\beta}, \widetilde{\beta}'\right)$. Moreover, the

probability that there will be an inefficient allocation of units in the equilibrium $\left(\widetilde{\beta}, \widetilde{\beta}'\right)$ *is strictly higher than in the equilibrium* $\left(\beta, \beta'\right)$.

(d) (decrease of the bidders' interim and ex-ante expected payoffs on every high valuation unit and on every low valuation unit)

$$\widetilde{P}_H(v) < P_H(v),\ \widetilde{P}_L(w) < P_L(w)$$

, for all v *in* $(\underline{c}, d]$ *and all* w *in* $\left(g^{-1}(\underline{c}), d\right]$ *,and*

$$\widetilde{EP}_H < EP_H,\ \widetilde{EP}_L < EP_L.$$

Remark that contrary to Theorem 6 there is no result in Theorem 7 pertaining to the average surplus AS. The reason for this is that the two effects referred to above go now in opposite directions. The increase of the ratio m/m' through the increase of the proportion of high valuation units still goes in the direction of a larger average surplus. However, from Theorem 7 (c) above, the efficiency of the equilibrium allocation decreases and this effect goes in the direction of a smaller average surplus. The net total effect is therefore ambiguous.

Other results of comparative statics could be obtained from our characterizations of the equilibrium. Using methods of proof similar to those used in Lebrun (1998), we could, for example, study the effects of changes in the function g.

7: Outline of the Proofs of Section 6

We sketch the proof of Theorem 6 (Section 5). The proof of Theorem 7 (Section 5) is similar and actually simpler.

Outline of the proof of Theorem 6 (1)

Assume thus that $m \geq m'$. First we establish in *Lemma A3-1* (in Appendix 3) that over the interval $(\underline{c}, \min(g(d), \eta))$, the two parts of the characterization in Theorem 4 (Section 3) of the symmetric regular equilibrium can be subsumed in the following single system of differential equations[17] :

$$\frac{d}{db}\ln F(\gamma(b)) = \min\left(\frac{1}{\sigma'(b) - b}, \frac{1}{\gamma(b) - b}\left\{1 + \frac{1}{m/m' - 1}\frac{1}{F(\gamma(b))}\right\}\right) \text{(7-1)}$$

$$\frac{d}{db}\ln H(\sigma'(b)) = \max\left(\frac{1}{\gamma(b) - b}\left\{1 + \left(\frac{m}{m'} - 1\right)\frac{F(\gamma(b))}{H(\sigma'(b))}\left(\frac{\sigma'(b) - \gamma(b)}{\sigma'(b) - b}\right)\right\}, 0\right) \text{(7-2).}$$

The first argument in the min operator above is a strictly decreasing function of $\sigma'(b)$, and the second argument is a strictly decreasing function of the ratio m/m'. The first argument of the max operator is a strictly decreasing function of $\gamma(b)$ and of the ratio m/m'. Thanks to these properties of the system above, we prove in *Lemma A3-4* that if there exists $\overline{b}$ in $\left(\underline{c}, \max\left(\eta', \widetilde{\eta}'\right)\right]$ such that $\gamma\left(\overline{b}\right) \leq \widetilde{\gamma}\left(\overline{b}\right)$ and $\sigma'\left(\overline{b}\right) \leq \widetilde{\sigma}'\left(\overline{b}\right)$, then $\gamma(b) \leq \widetilde{\gamma}(b)$ and $\sigma'(b) \leq \widetilde{\sigma}'(b)$, for all b in $\left(c, \overline{b}\right]$, and $\sigma'(b) < \widetilde{\sigma}'(b)$, for all b in $\left(c, \min\left(\eta', \widetilde{\eta}', \overline{b}\right)\right)$, where $\widetilde{\gamma}, \widetilde{\sigma}', \widetilde{\eta}'$ are defined as in Theorem 4 for the ratio $\widetilde{m}/\widetilde{m}'$ and γ, σ, η pertain to the ratio m/m' with $\widetilde{m}/\widetilde{m}' > m/m'$. We actually break down the proof of *Lemma A3-4* into two components. First we prove in *Lemma A3-2* that if $\overline{b}$ belongs to $\left[\min\left(\eta', \widetilde{\eta}'\right), \max\left(\eta', \widetilde{\eta}'\right)\right]$, then $\gamma(b) < \widetilde{\gamma}(b)$ and $\sigma'(b) < \widetilde{\sigma}'(b)$ for all b in $\left[\min\left(\eta', \widetilde{\eta}'\right), \overline{b}\right)$ (and $\eta' \geq \widetilde{\eta}'$). Then, we prove in *Lemma A3-3* that if $\overline{b}$ belongs to $\left(\underline{c}, \min\left(\eta', \widetilde{\eta}'\right)\right]$, then $\gamma(b) < \widetilde{\gamma}(b)$ and $\sigma'(b) < \widetilde{\sigma}'(b)$ for all b in $\left(\underline{c}, \overline{b}\right)$.

We show then in *Lemma A3-5* that when $r > c$ and thus $F(\underline{c}) > 0$, the inequalities $\gamma\left(\overline{b}\right) \leq \widetilde{\gamma}\left(\overline{b}\right)$ and $\sigma'\left(\overline{b}\right) \leq \widetilde{\sigma}'\left(\overline{b}\right)$ cannot hold simultaneously for any $\overline{b}$ in $\left(\underline{c}, \max\left(\eta', \widetilde{\eta}'\right)\right]$. If there existed such a $\overline{b}$ for which both inequalities held true, from the previous paragraph we would obtain $\gamma(b) \leq \widetilde{\gamma}(b)$ and $\sigma'(b) \leq \widetilde{\sigma}'(b)$, for all b in $\left(c, \overline{b}\right]$, and $\sigma'(b) < \widetilde{\sigma}'(b)$, for all b in $\left(c, \min\left(\eta', \widetilde{\eta}', \overline{b}\right)\right)$. From the first equation (3-2), which equation (7-1) reduces to, in the differential system of Theorem 4 (Section 3), we would have $\frac{d}{db}\ln F(\gamma(b)) \geq \frac{d}{db}\ln F(\widetilde{\gamma}(b))$ over $\left(c, \overline{b}\right]$ with a strict inequality over $\left(c, \min\left(\eta', \widetilde{\eta}', \overline{b}\right)\right)$. Consequently, $F(\widetilde{\gamma})/F(\gamma)$ would be nonincreasing over $\left(c, \overline{b}\right)$ and strictly decreasing over $\left(c, \min\left(\eta', \widetilde{\eta}', \overline{b}\right)\right)$. Thus $F(\widetilde{\gamma}(c))/F(\gamma(c)) > F\left(\widetilde{\gamma}\left(\overline{b}\right)\right)/F\left(\gamma\left(\overline{b}\right)\right) \geq 1$ and $F(\widetilde{\gamma}(c)) > F(\gamma(c))$. However, (3-3) implies $F(\widetilde{\gamma}(c)) = F(\gamma(c)) = F(c)$, and we would thus obtain a contradiction.

From *Lemma A3-5* introduced in the previous paragraph, we prove in *Lemma A3-6* that $\widetilde{\gamma}(b) < \gamma(b)$, for all b in $\left(\underline{c}, \max\left(\eta', \widetilde{\eta}'\right)\right]$. That this strict inequality holds true at $b = \max\left(\eta', \widetilde{\eta}'\right)$ is an immediate consequence of *Lemma A3-5* and $\sigma'\left(\max\left(\eta', \widetilde{\eta}'\right)\right) = \widetilde{\sigma}'\left(\max\left(\eta', \widetilde{\eta}'\right)\right) = g(d)$. Proving that the inequality holds true everywhere in the interval we must recognize that at any point b where $\widetilde{\gamma}(b) = \gamma(b)$, *Lemma A3-5* implies that $\widetilde{\sigma}'(b) < \sigma'(b)$. From equation (7-1), we thus have $\frac{d}{db}\ln F(\gamma(b)) \leq \frac{1}{\sigma'(b)-b} < \frac{1}{\widetilde{\sigma}'(b)-b} = \frac{d}{db}\ln F(\widetilde{\gamma}(b))$. The function $\widetilde{\gamma}$ is thus strictly smaller than γ at $\max\left(\eta', \widetilde{\eta}'\right)$ and has a strictly larger derivative if it is equal to γ. These properties imply that the function $\widetilde{\gamma}$ is always strictly smaller than γ to the left of $\max\left(\eta', \widetilde{\eta}'\right)$.

We then extend in *Lemma A3-7* the result of the previous paragraph (*Lemma A3-6*) to the case where the reserve price is nonbinding, or $r \leq c$ and thus $F(\underline{c}) = F(c) = 0$. As is apparent from our proofs of Theorems 2 and 4 (see Appendix 2) and from the continuity (under our assumptions) of the solution of a differential system with respect to the initial conditions, the inverse equilibrium bid functions γ and σ' are the limits of the solutions of differential equations (2) and (3) with the initial condition (4-35), such that the left-hand extremities of their maximal definition intervals are strictly larger than $\underline{c} = c$. However, such a solution $\left(\widetilde{\gamma}_\rho, \widetilde{\sigma}'_\rho\right)$ with a left-hand extremity $\rho > c$ of its maximal definition interval is simply the couple of inverse equilibrium bid function when the reserve price is equal to ρ. From the previous paragraph, we know that $\widetilde{\gamma}_\rho(b) < \gamma_\rho(b)$, for all b in $\left(\rho, \max\left(\eta'_\rho, \widetilde{\eta}'_\rho\right)\right]$. Taking the limit for $\rho \to c$, we thus find

$$\widetilde{\gamma}(b) \leq \gamma(b) \text{ (7-3)}$$

, for all b in $\left(\rho, \max\left(\eta', \widetilde{\eta}'\right)\right]$.

We then show that, as when $r > c$, the inequalities $\gamma\left(\overline{b}\right) \leq \widetilde{\gamma}\left(\overline{b}\right)$ and $\sigma'\left(\overline{b}\right) \leq \widetilde{\sigma}'\left(\overline{b}\right)$ cannot hold simultaneously for any $\overline{b}$ in $\left(\underline{c}, \max\left(\eta', \widetilde{\eta}'\right)\right]$. From (A4-3), it suffices to consider the case $\gamma\left(\overline{b}\right) = \widetilde{\gamma}\left(\overline{b}\right)$. If $\sigma'\left(\overline{b}\right) \leq \widetilde{\sigma}'\left(\overline{b}\right)$, we would obtain from *Lemma A3-4* presented above that $\sigma'(b) < \widetilde{\sigma}'(b)$, for all b in $\left(c, \overline{b}\right)$, which would imply by the equation (1) that $\gamma(b) < \widetilde{\gamma}(b)$, for all b in $\left(c, \overline{b}\right)$. This inequality obviously contradicts (A4-3). The rest of the proof from here on proceeds as in the case $r > c$.

So far, we have established that $\widetilde{\gamma}(b) < \gamma(b)$, for all b in $\left(\underline{c}, \max\left(\eta', \widetilde{\eta}'\right)\right]$. To end the proof of Theorem 6 (a), we prove in *Lemma A3-8* from the differential equation $\frac{d}{db} \ln F(\gamma(b)) = \frac{1}{\gamma(b)-b}\left\{1 + \frac{1}{m/m'-1}\frac{1}{F(\gamma(b))}\right\}$, which the system (7-1, 7-2) reduces to, that $\widetilde{\gamma}\left(\max\left(\eta', \widetilde{\eta}'\right)\right) < \gamma\left(\max\left(\eta', \widetilde{\eta}'\right)\right)$ implies that $\widetilde{\eta} > \eta$ and $\widetilde{\gamma} < \gamma$ over the interval $\left[\min\left(\eta', \widetilde{\eta}'\right), \max\left(\eta, \widetilde{\eta}\right)\right] = \left[\max\left(\eta', \widetilde{\eta}'\right), \eta\right]$. The function $\widetilde{\gamma}$ is thus strictly smaller than the function γ over the whole interval $(\underline{c}, \eta]$, so by taking the inverses we find (a) in Theorem 6.

Outline of the proof of Theorem 6 (c)

The link (3-5) in Theorem 4 between d' and η' can be rewritten as the equality

$$d' = \delta'(\eta'),$$

where δ' is a nonincreasing function from $[c, g(d)]$ to $[g(d), d]$ such that $\delta'(c) = d$ and $\delta'(g(d)) = g(d)$. From the boundary condition (3-4) in Theorem 4, we have

$$d' = \gamma(\eta')$$

and the couple (η', d') is thus the solution of the system that the two last equations above form. We observe in *Lemma A4-1* (in Appendix 4) that the function δ' is a nondecreasing function of m/m'. Moreover, from Theorem 6 (a) we know that $\widetilde{\gamma}(b) < \gamma(b)$, for all b in $(\underline{c}, \eta]$. Consequently, as we can see from Figure 1, we have

$$\widetilde{d}' \leq d'.$$

In the course of the proof (in Appendix 2) of Theorems 2 and 5 we notice (in *Lemma A2-14*) that the couple $(\beta, \varphi) = (\beta, \sigma' \circ \beta)$ is a solution over $(\underline{c}, \gamma(\eta')]$ of the differential system (7-4, 7-5) with boundary conditions (7-6) below:

$$\frac{d}{dv}\beta(v) = \frac{\varphi(v) - \beta(v)}{F(v)} \text{(7-4)}$$

$$\frac{d}{dv}\varphi(v) = \frac{H(\varphi(v))}{F(v)}\frac{\varphi(v) - \beta(v)}{v - \beta(v)}\left\{1 + \left(\frac{m}{m'} - 1\right)\frac{F(v)}{H(\varphi(v))}\left(\frac{\varphi(v) - v}{\varphi(v) - \beta(v)}\right)\right\} \text{(7-5)}$$

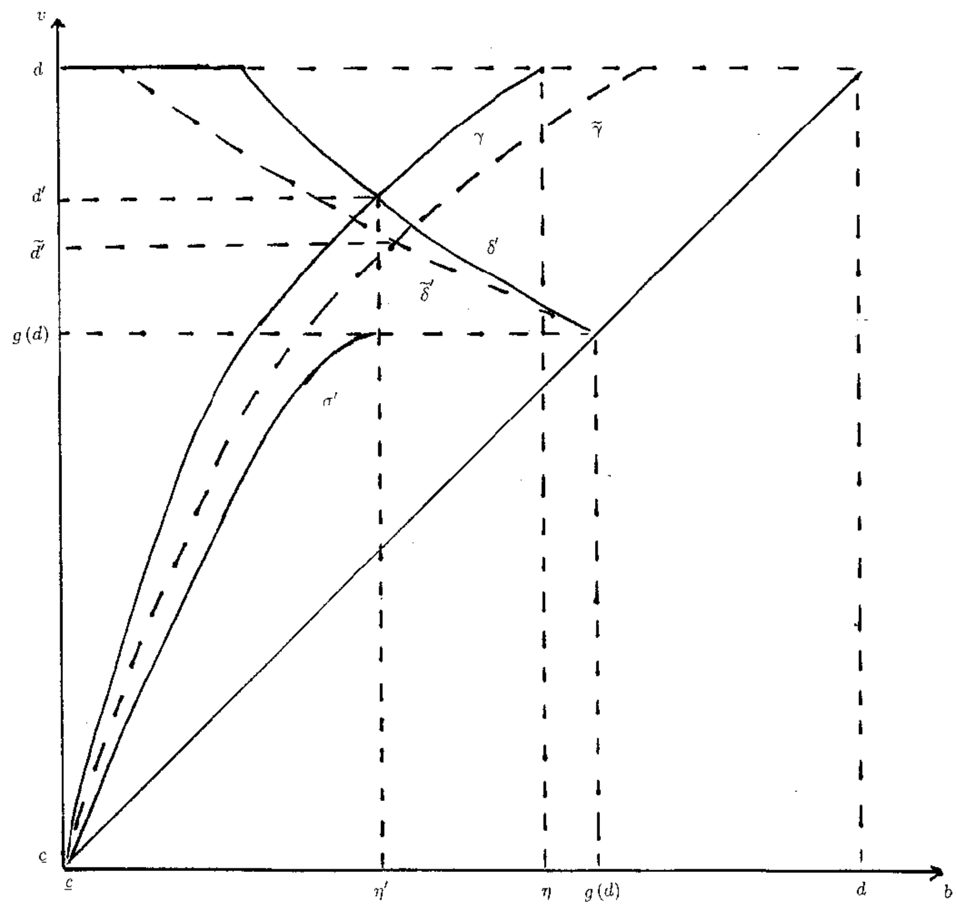

Figure 1

Figure 1:

$$\beta(c) = \varphi(c) = c, \beta(d') = \eta', \varphi(d') = g(d) \text{ (7-6)}.$$

The equations (7-4, 7-5) are actually immediate consequences of the equations (3-2) and (3-3) in Theorem 4, and the conditions (7-6) are immediate consequences of the boundary conditions (3-4). When $\widetilde{d'} < d'$, we immediately see from Figure 2 that $\widetilde{\varphi}(v) > \varphi(v)$ to the left of $\widetilde{d'}$, that is, for all v in an interval $\left(\widetilde{d'} - \varepsilon, \widetilde{d'}\right)$. From the equations (7-4, 7-5) and the initial condition at d' in the boundary condition (7-6), we prove in *Lemma A4-2* that this inequality always holds true even when $\widetilde{d'} = d'$ (which is the case, for example, if $g(d) = d$ since then $\widetilde{d'} = d' = d$). Because the R.H.S. of the equation (7-5) is a strictly decreasing function of $\beta(v)$ and of the ratio m/m', and because, from Theorem 6 (a) $\widetilde{\beta} > \beta$ over the interval $(\underline{c}, d]$, we are able to show in *Lemma A4-3* that the inequality $\widetilde{\varphi} > \varphi$ which holds true over $\left(\widetilde{d'} - \varepsilon, \widetilde{d'}\right)$ must also hold true over the whole interval $\left(\underline{c}, \widetilde{d'}\right)$. Theorem 6 (c) is thus proved.

8. Conclusion

We have studied the discriminatory auction or pay-your-bid auction in a model with multi-unit supply and demands, with two homogenous bidders, and with independent private values. We have proved the existence and uniqueness of the symmetric equilibrium satisfying regularity conditions. We have derived characterizations and properties of the equilibrium. Among these properties, we proved the property of lumpy bidding according to which bidders bid identical bids on units of identical valuations. We have also proved the property of more aggressive bidding on low valuation units than on high valuation units. We have exhibited links between the multi-unit discriminatory and Vickrey auctions with homogenous bidders, on the one hand, and the single-unit first price and second price auctions with heterogenous bidders, on the other. We have proved that, contrary to the Vickrey auction, bundling does not necessarily increase revenues in the discriminatory auction. We have showed that there is no general ranking between the expected revenues at the Vickrey auction and discriminatory auction with homogenous bidders. We have also showed how a change of the bidders' possible demand curves affects the equilibrium strategies, the efficiency of the equilibrium allocation, and the bidders' expected payoffs.

Appendix 1

Lemma A1-1(Proof of Theorem 1 (c)): *If* $(\beta_1, ..., \beta_n; \beta_1, ..., \beta_n)$ *is a symmetric regular equilibrium, then* $\beta_1(\underline{c}) = ... = \beta_m(\underline{c}) = \beta_{m+1}\left(g^{-1}(\underline{c})\right) = ... = \beta_n\left(g^{-1}(\underline{c})\right) = \underline{c}$, *where* $\underline{c} = \max(r, c)$.

Proof: Assume first that $r \geq c$, that is, in a sense that the reserve price r is "binding". Since bids do not exceed valuations, we have $\beta_j(r) \leq r$, for all

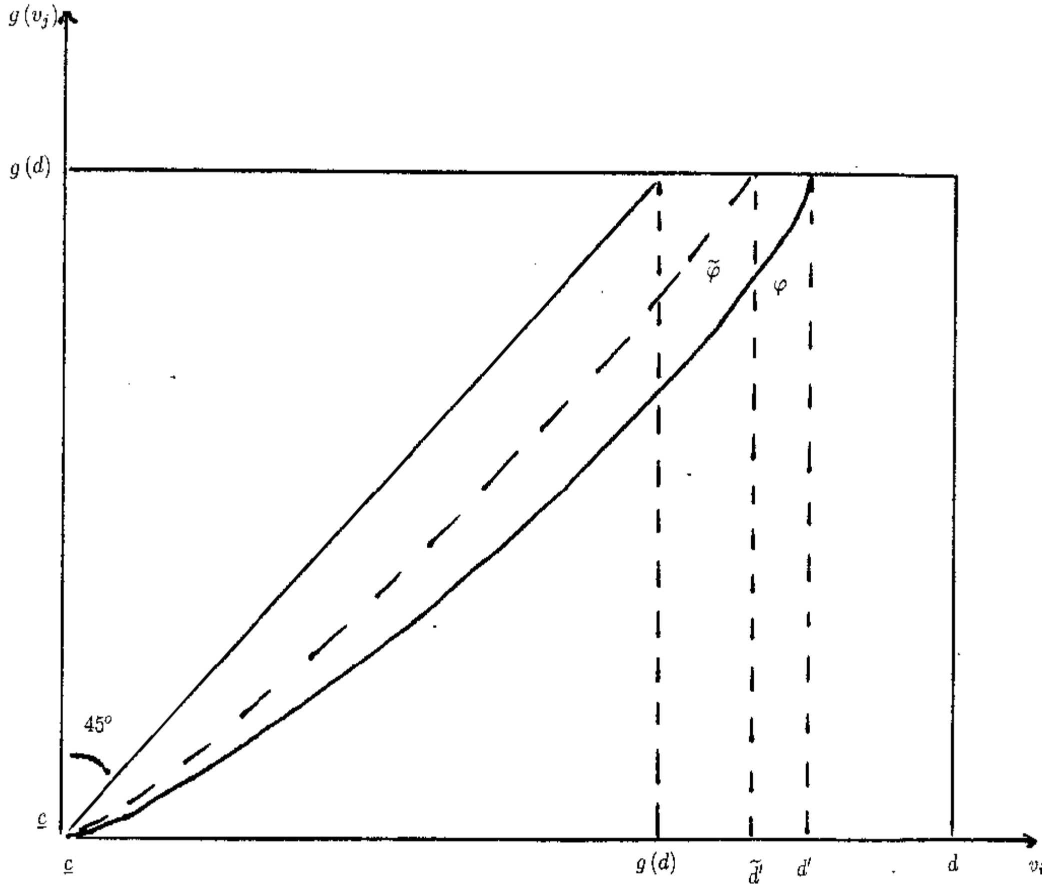

Figure 2

Figure 2:

j. Suppose there exists a type v bidder who submits a bid strictly smaller than r on one of his units of valuation strictly larger than r. Let j be the smallest index of such a unit, that is, let j be the index of the largest such bid. Then, the bidder would increase strictly his expected payoff if he submitted instead a jth bid strictly between r and the minimum of his valuation on his jth unit and his $(j-1)$th bid. In fact, this new jth bid will win with a strictly positive probability since his opponent's valuation on his $(n-j+1)$th unit and thus his bid on this unit will be strictly smaller with a strictly positive probability. Thus, $\beta_i(v) > r$, for all $v > r$ and $i \leq m$, and $\beta_i(v) > r$, for all $v > g^{-1}(r)$ and all $i \geq m+1$. By continuity, we then find the equalities in the statement of the lemma.

Assume that $r < c$, that is, that the reserve price is not binding. From the definition of regular strategies, we immediately obtain $\beta_1(c), ..., \beta_n(c) \leq c$. Suppose that not all equalities in $\beta_1(c) = ... = \beta_n(c)$ hold true. Then, there will exist $j \leq (n+1)/2$ such that $\beta_{j-1}(c) > \beta_j(c) = ... = \beta_{n-j+1}(c)$ or $\beta_j(c) = ... = \beta_{n-j+1}(c) > \beta_{n-j+2}(c)$ (or both). In the former case, it will be more advantageous for a type c bidder to change his jth, ..., $(n-j+1)$th bids from $\beta_j(c) = ... = \beta_{n-j+1}(c)$, which wins a probability zero, to any strictly higher bid $b \leq \beta_{j-1}(c)$. This former case is thus impossible and we must have $\beta_{j-1}(c) = \beta_j(c) = ... = \beta_{n-j+1}(c)$. Then in the latter case, it will be strictly profitable to a type v bidder, with v close to c, to increase his $(n-j+2)$th bid from $\beta_{n-j+2}(v) < \beta_{n-j+1}(c)$ to, for example, $\left(\beta_{n-j+2}(v) + \beta_{n-j+1}(c)\right)/2$ since this bid is strictly larger than $\beta_{j-1}(c)$ and thus wins with a strictly positive probability. ||

Lemma A1-2: *If* $(\sigma_1, \sigma_2) = (\beta_1, ..., \beta_n; \beta_1, ..., \beta_n)$ *is a regular symmetric equilibrium then* $\beta_1(d) = ... = \beta_m(d)$ *and* $\beta_{m+1}(d) = ... = \beta_n(d)$.

Proof: Let $(\sigma_1, \sigma_2) = (\beta_1, ..., \beta_n; \beta_1, ..., \beta_n)$ be a regular symmetric equilibrium. We first show that $\beta_1(d) = ... = \beta_m(d)$. Assume this is not the case and let j be defined as the smallest index such that $\beta_j(d) < \beta_1(d)$. We have $1 < j \leq m$. We then see that for all v in $\left[\gamma_{n-j+2}\left(\beta_j(d)\right), d\right]$ an $(n-j+1)$th bid b_{n-j+1} equal to $\beta_{n-j+2}(v)$ is strictly better than any strictly higher bid. Consequently, $\beta_{n-j+1}(v) = \beta_{n-j+2}(v)$ for all v in $\left[\gamma_{n-j+2}\left(\beta_j(d)\right), d\right]$ and thus $\gamma_{n-j+1}(b) = \gamma_{n-j+2}(b)$, for all b in $\left[\beta_j(d), \beta_1(d)\right]$, where $\eta = \beta_1(d)$. From Lemma 1 (Section 3), $\beta_{j-1}(d) = \beta_1(d)$ maximizes $(d-b)\, F\left(\gamma_{n-j+2}(b)\right)$ over $\left[\beta_j(d), \beta_{j-1}(d) = \beta_1(d)\right]$ and thus $(d-\eta) \geq \left(d - \beta_j(d)\right) F\left(\gamma_{n-j+2}\left(\beta_j(d)\right)\right) = \left(d - \beta_j(d)\right) F\left(\gamma_{n-j+1}\left(\beta_j(d)\right)\right)$. From the same lemma, $\beta_j(d)$ maximizes $(d - b_j)\, F\left(\gamma_{n-j+1}(b_j)\right)$ over $\left[\beta_j(d), \beta_{j-1}(d) = \eta\right]$ and thus $\left(d - \beta_j(d)\right) F\left(\gamma_{n-j+1}\left(\beta_j(d)\right)\right) \geq (d - \beta_1(d))$. The two last inequalities imply

$$\left(d - \beta_j(d)\right) F\left(\gamma_{n-j+1}\left(\beta_j(d)\right)\right) = (d - \beta_1(d))\, (9)(\text{A1-1}).$$

For all v in $\left[\gamma_{j-1}\left(\beta_j(d)\right), d\right)$, $\beta_{j-1}(v)$ maximizes $(v-b)\, F\left(\gamma_{n-j+2}(b)\right)$ over $\left[\beta_j(v), \beta_{j-1}(v)\right]$. Notice that, for all such v, $\beta_j(v) < \beta_j(d) \leq \beta_{j-1}(v)$.

Consequently, $\left[\frac{d_r}{db_{j-1}} \ln(v - b_{j-1}) F\left(\gamma_{n-j+2}(b_{j-1})\right)\right]_{b_{j-1}=\beta_{j-1}(v)} \geq 0$, for all such v, or, equivalently, $\left[\frac{d_r}{db_{j-1}} \ln(v - b_{j-1}) F\left(\gamma_{n-j+2}(b_{j-1})\right)\right]_{v=\gamma_{j-1}(b)} \geq 0$, for all b in $\left[\beta_j(d), \eta\right)$. We thus have

$$\frac{d_r}{db} \ln F\left(\gamma_{n-j+2}(b)\right) \geq \frac{1}{\gamma_{n-j+2}(b) - b}$$

for all b in $\left[\beta_j(d), \eta\right)$. However, $\frac{d_r}{db} \ln(d-b) F\left(\gamma_{n-j+2}(b)\right) = \frac{d_r}{db} \ln(d-b) F\left(\gamma_{n-j+1}(b)\right)$ is equal to $\frac{d_r}{db} \ln F\left(\gamma_{n-j+1}(b)\right) - \frac{1}{d-b}$. Consequently,

$$\frac{d_r}{db} \ln(d-b) F\left(\gamma_{n-j+1}(b)\right) \geq \frac{1}{\gamma_{n-j+2}(b) - b} - \frac{1}{d-b} > 0$$

for all b in $\left[\beta_j(d), \eta\right)$, since $\gamma_{n-j+2}(b) < d$ for all such b. The function $(d-b) F\left(\gamma_{n-j+1}(b)\right)$ is thus strictly increasing over $\left[\beta_j(d), \beta_1(d)\right)$ and (A1-1) is impossible. We have thus proved $\beta_1(d) = ... = \beta_m(d)$. The proof of $\beta_{m+1}(d) = ... = \beta_n(d)$ is similar. ||

Lemma A1-3(Proof of Theorem 1 (d)): *If $g(d) = d$ or if $m \leq n/2$,*

then $\beta_1(d) = ... = \beta_m(d) = \beta_{m+1}(d) = ... = \beta_n(d)$.

Proof: Assume this is not the case. Then $\beta_1(d) = ... = \beta_m(d) > \beta_{m+1}(d) = ... = \beta_n(d)$. Consider first the case $m \leq n/2$. Then $n-m+1 > m$ and a bidder with type d would strictly increase his payoff if he submitted instead a mth bid equal to $\beta_{m+1}(d)$. This contradicts lemma 1 in Section 3 and the lemma is proved in this case.

Consider now the case $m > n/2$ and $g(d) = d$. Reasoning as in the previous proof, we see that $\beta_{n-m}(v) = \beta_{n-m+1}(v)$, for all v in $\left[\gamma_{n-m+1}\left(\beta_{m+1}(d)\right), d\right]$, and thus $\gamma_{n-m}(b) = \gamma_{n-m+1}(b)$, for all b in $\left[\beta_{m+1}(d), \beta_1(d)\right]$. Since $\beta_m(d)$ must be the best mth bid, we have

$$(d - \beta_1(d)) \geq \left(d - \beta_{m+1}(d)\right) F\left(\gamma_{n-m+1}\left(\beta_{m+1}(d)\right)\right) = \left(d - \beta_{m+1}(d)\right) F\left(\gamma_{n-m}\left(\beta_{m+1}(d)\right)\right)$$

and since $\beta_{m+1}(d)$ must be the best $(m+1)$th bid, we have

$$\left(g(d) - \beta_{m+1}(d)\right) F\left(\gamma_{n-m}\left(\beta_{m+1}(d)\right)\right) = \left(d - \beta_{m+1}(d)\right) F\left(\gamma_{n-m+1}\left(\beta_{m+1}(d)\right)\right) \geq (d - \beta_1(d))$$

These two last inequalities imply

$$\left(d - \beta_{m+1}(d)\right) F\left(\gamma_{n-m+1}\left(\beta_{m+1}(d)\right)\right) = d - \beta_1(d) \text{ (A1-2)}$$

.

For all v in $\left[\gamma_m\left(\beta_{m+1}(d)\right), d\right)$, $\beta_m(v)$ maximizes $(v - b_m) F\left(\gamma_{n-m+1}(b_m)\right)$ over $\left[\beta_{m+1}(v), \beta_m(v)\right]$. Notice that, for all such v, $\beta_{m+1}(v) < \beta_{m+1}(d) \leq$

$\beta_m(v)$. Consequently, $\left[\frac{d_l}{db_m}\ln(v-b_m)F\left(\gamma_{n-m+1}(b_m)\right)\right]_{b_m=\beta_m(v)} \geq 0$, for all such v, or, equivalently, $\left[\frac{d_l}{db_m}\ln(v-b_m)F\left(\gamma_{n-m+1}(b_m)\right)\right]_{v=\gamma_m(b)} \geq 0$, for all b in $\left[\beta_{m+1}(d),\beta_1(d)\right]$. We thus have

$$\frac{d_l}{db}\ln F\left(\gamma_{n-m+1}(b)\right) \geq \frac{1}{\gamma_{n-m+1}(b)-b}$$

for all b in $\left[\beta_{m+1}(d),\eta\right)$. However, $\frac{d_l}{db}\ln(g(d)-b)F\left(\gamma_{n-m}(b)\right) = \frac{d_l}{db}\ln(d-b)F\left(\gamma_{n-m+1}(b)\right)$ is equal to $\frac{d_l}{db}\ln F\left(\gamma_{n-m+1}(b)\right) - \frac{1}{d-b}$. Consequently,

$$\frac{d_l}{db}\ln(d-b)F\left(\gamma_{n-m+1}(b)\right) \geq \frac{1}{\gamma_{n-m+1}(b)-b} - \frac{1}{d-b} > 0$$

for all b in $\left[\beta_{m+1}(d),\beta_1(d)\right)$, since $\gamma_{_{n-m+1}}(b) < d$ for all such b. The function $(d-b)F\left(\gamma_{n-j+1}(b)\right)$ is thus strictly increasing over $\left[\beta_{m+1}(d),\beta_1(d)\right)$ and (A1-2) is impossible. ||

We denote by η the common maximum of the bid functions $\beta_1,...,\beta_m$ and by η' the common maximum of the bid functions $\beta_{m+1},...,\beta_n$, that is,

$$\begin{aligned}\eta &= \beta_1(d) = ... = \beta_m(d)\\ \eta' &= \beta_{m+1}(d) = ... = \beta_n(d)\end{aligned}$$

Lemma A1-4: *Assume that the equalities $\beta_1(v) = ... = \beta_m(v)$, for all v in $\left[\gamma_m\left(\beta_{m+1}(d)\right),d\right]$ do not simultaneously hold. Then there exist $\beta_{m+1}(d) < b^* \leq \beta_m(d) = \eta$, $\delta > 0$, $I = \{i_1,...,i_t\} \subseteq \{1,...,m-1\}$, $i_1 < ... < i_t$, such that*

$$\gamma_1(b) = ... = \gamma_{i_1}(b) < \gamma_{i_1+1}(b) = ... = \gamma_{i_2}(b) < ... < \gamma_{i_t+1}(b) = ... = \gamma_m(b) \quad \text{(A1-3)}$$

, for all b in $(b^*-\delta,b^*)$, the functions $\gamma_1,...,\gamma_m$ are differentiable over $(b^*-\delta,b^*)$, and there exists $1 \leq t^* \leq t-1$ such that $\gamma_{i_{t^*}}(b^*) = \gamma_{i_{t^*}+1}(b^*)$.

Proof: Assume that the equalities $\beta_1(v) = ... = \beta_m(v)$, for all v in $\left[\gamma_m\left(\beta_{m+1}(d)\right),d\right]$ do not simultaneously hold. Then there would exist $\eta > b' \geq \beta_{m+1}(d)$ and $i \neq m$ such that $\gamma_i(b') < \gamma_{i+1}(b')$. By continuity, we can assume that $b' > \beta_{m+1}(d)$. Let $b^{(1)} = b'$ and let I_1 be the set of indices defined as follows:

$$I_1 = \left\{i \in \{1,...,m-1\} \mid \gamma_i\left(b^{(1)}\right) < \gamma_{i+1}\left(b^{(1)}\right)\right\}$$

We thus have $I_1 \neq \emptyset$ and $\gamma_j\left(b^{(1)}\right) = \gamma_{j+1}\left(b^{(1)}\right)$, for all $j \in \{1,...,m-1\} \setminus I_1$.

Let ε_1 be a strictly positive number such that $\varepsilon_1 < \min\left(\eta - b^{(1)}, b^{(1)} - \beta_{m+1}(d)\right)$ and such that $\gamma_i(b) < \gamma_{i+1}(b)$, for all b in the interval $V_1 = \left(b^{(1)}-\varepsilon_1, b^{(1)}+\varepsilon_1\right)$. Let $j_1,...,j_k$ be the elements of $\{1,...,m-1\} \setminus I_1$. Either $\gamma_{j_1}(b) = \gamma_{j_1+1}(b)$, for all b in V_1or there exists b'' in V_1 such that $\gamma_{j_1}(b'') < \gamma_{j_1+1}(b'')$. In the former case, let $b^{(2)} = b^{(1)}$, $I_2 = I_1$, $\varepsilon_2 = \varepsilon_1$, $V_2 = V_1$ and in the latter case let $b^{(2)} = b''$, $I_2 = I_1 \cup \{j_1\}$, and $0 < \varepsilon_2 < \varepsilon_1$ be such that $\gamma_i(b) < \gamma_{i+1}(b)$, for all b in the interval $V_2 = \left(b^{(2)} - \varepsilon_2, b^{(2)} + \varepsilon_2\right)$. Proceed then similarly for j_2. Continuing in this manner, after $k = \#(\{1, ..., m-1\} \setminus I_1)$ steps we obtain $b^{(k)}, I_k, \varepsilon_k, V_k = \left(b^{(k)} - \varepsilon_k, b^{(k)} + \varepsilon_k\right)$such that $\gamma_i(b) < \gamma_{i+1}(b)$, for all b in V_k and all i in $I_k \neq \emptyset$, $\gamma_j(b) = \gamma_{j+1}(b)$, for all b in V_k and all j in $\{1, ..., m-1\} \setminus I_k$, and .

Define b^* as follows:

$$b^* = \sup \left\{ \begin{array}{c} \bar{b} \geq b^{(k)} \mid \gamma_i(b) < \gamma_{i+1}(b), \gamma_j(b) = \gamma_{j+1}(b), \\ \text{for all b in } \left(b^{(k)}, \bar{b}\right), \text{ all i in } I_k, \text{ and all } j \text{ in } \{1, ..., m-1\} \setminus I_k \end{array} \right\}$$

From the previous paragraph and Lemma A1-2, we have $b^{(k)} + \varepsilon_k \leq b^* \leq \eta$. By continuity, there exists l in I_k such that $\gamma_l(b^*) = \gamma_{l+1}(b^*)$.

Let I_k be equal to $\{i_1, ..., i_t\}$, where $t = \#I_k$ and with $i_1 < ... < i_t \leq m-1$. Since $\gamma_1 \leq ... \leq \gamma_n$, the set $\{1, ..., m-1\} \backslash I_k$ is equal to $\left(\left([1, i_1 - 1] \cup \bigcup_{s=1}^{t} [i_s + 1, i_{s+1} - 1] \cup [i_t + 1, m-1]\right) \cap\right.$ From the definition of b^* and by continuity, we have $\gamma_i(b) < \gamma_{i+1}(b)$, $\gamma_j(b) = \gamma_{j+1}(b)$,for all b in $\left(b^{(k)}, b^*\right)$, all i in I_k, and all j in $\{1, ..., m-1\} \setminus I_k$. Since $\gamma_1, ..., \gamma_n$ are piecewise differentiable over $(\underline{c}, \eta]$, there exists $\delta > 0$ such that $\gamma_1, ..., \gamma_n$ are differentiable over $(b^* - \varepsilon, b^*)$. The lemma is then proved by taking $I = I_k$. ||

Lemma A1-5: $\gamma_i(b^*) < \gamma_{i+1}(b^*)$ *if and only if* $\gamma_{n-i}(b^*) < \gamma_{n-i+1}(b^*)$, *for all* b^* *in* $(\underline{c}, \eta]$ *such that* $b^* \neq \eta'$ *and all* i *such that* $i+1 \leq m$ *and* $n-i+1 \leq m$.

Proof: Assume there exist i such that $i + 1 \leq m$ and b^* such that b^* in $(\underline{c}, \eta]$ such that $b^* \neq \eta'$ and $\gamma_i(b^*) < \gamma_{i+1}(b^*)$. From Lemma A1-2, we also have $b^* < \eta$. Let (b', b'') be an open interval such that $b^* \in (b', b'')$, $\eta' \notin (b', b'')$, $b'' \leq \eta$, and $\gamma_i(b) < \gamma_{i+1}(b)$, for all b in (b', b''). Since $\eta' \notin (b', b'')$ and $b'' \leq \eta$, the definition of regular strategies imply that all functions $\gamma_1, ..., \gamma_n$ are differentiable over (b', b''). Since $b_i = b$ maximizes $(\gamma_i(b) - b_i) F\left(\gamma_{n-i+1}(b_i)\right)$ over $\left[\beta_{i+1}(\gamma_i(b)), b\right]$ and since $\beta_{i+1}(\gamma_i(b)) < \beta_{i+1}\left(\gamma_{i+1}(b)\right) = b$, we have $\left[\frac{d}{db_i} \ln(\gamma_i(b) - b_i) F\left(\gamma_{n-i+1}(b_i)\right)\right]_{b_i = b} \geq 0$ and thus

$$\frac{d}{db} \ln F\left(\gamma_{n-i+1}(b)\right) \geq \frac{1}{\gamma_i(b) - b}$$

for all b in (b', b''). Similarly, $b_{i+1} = b$ maximizes $\left(\gamma_{i+1}(b) - b_{i+1}\right) F\left(\gamma_{n-i}(b_{i+1})\right)$ over $\left[b, \beta_i\left(\gamma_{i+1}(b)\right)\right]$ and since $\beta_i\left(\gamma_{i+1}(b)\right) > \beta_i(\gamma_i(b)) = b$, we have $\left[\frac{d}{db_{i+1}} \ln\left(\gamma_{i+1}(b) - b_{i+1}\right) F\left(\gamma_{n-i}(b_{i+1})\right)\right.$ 0 and thus

$$\frac{d}{db} \ln F\left(\gamma_{n-i}(b)\right) \leq \frac{1}{\gamma_{i+1}(b) - b}$$

for all b in (b', b'') . Since $\gamma_i(b) < \gamma_{i+1}(b)$, we obtain

$$\frac{d}{db} \ln F\left(\gamma_{n-i}(b)\right) < \frac{d}{db} \ln F\left(\gamma_{n-i+1}(b)\right) \text{ (A1-4)}$$

for all b in (b', b''). Consequently, $\gamma_{n-i}(b) \neq \gamma_{n-i+1}(b)$, for all b in (b', b''). In fact, if there existed $\widetilde{b}$ in (b', b'') such that $\gamma_{n-i}\left(\widetilde{b}\right) = \gamma_{n-i+1}\left(\widetilde{b}\right)$(A1-4) would imply $\frac{d}{db}\gamma_{n-i}\left(\widetilde{b}\right) < \frac{d}{db}\gamma_{n-i+1}\left(\widetilde{b}\right)$ and γ_{n-i} would be strictly larger than γ_{n-i+1}over a left-hand neighborhood of $\widetilde{b}$ which is impossible. We have thus proved that $\gamma_i(b^*) < \gamma_{i+1}(b^*)$ implies $\gamma_{n-i}(b^*) < \gamma_{n-i+1}(b^*)$, for all $b^* > c$ and all $1 \leq i \leq m-1$. The proof of the Lemma A1-5 is complete if we then apply this result to $i' = n-i$ in the statement of the lemma. ||

Lemma A1-6: $\gamma_1(b) = ... = \gamma_m(b)$, *for all b in* $[\eta', \eta]$.

Proof: Assume that some of the equalities in the statement of the lemma do not true over the interval $[\eta', \eta]$. Then, from Lemma A1-4 there would exist a bid b^* in $(\eta', \eta]$, $\delta > 0$, and two consecutive groups (possibly counting only one element) of inverse bid functions which "separate" at b^* , that is, there would exist i, k, j such that $1 \leq i \leq k < k+1 \leq j \leq m$,

$$\gamma_{i-1}(b) < \gamma_i(b) = ... = \gamma_k(b) < \gamma_{k+1}(b) = ... = \gamma_j(b) < \gamma_{j+1}(b) \text{ (A1-5)}$$

for all b in $(b^* - \delta, b^*)$, and

$$\gamma_i(b^*) = ... = \gamma_k(b^*) = \gamma_{k+1}(b^*) = ... = \gamma_j(b^*) \text{ (A1-6)}$$

. If $n-k+1 \geq m+1$, (A1-5) is clearly impossible. In fact, otherwise the type $\gamma_k(b)$ bidder would strictly increase his expected payoff if he decreased his kth bid b to, for example, $\max\left(\beta_{k+1}(b), \beta_{n-k+1}(b) = \eta'\right) < b$.

Assume next that $n-k+1 < m+1$ or, equivalently, that $n-k < m$.From Lemma A1-5, (A1-5) and (A1-6) imply

$$\begin{aligned} \gamma_{n-j}(b) &< \gamma_{n-j+1}(b) = ... = \gamma_{n-k}(b) < \gamma_{n-k+1}(b) = ... = \gamma_{\max(n-i+1,m)}(b) \\ &< \gamma_{\max(n-i+1,m)+1}(b) \text{ (A1-7)} \end{aligned}$$

for all b in $(b^* - \delta, b^*)$ and

$$\gamma_{n-j+1}(b^*) = ... = \gamma_{n-k}(b^*) = \gamma_{n-k+1}(b^*) \text{ (A1-8)}$$

. Since from (A1-5) small equal and simultaneous changes of the $(k+1)$th bid to the jth bid are feasible they must not be profitable and we find the first order condition $\frac{d}{db}\ln F\left(\gamma_{n-k}(b)\right) = \frac{1}{\gamma_{k+1}(b)-b}$, for all b in $(b^* - \delta, b^*)$. Similarly from (A1-7) simultaneous changes of the $(n-j+1)$th bid to the $(n-k)$th bid are feasible and hence unprofitable. We thus have also $\frac{d}{db}\ln F\left(\gamma_{k+1}(b)\right) = \frac{1}{\gamma_{n-k}(b)-b}$, for all b in $(b^* - \delta, b^*)$. From (A1-7), simultaneous changes of the $(n-k+1)$th bid to the $\min(n-i+1, m)$th bid are feasible and thus we find $\frac{d}{db}\ln F(\gamma_k(b)) = \frac{1}{\gamma_{n-k+1}(b)-b}$, for all b in $(b^* - \delta, b^*)$. From (A1-5), small decreases of the kth bid are possible and are thus unprofitable It implies $\frac{d}{db}\ln F\left(\gamma_{n-k+1}(b)\right) \geq \frac{1}{\gamma_k(b)-b}$, for all b in $(b^* - \delta, b^*)$. Summing up, over the interval $(b^* - \delta, b^*)$ $\left(\gamma_{k+1}, \gamma_{n-k}\right)$ satisfies the system of differential equations (A1-9) and $\left(\gamma_k, \gamma_{n-k+1}\right)$ satisfies the system of differential inequations (A1-10) below:

$$\begin{aligned}
\frac{d}{db} \ln F\left(\gamma_{k+1}(b)\right) &= \frac{1}{\gamma_{n-k}(b) - b}, \frac{d}{db} \ln F\left(\gamma_{n-k}(b)\right) = \frac{1}{\gamma_{k+1}(b) - b} \text{(A1-9)} \\
\frac{d}{db} \ln F\left(\gamma_k(b)\right) &= \frac{1}{\gamma_{n-k+1}(b) - b}, \frac{d}{db} \ln F\left(\gamma_{n-k+1}(b)\right) \geq \frac{1}{\gamma_k(b) - b} \text{(A1-10)}
\end{aligned}$$

Moreover, from (A1-6) and (A1-8) $\left(\gamma_{k+1}, \gamma_{n-k}\right)$ and $\left(\gamma_k, \gamma_{n-k+1}\right)$ satisfy the same initial condition at b^*:

$$\gamma_{k+1}(b^*) = \gamma_k(b^*), \gamma_{n-k}(b^*) = \gamma_{n-k+1}(b^*)$$

. As it can then be easily seen, it implies that the inequalities $\gamma_k(b) \leq \gamma_{k+1}(b)$ and $\gamma_{n-k+1}(b) \leq \gamma_{n-k}(b)$ hold true over $[b^* - \delta, b^*]$. The latter inequality contradicts the second inequality in (A1-7) and we have thus proved $\gamma_1(b) = \ldots = \gamma_m(b)$, for all b in $[\eta', \eta]$. ||

Lemma A1-7: *$\gamma_i(b^*) < \gamma_{i+1}(b^*)$ implies $\gamma_{n-i}(b^*) < \gamma_{n-i+1}(b^*)$, for all b^* in $(\underline{c}, \eta']$ and all $i \neq m$.*

Proof: Assume there exist $i \leq m-1$ and b^* in $(\underline{c}, \eta']$ such that $\gamma_i(b^*) < \gamma_{i+1}(b^*)$. From the previous lemma, we also have $b^* < \eta'$. The rest of the proof is then similar to the proof of the previous lemma. Assume then that there exists $i \geq m+1$ and b^* in $(\underline{c}, \eta']$ such that $\gamma_i(b^*) < \gamma_{i+1}(b^*)$. From Lemma A1-2, $b^* < \eta'$. Let (b', b'') be an open interval such that $b^* \in (b', b'')$, $b'' \leq \eta'$, and $\gamma_i(b) < \gamma_{i+1}(b)$, for all b in (b', b''). Since $\underline{c}, \eta', \eta \notin (b', b'')$, the definition of regular strategies imply that all functions $\gamma_1, \ldots, \gamma_n$ are differentiable over (b', b''). Since $b_i = b$ maximizes $\left(g\left(\gamma_i(b)\right) - b_i\right) F\left(\gamma_{n-i+1}(b_i)\right)$ over $\left[\beta_{i+1}\left(\gamma_i(b)\right), b\right]$ and since $\beta_{i+1}\left(\gamma_i(b)\right) < \beta_{i+1}\left(\gamma_{i+1}(b)\right) = b$, we have $\left[\frac{d}{db_i} \ln\left(g\left(\gamma_i(b)\right) - b_i\right) F\left(\gamma_{n-i+1}(b_i)\right)\right]_{b_i = b} \geq 0$ and thus

$$\frac{d}{db} \ln F\left(\gamma_{n-i+1}(b)\right) \geq \frac{1}{g\left(\gamma_i(b)\right) - b}$$

for all b in (b', b''). Similarly, $b_{i+1} = b$ maximizes $\left(g\left(\gamma_{i+1}(b)\right) - b_{i+1}\right) F\left(\gamma_{n-i}(b_{i+1})\right)$ over $\left[b, \beta_i\left(\gamma_{i+1}(b)\right)\right]$ and since $\beta_i\left(\gamma_{i+1}(b)\right) > \beta_i\left(\gamma_i(b)\right) = b$, we have $\Big[\frac{d}{db_{i+1}} \ln\left(g\left(\gamma_{i+1}(b)\right) - b_{i+1}\right) F\left(\gamma_{n-i}(b_{i+}\right.$
0 and thus

$$\frac{d}{db} \ln F\left(\gamma_{n-i}(b)\right) \leq \frac{1}{g\left(\gamma_{i+1}(b)\right) - b}$$

for all b in (b', b''). Since $\gamma_i(b) < \gamma_{i+1}(b)$, we obtain

$$\frac{d}{db} \ln F\left(\gamma_{n-i}(b)\right) < \frac{d}{db} \ln F\left(\gamma_{n-i+1}(b)\right) \text{(A1-11)}$$

for all b in (b', b''). Consequently, $\gamma_{n-i}(b) \neq \gamma_{n-i+1}(b)$, for all b in (b', b''). In fact, if there existed $\widetilde{b}$ in (b', b'') such that $\gamma_{n-i}\left(\widetilde{b}\right) = \gamma_{n-i+1}\left(\widetilde{b}\right)$(A1-11) would imply $\frac{d}{db}\gamma_{n-i}\left(\widetilde{b}\right) < \frac{d}{db}\gamma_{n-i+1}\left(\widetilde{b}\right)$ and γ_{n-i} would be strictly larger than γ_{n-i+1}over a left-hand neighborhood of $\widetilde{b}$ which is impossible. The proof of the lemma is complete. ||

Lemma A1-8: $\beta_m(v) > \beta_{m+1}(v)$, *for all* v *in* $(\underline{c}, d)$.

Proof: If there existed v in this interval such that $\beta_m(v) = \beta_{m+1}(v)$, then we would have $\gamma_m(b^*) = \gamma_{m+1}(b^*)$ with $b^* = \beta_m(v) = \beta_{m+1}(v)$ and thus $\underline{c} < b^* < \eta'$. Let i be the smallest index not larger than m such that $\beta_i(v) = b^*$ and let j be the largest index not smaller than $(m+1)$ such that $\beta_j(v) = b^*$. From the definitions of i and j, we have

$$\gamma_{i-1}(b^*) < v = \gamma_i(b^*) = ... = \gamma_m(b^*) = \gamma_{m+1}(b^*) = ... = \gamma_j(b^*) < \gamma_{j+1}(b^*)$$

From the definition of regular strategies, all inverse bid functions are differentiable at b^*. Since an increase of the ith bid alone is feasible, it must not be profitable and we obtain the inequality below

$$\frac{d}{db}\ln F\left(\gamma_{n-i+1}(b^*)\right) \leq \frac{1}{v-b^*}$$

A decrease of the jth bid is also feasible and thus cannot be profitable. We find

$$\frac{d}{db}\ln F\left(\gamma_{n-j+1}(b^*)\right) \geq \frac{1}{g(v)-b^*}$$

and consequently $\frac{d}{db}\ln F\left(\gamma_{n-j+1}(b^*)\right) > \frac{d}{db}\ln F\left(\gamma_{n-i+1}(b^*)\right)$. Since their derivatives are different at b^*, γ_{n-j+1} and γ_{n-i+1} also differ at this bid since otherwise the two functions would cross. We thus have

$$\gamma_{n-j+1}(b^*) < \gamma_{n-i+1}(b^*) \text{ (A1-12)}$$

From Lemma A1-7 and $\gamma_i(b^*) = \gamma_j(b^*)$, we must have $n-j+1 \leq m$ and $m+1 \leq n-i+1$. The first inequality implies $(n+1)/2 < j$ and the second inequality implies $i < (n+1)/2$. Since $(n+1)/2$ is strictly between i and j, if $n-i+1$ was strictly larger than j, Lemma A1-7 and $\gamma_j(b^*) < \gamma_{j+1}(b^*)$ would imply $\gamma_{n-j}(b^*) < \gamma_{n-j+1}(b^*)$, with $i \leq n-j \leq n-j+1 \leq j$, which would contradict the definition of i and j. Similarly, we can prove that $n-i+1 < j$ is impossible. We thus find $n-j+1 = i$ or, equivalently, $n-i+1 = j$. The inequality (A1-12) then contradicts $\gamma_i(b^*) = \gamma_j(b^*)$. ||

Lemma A1-9 below can be considered as an extension of Lemma A1-4.

Lemma A1-9: *Assume that the equalities* $\beta_1(v) = ... = \beta_m(v)$ *and* $\beta_{m+1}(v) = ... = \beta_n(v)$, *for all* v *in* $[\underline{c}, d]$ *do not simultaneously hold. Then* *there exist* $c < b^* \leq \eta'$, $\delta > 0$, $I = \{i_1, ..., i_t\} \subseteq \{1, ..., m-1\}$, $i_1 < ... < i_t$, $K = \{k_1, ..., k_l\} \subseteq \{m+1, ..., n-1\}$, $k_1 < ... < k_l$, *such that*

$$\gamma_1(b) = ... = \gamma_{i_1}(b) < \gamma_{i_1+1}(b) = ... = \gamma_{i_2}(b) < ... < \gamma_{i_t+1}(b) = ... = \gamma_m(b) \text{ (A1-13)}$$

$$\gamma_{m+1}(b) = ... = \gamma_{k_1}(b) < \gamma_{k_1+1}(b) = ... = \gamma_{k_2}(b) < ... < \gamma_{k_l+1}(b) = ... = \gamma_m(b) \text{ (A1-14)}$$

, *for all b in* $(b^* - \delta, b^*)$, *the functions* $\gamma_1, ..., \gamma_n$ *are differentiable over* $(b^* - \delta, b^*)$, *and there exists* $1 \leq t^* \leq t-1$ *such that* $\gamma_{i_{t^*}}(b^*) = \gamma_{i_{t^*}+1}(b^*)$ *or there exists* $1 \leq l^* \leq l-1$ *such that* $\gamma_{k_{l^*}}(b^*) = \gamma_{k_{l^*}+1}(b^*)$.

Proof: Assume that the equalities $\beta_1(v) = ... = \beta_m(v)$ and $\beta_{m+1}(v) = ... = \beta_n(v)$, for all v in $[\underline{c}, d]$ do not simultaneously hold. Then, from Lemma A1-6 there would exist $\eta' > b' > \underline{c}$ and $i \neq m$ such that $\gamma_i(b') < \gamma_{i+1}(b')$. It suffices then to make a construction as in the proof of Lemma A1-4 with the n functions $\gamma_1, ..., \gamma_n$ over the set of bids $[\underline{c}, \eta']$. From Lemma A1-8, the functions γ_m and γ_{m+1} will belong to different groups. If the bid b^* one obtains through this construction is strictly smaller than η', from Lemma A1-8 again the two consecutive groups which coincide at b^* must be either included in $\{\gamma_1, ..., \gamma_m\}$ or in $\{\gamma_{m+1}, ..., \gamma_n\}$ and the statement of the lemma holds true. If $b^* = \eta'$, this statement also holds true since, for example, at η' all functions $\gamma_{m+1}, ..., \gamma_n$ coincide (from Lemma A1-2). ||

Lemma A1-10: *Let* $(\sigma_1, \sigma_2) = (\beta_1, ..., \beta_n; \beta_1, ..., \beta_n)$ *be a regular symmetric equilibrium. Assume there exist* $c < b^* \leq \eta'$, $\delta > 0$, $I = \{i_1, ..., i_t\} \subseteq \{1, ..., m-1\}$, $i_1 < .. < i_t$, $K = \{k_1, ..., k_l\} \subseteq \{m+1, ..., n-1\}$, $k_1 < ... < k_l$, *and* t^* *as in the previous lemma. Then* $n - i_{t^*} = m$.

Proof: By assumption, we have (A1-13) and (A1-14) for all b in $(b^* - \delta, b^*)$ and two groups both in the same line coincide at b^*. Assume for example that these two groups belong to the first line and there thus exists $1 \leq t^* \leq t-1$ such that

$$\gamma_{i_{t^*-1}}(b^*) = ... = \gamma_{i_{t^*}}(b^*) = \gamma_{i_{t^*}+1}(b^*) = ... = \gamma_{i_{t^*+1}}(b^*)$$

From Lemma A1-8 we know that $\gamma_m(b) < \gamma_{m+1}(b)$, for all b in $(b^* - \delta, b^*)$.

To simplify the notations, denote i_{t^*-1} by i, i_{t^*} by k, and i_{t^*+1} by j. Summing up, we have $\underline{c} < b^* \leq \eta'$, $i < k < k+1 < j \leq m$, and

$$\gamma_{i-1}(b) < \gamma_i(b) = ... = \gamma_k(b) < \gamma_{k+1}(b) = ... = \gamma_j(b) < \gamma_{j+1}(b) \text{ (A1-15)}$$

$$\gamma_m(b) < \gamma_{m+1}(b) \text{ (A1-16)}$$

, for all b in $(b^* - \delta, b^*)$, and

$$\gamma_i(b^*) = ... = \gamma_k(b^*) = \gamma_{k+1}(b^*) = ... = \gamma_j(b^*) \text{ (A1-17)}$$

. Assume first that $n - k \geq m+1$. Then, from Lemma A1-7 we see that (A1-15) and (A1-17) imply

$$\begin{aligned} \gamma_{\max(m+1,n-j+1)-1}(b) &< \gamma_{\max(m+1,n-j+1)}(b) = ... = \gamma_{n-k}(b) \\ &< \gamma_{n-k+1}(b) = ... = \gamma_{n-i+1}(b) < \gamma_{n-i+2}(b) \text{ (A1-18)} \end{aligned}$$

for all b in $(b^* - \delta, b^*)$, and

$$\gamma_{\max(m+1,n-j+1)}(b^*) = ... = \gamma_{n-k}(b^*) = \gamma_{n-k+1}(b^*) = ... = \gamma_{n-i+1}(b^*)$$

. Equal deviations of the $(n-k+1)$th bid to the $(n-i+1)$th bid are feasible and thus unprofitable. and we find $\frac{d}{db}\ln F(\gamma_k(b)) = \frac{1}{g(\gamma_{n-k+1}(b))-b}$, for all b in $(b^* - \delta, b^*)$. Similarly, equal deviations of the th bid to the kth bid are feasible and must thus be unprofitable. We find $\frac{d}{db}\ln F(\gamma_{n-k+1}(b)) = \frac{1}{\gamma_k(b)-b}$, for all b in $(b^* - \delta, b^*)$. Simultaneous deviations of the $\max(m+1, n-j+1)$th bid to the $(n-k)$th bid are also possible and cannot be profitable and thus $\frac{d}{db}\ln F(\gamma_{k+1}(b)) = \frac{1}{g(\gamma_{n-k}(b))-b}$, for all b in $(b^* - \delta, b^*)$. Small increases of the $(k+1)$th bid are also possible and we thus obtain the condition $\frac{d}{db}\ln F(\gamma_{n-k}(b)) \leq \frac{1}{\gamma_{k+1}(b)-b}$, for all b in $(b^* - \delta, b^*)$. Over the interval $(b^* - \delta, b^*)$, $(\gamma_k, \gamma_{n-k+1})$ is thus a solution of the system (A1-19) of differential equations and $(\gamma_{k+1}, \gamma_{n-k})$ is a solution of the system (A1-20) of differential inequations below:

$$\frac{d}{db}\ln F(\gamma_k(b)) = \frac{1}{g(\gamma_{n-k+1}(b))-b}, \frac{d}{db}\ln F(\gamma_{n-k+1}(b)) = \frac{1}{\gamma_k(b)-b} \quad \text{(A1-19)}$$

$$\frac{d}{db}\ln F(\gamma_{k+1}(b)) = \frac{1}{g(\gamma_{n-k}(b))-b}, \frac{d}{db}\ln F(\gamma_{n-k}(b)) \leq \frac{1}{\gamma_{k+1}(b)-b} \quad \text{(A1-20)}$$

Moreover, at $b = b^*$ they satisfy the same initial condition, that is:

$$\gamma_k(b^*) = \gamma_{k+1}(b^*), \gamma_{n-k+1}(b^*) = \gamma_{n-k}(b^*)$$

As it can be easily seen, it then implies that $\gamma_{k+1}(b) \geq \gamma_k(b)$ and $\gamma_{n-k}(b) \geq \gamma_{n-k+1}(b)$, for all b in $(b^* - \delta, b^*]$. The latter inequality contradicts the second inequality in (A1-18) and we have thus ruled out the case $n-k \geq m+1$. Ruling out the case $n-k+1 \leq m$ is similar. ||

Lemma A1-11: *Let $(\sigma_1, \sigma_2) = (\beta_1, ..., \beta_n; \beta_1, ..., \beta_n)$ be a regular symmetric equilibrium. Assume there exist $c < b^* \leq \eta'$, $\delta > 0$, $I = \{i_1, ..., i_t\} \subseteq \{1, ..., m-1\}$, $i_1 < .. < i_t$, $K = \{k_1, ..., k_l\} \subseteq \{m+1, ..., n-1\}$, $k_1 < ... < k_l$, and l^* as in the previous lemma. Then $n - k_{l^*} = m$.*

Proof: Similar to the proof of Lemma A1-10. ||

Proof of Theorem 1 (a) (Section 3): If the equalities in Theorem 1

(a) did not hold simultaneously, then from Lemmas A1-6, A1-9 there would exist $\delta > 0$, $I = \{i_1, ..., i_t\} \subseteq \{1, ..., m-1\}$, $i_1 < .. < i_t$, $K = \{k_1, ..., k_l\} \subseteq \{m+1, ..., n-1\}$, $k_1 < ... < k_l$, and t^* or l^* and b^* as in the statement of Lemma A1-9. Assume there exists t^* as in the statement of Lemma A1-9 (the proof when there exists l^* as in the statement of Lemma A1-9 is similar). From Lemma A1-10, then $n - i_{t^*} = m$ (and thus $m > (n+1)/2$). Lemma A1-7 then implies $n-i_{t^*-1}+1=k_1$. Simplifying the notations as in the proof of Lemma A1-10, we find

$$\begin{aligned}\gamma_{i-1}(b) &< \gamma_i(b)=...=\gamma_{n-m}(b)<\gamma_{n-m+1}(b)=...=\gamma_m(b)\\ &< \gamma_{m+1}(b)=...=\gamma_{n-i+1}(b)<\gamma_{n-i+2}(b)\,\text{(A1-21)}\end{aligned}$$

for all b in $(b^*-\delta,b^*)$, where $i\le n-m<m$. We can rewrite then $\gamma_{i_{t^*}}(b^*)=\gamma_{i_{t^*}+1}(b^*)$ in Lemma A1-9 as (A1-22) below

$$\gamma_i(b^*)=...=\gamma_{n-m}(b^*)=\gamma_{n-m+1}(b^*)=...=\gamma_m(b^*)\,\text{(A1-22)}$$

. Since we have ruled out any other possible "separation", the inequality (A1-21) must hold over $(\underline{b},b^*)$ and $\gamma_1(\underline{b})=...=\gamma_m(\underline{b})$ and $\gamma_{m+1}(\underline{b})=...=\gamma_n(\underline{b})$, with $\underline{b}\le b^*-\delta$. Such a $\underline{b}$ must exist, since from Theorem 1 (c), which we already proved, we have $\gamma_1(\underline{c})=...=\gamma_m(\underline{c})=\underline{c}$ and $\gamma_{m+1}(\underline{c})=...=\gamma_n(\underline{c})=g^{-1}(\underline{c})$. Since equal changes of all bids from the $(n-m+1)$th bid to the mth bid are feasible and thus unprofitable, we obtain the first order condition below $\frac{d}{db}\ln F(\gamma_m(b))=\frac{1}{\gamma_m(b)-b}$, for all b in $(\underline{b},b^*)$. Similarly, by considering equal changes of all bids from the $(m+1)$th bid to the $(n-i+1)$th bid we find $\frac{d}{db}\ln F\left(\gamma_{n-m}(b)\right)=\frac{1}{g(\gamma_{m+1}(b))-b}$ and by considering equal changes of all bids from the ith bid to the $(n-m)$th bid we find $\frac{d}{db}\ln F\left(\gamma_{m+1}(b)\right)=\frac{1}{\gamma_{n-m}(b)-b}$, for all b in $(\underline{b},b^*)$. Moreover, since $\gamma_{n-m}(b^*)=\gamma_m(b^*)$ and since γ_{n-m} can never be strictly larger than γ_m, the left-hand derivative of $\ln F\left(\gamma_{n-m}\right)$ at b^* is at least as large as the left-hand derivative of $\ln F(\gamma_m)$ at b^*. Since these left-hand derivatives are the limits of the corresponding two-sided derivatives at b for b tending from below to b^*, the previous differential equations imply $g\left(\gamma_{m+1}(b^*)\right)\le\gamma_m(b^*)$. Denote by $\underline{v}$ the common value of γ_{n-m} and γ_m at $\underline{b}$. Summing up our conclusions, we know that γ_m is the solution over $(\underline{b},b^*]$ of the differential equation (A1-23) with (partial) initial condition (A1-24) below

$$\frac{d}{db}\ln F(\gamma_m(b))=\frac{1}{\gamma_m(b)-b}\text{(A1-23)}$$

$$\gamma_m(\underline{b})=\underline{v}\text{(A1-24)}$$

and we know that $\left(\gamma_{n-m},\gamma_{m+1}\right)$ is the solution over $(\underline{b},b^*]$ of the system of differential equations (A1-25) with (partial) initial condition (A1-26) below

$$\frac{d}{db}\ln F\left(\gamma_{n-m}(b)\right)=\frac{1}{g\left(\gamma_{m+1}(b)\right)-b},\frac{d}{db}\ln F\left(\gamma_{m+1}(b)\right)=\frac{1}{\gamma_{n-m}(b)-b}\text{(A1-25)}$$

$$\gamma_{n-m}(\underline{b})=\underline{v}\text{(A1-26)}$$

. Moreover we know that (A1-27) below holds true

$$\gamma_{n-m}(b^*)\le\gamma_m(b^*)\,,\,g\left(\gamma_{m+1}(b^*)\right)\le\gamma_m(b^*)\,\text{(A1-27)}$$

From Lemma A6-2, it is a property of the equation (A1-23) and the system (A1-25) that no such solutions can exist. We have thus ruled out the last possible case of "separation" and Theorem 1 (a) is proved. ||

Appendix 2

We denote the objective function in the maximization problem of Lemma 1 in Section 3 by $\mathcal{P}(v; b_1, ..., b_n)$. Since the expression for the objective function is meaningful over the product $\mathcal{R}^n$ and because it will be convenient in our proofs, we consider that the function $\mathcal{P}$ is defined over this product .

Lemma A2-1: *Assume that $m > m'$. Let (β, β') define a regular strategy and let η, η', and d' be defined as follows, $\eta = \beta(d), \eta' = \beta'(d), d' = \beta^{-1}(\eta')$. Then (β, β') is a symmetric regular equilibrium if and only if*

(1) $g(d) \leq d'$; $\eta' < g(d)$,

$$\eta' \leq g(d) + (g(d) - d')\left(\frac{m}{m'} - 1\right) F(d') \text{ (A2-1)}$$

and if $\eta' < \eta$ then

$$\eta' = g(d) + (g(d) - d')\left(\frac{m}{m'} - 1\right) F(d') \, 200\text{(A2-2)}$$

(2) for all v in $[d', d]$

$$\beta(v) = v - \frac{(d' - \eta')(m' + (m - m') F(d')) + \int_{d'}^{v} (m' + (m - m') F(u)) \, du}{m' + (m - m') F(v)}$$

and

$$\beta(v) = v - \frac{(d' - g(d))(1 + (m/m' - 1) F(d'))^2 + \int_{d'}^{v} (1 + (m/m' - 1) F(u)) \, du}{1 + (m/m' - 1) F(v)}$$

(3) $\gamma = \beta^{-1}$ and $\gamma' = \beta'^{-1}$ satisfy over $[\underline{c}, \eta']$ the system (A2-3,A2-4) considered on the domain $D' = \{(b, \gamma, \gamma') \mid d \geq \gamma > b, g(\gamma') > b, \gamma > c, d \geq \gamma' > c\}$ with initial conditions (A2-5) and (A2-6) below

$$\frac{d}{db} \ln F(\gamma(b)) = \frac{1}{g(\gamma'(b)) - b} \text{(A2-3)}$$

$$\frac{d}{db} \ln F(\gamma'(b)) = \frac{1}{\gamma(b) - b}\left\{1 + \left(\frac{m}{m'} - 1\right) \frac{F(\gamma(b))}{F(\gamma'(b))} \left(\frac{g(\gamma'(b)) - \gamma(b)}{g(\gamma'(b))(b) - b}\right)\right\} \text{(A2-4)}$$

$$\gamma(\eta') = d', \gamma'(\eta') = d\text{(A2-5)}$$

$$\gamma(\underline{c}) = \underline{c}, \gamma'(\underline{c}) = g^{-1}(\underline{c}) \text{ (A2-6)}$$

such that

$$g(\gamma'(b)) \leq \gamma(b) \text{ (A2-7)}$$

, for all b in $[\underline{c}, \eta']$.

Proof: We first prove the necessity of the listed properties. Here, $\mathcal{P}(v;b_1,...,b_n)$ is given by the following equation:

$$\mathcal{P}(v;b_1,...,b_n) = \sum_{i=1}^{m'}(v-b_i)F(\gamma'(b_i)) + \sum_{i=m'+1}^{m}(v-b_i)F(\gamma(b_i)) + \sum_{i=m+1}^{n}(g(v)-b_i)F(\gamma(b_i)) \quad \text{(A2-8)}$$

From Lemma 1 and Theorem 1 in Section 3, we have

$$\left(\beta(v),...,\beta(v),\beta'(v)...,\beta'(v)\right) \in \arg\max_{\underline{c}\le b_1...\le b_n} \mathcal{P}(v;b_1,...,b_n) \quad \text{(A2-9)}$$

for all v in $[\underline{c},d]$ and thus in particular

$$\left(\beta(v),\beta'(v)\right) \in \arg\max_{\underline{c}\le b\le b'} \mathcal{P}(v;b,...,b,b',...,b')$$

for all v in $[\underline{c},d]$. For $\underline{c}\le b,b'\le\eta'$ $\mathcal{P}(v;b,...,b,b',...,b')$ is equal to

$$m'(v-b)F(\gamma'(b)) + (m-m')(v-b)F(\gamma(b)) + m'(g(v)-b')F(\gamma(b'))$$

From Theorem 1 (b) in Section 3, we have $\beta(v)<\beta'(v)$, for all v in $(\underline{c},d)$. Consequently, we have $\frac{\partial}{\partial b'}\mathcal{P}(v;b,...,b,b',...,b') = -m'F(\gamma(b'))+m'(g(v)-b')f(\gamma(b'))\frac{d}{db}\gamma(b') = 0$ at $b'=\beta'(v)$, for all v in $\left(g^{-1}(\underline{c}),d\right)$, or, equivalently,

$$\frac{d}{db}\ln F(\gamma(b)) = \frac{1}{g(\gamma'(b))-b}$$

for all v in $(\underline{c},\eta']$, if the derivative when $b=\eta'$ is a left-hand derivative. We thus proved (A2-3) above. Moreover, $\frac{\partial}{\partial b}\pi(v;b,...,b,b',...,b')$ is given by the expression below

$$-m'F(\gamma'(b)) - (m-m')F(\gamma(b)) + m'(v-b)f(\gamma'(b))\frac{d}{db}\gamma'(b) + (m-m')(v-b)f(\gamma(b))\frac{d}{db}\gamma(b)$$

It must be equal to 0 at $b=\beta(v)$, for all v in $(\underline{c},d')$, and we find the differential equation below:

$$m'F(\gamma'(b)) + (m-m')F(\gamma(b)) = m'(\gamma(b)-b)f(\gamma'(b))\frac{d}{db}\gamma'(b) + (m-m')(\gamma(b)-b)f(\gamma(b))\frac{d}{db}\gamma(b) \quad \text{(A2-10)}$$

for all b in $(\underline{c},\eta')$. Substituting in (A2-10) $f(\gamma(b))\frac{d}{db}\gamma(b)$ by its value from (A2-3) and rearranging, we find that the equation (A2-4) above holds true for all b in $(\underline{c},\eta')$, and thus all b in $(\underline{c},\eta']$ if the derivative at η' is a left-hand derivative.

The initial condition (A2-5) is an immediate consequence of the definitions and (A2-6) follows from Theorem 1 (c) (Section 3).

From the definition of a regular strategy, we have $\frac{d}{db}\ln F\left(\gamma'\left(\eta'\right)\right)\geq 0$. Substituting η' to b in (A2-4), we find $\frac{1}{d'-\eta'}\left\{1+\left(\frac{m}{m'}-1\right)F\left(d'\right)\frac{g(d)-d'}{g(d)-\eta'}\right\}\geq 0$. Rearranging the factor between braces, we find (A2-1).

From (A2-9), the inequality $\frac{\partial}{\partial b_1}\mathcal{P}\left(v;\beta\left(v\right),...,\beta\left(v\right),\beta'\left(v\right),...,\beta'\left(v\right)\right)\leq 0$ has to hold for all v in $(\underline{c},d)$, and, in particular, for all v in $(\underline{c},d')$, and we obtain through the change of variables $v=\gamma\left(b\right)$, after diving by $F\left(\gamma'\left(b\right)\right)\left(\gamma'\left(b\right)-b\right)$, and after substituting $\frac{d}{db}\ln F\left(\gamma'\left(b\right)\right)$ by its value in (A2-4):

$$\left(\frac{m}{m'}-1\right)\frac{F\left(\gamma\left(b\right)\right)}{F\left(\gamma'\left(b\right)\right)}\left(\frac{g\left(\gamma'\left(b\right)\right)-\gamma\left(b\right)}{g\left(\gamma'\left(b\right)\right)-b}\right)\leq 0$$

and consequently we must have the inequality (A2-7) for all b in $(\underline{c},\eta')$ and thus for all b in $[\underline{c},\eta']$.

For all b in $[\eta',\eta]$, $\gamma'\left(b\right)=d$ and, from (A2-8), the derivative $\frac{\partial}{\partial b}\mathcal{P}\left(v;b,...,b,b',...,b'\right)$ is equal to $-\left[m'+\left(m-m'\right)F\left(\gamma\left(b\right)\right)\right]+\left(v-b\right)\left(m-m'\right)f\left(\gamma\left(b\right)\right)\frac{d}{db}\gamma\left(b\right)$. It must be equal to 0 at $b=\beta\left(v\right)$, for all v in (d',d), or, equivalently, $v=\gamma\left(b\right)$, for all b in (η',η), and we see that γ is a solution over (η',η) of the following differential equation considered in the domain $\mathcal{E}=\{(b,\gamma)\mid\gamma>c,\gamma>b\}$:

$$\frac{d}{db}\ln F\left(\gamma\left(b\right)\right)=\frac{1}{\gamma\left(b\right)-b}\left[1+\frac{m'}{m-m'}\frac{1}{F\left(\gamma\left(b\right)\right)}\right]\text{(A2-11)}$$

Setting $b=\beta\left(v\right)$ and rearranging, we find

$$\frac{d}{dv}\left\{\left(v-\beta\left(v\right)\right)\left(m'+\left(m-m'\right)F\left(v\right)\right)\right\}=\left(m'+\left(m-m'\right)F\left(v\right)\right)\text{(A2-12)}$$

, for all v in (d',d). Integrating from d' to v in (d',d) and using the condition $\beta\left(d'\right)=\eta'$, we find the equation in (2).

Substituting η' to b in (A2-7), we find the first inequality $g\left(d\right)\leq d'$ in (1). The second inequality $\eta'<g\left(d\right)$ is an immediate consequence of the definition of regular strategies. If $\eta'<\eta$, from Lemma 1 in Section 3 a type d bidder cannot increase his payoff $\mathcal{P}$ by submitting an $(m+1)\,th$ bid strictly larger than η' and thus the right-hand derivative of $\ln\left(g\left(d\right)-b\right)F\left(\gamma\left(b\right)\right)$ must be nonpositive at $b=\eta'$. However, from $\frac{d}{db}\ln\left(g\left(d\right)-b\right)F\left(\gamma\left(b\right)\right)$ is equal to $-1/\left(g\left(d\right)-b\right)+\frac{d}{db}\ln\left(F\left(\gamma\left(b\right)\right)\right)$ and by substituting to the last term its value given in (A2-11), we find that $\frac{d}{db}\ln\left(g\left(d\right)-b\right)F\left(\gamma\left(b\right)\right)$ is equal to $-1/\left(g\left(d\right)-b\right)+\left[1+\left(m'/\left(m-m'\right)\right)1/F\left(\gamma\left(b\right)\right)\right]/\left(\gamma\left(b\right)-b\right)$. Consequently, we must have $-1/\left(g\left(d\right)-\eta'\right)+\left(1+\left(m'/\left(m-m'\right)\right)1/F\left(d'\right)\right)/\left(d'-\eta'\right)\leq 0$ and after rearranging we find $\eta'\geq g\left(d\right)+\left(g\left(d\right)-d'\right)\left(\frac{m}{m'}-1\right)F\left(d'\right)$. Together with (A2-1), this inequality implies (A2-2) and we have completed the necessity part of the proof.

We now prove the sufficiency of (1),(2),and (3), that is, from Lemma 1 in Section 3, that if $\left(\beta,\beta'\right)$ is a regular strategy and thus if β and β' are nondecreasing and $\beta \geq \beta'$, the conditions (1), (2), and (3) imply (A2-9). We can separate $\mathcal{P}(v; b_1, ..., b_n)$ into the three components $\kappa(v; b_1, ..., b_{m'}) = \sum_{i=1}^{m'} (v - b_i) F(\gamma'(b_i))$, $\lambda(v; b_{m'+1}, ..., b_n) = \sum_{i=m'+1}^{m} (v - b_i) F(\gamma(b_i))$, and $\mu(v; b_{m+1}, ..., b_n) = \sum_{i=m+1}^{n} (g(v) - b_i) F(\gamma(b_i))$. First we show that $\beta'(v)$ is a solution of the unconstrained problem $\max_{\underline{c} \leq b} (g(v) - b) F(\gamma(b))$, for all v in $[\underline{c}, d]$. Using a standard argument from auction theory, it is simple to prove that because the equation (A2-3) holds true and because $\beta = \gamma^{-1}$ is strictly increasing, $\beta'(v) = \gamma'^{-1}(v)$ is a solution of the problem $\max_{b \in [\underline{c}, \eta']} (g(v) - b) F(\gamma(b))$, for all v in $[\underline{c}, d]$. Suppose $\eta' < \eta$. Then from (A2-11), which is implied by (2), $\frac{d}{db} \ln(g(v) - b) F(\gamma(b))$ is equal to $\frac{-1}{g(v)-b} + \frac{1 + \frac{m'}{m-m'} \frac{1}{F(\gamma(b))}}{\gamma(b) - b}$ and thus to

$$\left[\frac{g(v) - g(d)}{g(d) - b}\right] + \left[\frac{-1}{g(d) - b} + \frac{1 + \frac{m'}{m-m'} \frac{1}{F(\gamma(b))}}{\gamma(b) - b}\right] \text{(A2-13)}$$

, for all b in $(\eta', \eta]$. The first term between brackets in (A2-13) is a nonpositive since $v \leq d$. The second term between brackets is the same sign as $-\left(1 + \frac{\gamma(b) - g(d)}{g(d) - b}\right) + \frac{m'}{m-m'} \frac{1}{F(\gamma(b))}$. This expression is thus strictly decreasing in γ and is a noninceasing function of b since $\gamma(b) \geq b \geq d' \geq g(d)$, for all b$\geq d'$. At $b = \eta'$, this second term of (A2-13) is equal up to the factor $(g(d) - \eta')(d' - \eta') \frac{F(d')}{m-m'}$ to $g(d) + (g(d) - d') F(d') \left(\frac{m}{m'} - 1\right) - \eta'$. From the assumption (A2-1), it is thus equal to zero at $b = \eta'$ and consequently the second term in (A2-13), the entire expression in (A2-13), and $\frac{d}{db} \ln(g(v) - b) F(\gamma(b))$ are nonpositive, for all b $\geq \eta'$ and all v in $[\underline{c}, d]$. Consequently, $\beta(v) = \gamma'^{-1}(v)$ is a solution of the problem $\max_{\underline{c} \leq b} (g(v) - b) F(\gamma(b))$, for all v in $[c, d]$, and thus the m'-tuple $(\beta'(v), ..., \beta'(v))$ is a solution of the problem $\max_{\underline{c} \leq b_{m+1}, ..., b_n} \mu(v; b_{m+1}, ..., b_n)$ and we obtain

$$\max_{\underline{c} \leq b_{m+1}, ..., b_n} \mu(v; b_{m+1}, ..., b_n) = \mu(v; \beta'(v), ..., \beta'(v)) \text{ (A2-14)}$$

Let $\left(\widetilde{b}_1, ..., \widetilde{b}_m\right)$ be an element of $\arg\max_{\underline{c} \leq b_m \leq ... \leq b_1} (\kappa(v; b_1, ..., b_{m'}) + \lambda(v; b_{m'+1}, ..., b_n))$ and let $\overline{b}$ and $\widehat{b}$ be elements of $\arg\max_{b \in [\widetilde{b}_{m'}, \widetilde{b}_1]} (v - b) F(\gamma'(b))$ and $\arg\max_{b \in [\widetilde{b}_n, \widetilde{b}_{m'+1}]} (v - b) F(\gamma(b))$, respectively. Then $\kappa\left(v; \widetilde{b}_1, ..., \widetilde{b}_{m'}\right) + \lambda\left(v; \widetilde{b}_{m'+1}, ..., \widetilde{b}_n\right) \leq \kappa\left(v; \overline{b}, ..., \overline{b}\right) + \lambda\left(v; \widehat{b}, ..., \widehat{b}\right)$ and we have proved

$$\begin{aligned} & \max_{\underline{c} \leq b_m \leq ... \leq b_1} (\kappa(v; b_1, ..., b_{m'}) + \lambda(v; b_{m'+1}, ..., b_n)) \\ = & \max_{\underline{c} \leq b'' \leq b} (\kappa(v; b, ..., b) + \lambda(v; b'', ..., b'')) \\ = & \max_{\underline{c} \leq b'' \leq b} m'(v - b) F(\gamma'(b)) + (m - m')(v - b'') F(\gamma(b'')) \end{aligned}$$

From (A2-4), for all $1 \leq i \leq m$, $\frac{d}{db}(v - b) F(\gamma'(b))$ is equal to $F(\gamma'(b))(v - b)$ $\left[\frac{-1}{v-b} + \frac{1}{\gamma(b)-b} + \left(\frac{m}{m'} - 1\right) \frac{1}{\gamma(b)-b} \frac{F(\gamma(b))}{F(\gamma'(b))} \left(\frac{g(\gamma'(b)) - \gamma(b)}{g(\gamma'(b)) - b}\right)\right]$ and since from (A2-6) the third term between the brackets is nonpositive we have

$$\frac{d}{db}(v-b)\,F(\gamma'(b)) \leq 0 \text{(A2-15)}$$

for all $1 \leq i \leq m$,all $b \geq \beta(v)$ or, equivalently, all b such that $\gamma(b) \geq v$.

From (A2-3), $\frac{d}{db}(v-b)\,F(\gamma(b))$ is equal to $F(\gamma(b))\,(v-b)\left[\frac{-1}{v-b}+\frac{1}{g(\gamma'(b))-b}\right]$. However, from (A2-7) we have $v \geq g(\gamma'(b))$, for all b such that $\beta(v) \geq b$ or, equivalently, such that $v \geq \gamma(b)$. Consequently, we find

$$\frac{d}{db}(v-b)\,F(\gamma(b)) \geq 0 \text{(A2-16)}$$

for all b such that $\beta(v) \geq b$.

Let (b'', b) be an element of $\arg\max_{\underline{c} \leq b'' \leq b} m'(v-b)\,F(\gamma'(b)) + (m-m')(v-b'')\,F(\gamma(b''))$. If $\beta(v) \geq b$ and thus $\beta(v) \geq b''$, (A2-16) implies $m'(v-b)\,F(\gamma'(b)) + (m-m')(v-b'')\,F(\gamma(b''))$ $\leq m'(v-b)\,F(\gamma'(b)) + (m-m')(v-b)\,F(\gamma(b))$. If $\beta(v) \leq b''$ and thus $\beta(v) \leq b$, (A2-15) implies $m'(v-b)\,F(\gamma'(b)) + (m-m')(v-b'')\,F(\gamma(b''))$ $\leq m'(v-b'')\,F(\gamma'(b'')) + (m-m')(v-b'')\,F(\gamma(b''))$. If $b'' \leq \beta(v) \leq b$, (A2-15) and (A2-16) imply $m'(v-b)\,F(\gamma'(b)) + (m-m')(v-b)\,F(\gamma(b)) \leq$ $m'(v-\beta(v))\,F(\gamma'(\beta(v))) + (m-m')(v-\beta(v))\,F(\gamma(\beta(v)))$. In all cases, we thus have

$$\begin{aligned} & \max_{\underline{c} \leq b'' \leq b} m'(v-b)\,F(\gamma'(b)) + (m-m')(v-b'')\,F(\gamma(b'')) \\ = \; & \max_{\underline{c} \leq b} m'(v-b)\,F(\gamma'(b)) + (m-m')(v-b)\,F(\gamma(b)) \end{aligned}$$

.

From (A2-3) and (A2-4), up to the strictly positive factor $F(\gamma'(b))\,F(\gamma(b))$ the derivative$\frac{d}{db}\{m'(v-b)\,F(\gamma'(b)) + (m-m')(v-b)\,F(\gamma(b))\}$ is thus equal to

$$\begin{aligned} -\frac{m'}{F(\gamma'(b))} - \frac{\left(m-m'\right)}{F(\gamma(b))} + \frac{m'(v-b)}{F(\gamma(b))(\gamma(b)-b)}\left\{1 + \left(\frac{m}{m'}-1\right)\frac{F(\gamma(b))}{F(\gamma'(b))}\left(\frac{g\left(\gamma'(b)\right)-\gamma(b)}{g(\gamma'(b))-b}\right)\right\} \\ + \frac{\left(m-m'\right)(v-b)}{F(\gamma'(b))}\frac{1}{g(\gamma'(b))-b} \end{aligned}$$

Obviously, this expression vanishes at $v = \gamma(b)$. Moreover, since $\left(\beta, \beta'\right)$ is a regular strategy $\frac{d}{db}\gamma'$ is nonnegative over $(\underline{c}, \eta')$ and the factor between braces in the R.H.S. of (A2-4) is nonnegative. Consequently, the expression between brackets above is a nondecreasing function of v and thus it is nonnegative if $v \geq \gamma(b)$ or, equivalently, $\beta(v) \leq b$ and it is nonpositive if $v \leq \gamma(b)$ or, equivalently, $\beta(v) \leq b$. We have thus proved that $\beta(v)$ is a solution of $\max_{\underline{c} \leq b} m'(v-b)\,F(\gamma'(b)) + (m-m')(v-b)\,F(\gamma(b))$.

Consequently we have

$$\max_{\underline{c} \leq b_n \leq \ldots \leq b_1} \pi(v; b_1, ..., b_n) = \pi\left(v; \beta(v), ..., \beta(v), \beta'(v), ..., \beta'(v)\right)$$

and the proof is complete. ||

Lemma A2-2: *Let (β, β') be a symmetric regular equilibrium and let η' and d' be as in Lemma A2-1. If $g(d) < d$ then $g(d) < d'$.*

Proof: Assume that $g(d) < d$. We already know that $\eta' < g(d)$. Suppose that $g(d) = d'$. Then we would have $d' < d$ and thus $\eta' < \eta$ and the equation (A2-2) applies, that is, in this case $\eta' = g(d)$ and we would obtain a contradiction. ||

By rewriting the differential equations (A2-3,A2-4), the initial condition (A2-5), and the condition (A2-7) in the valuation spaces, we see that (3) in Lemma A2-1 above is equivalent (3)' below:

(3)' $\gamma = \beta^{-1}$ and $\sigma' = g \circ \beta'^{-1}$ satisfy over $[c, \eta']$ the system (A2-17, A2-18) considered on the domain $\mathcal{D}' = \{(b, \gamma, \sigma') \mid d \geq \gamma > b, \sigma' > b, \gamma > c, g(d) \geq \sigma' > c\}$ with initial conditions (A2-18) and (A2-20) below

$$\frac{d}{db} \ln F(\gamma(b)) = \frac{1}{\sigma'(b) - b} \text{(A2-17)}$$

$$\frac{d}{db} \ln H(\sigma'(b)) = \frac{1}{\gamma(b) - b} \left\{ 1 + \left(\frac{m}{m'} - 1 \right) \frac{F(\gamma(b))}{H(\sigma'(b))} \left(\frac{\sigma'(b) - \gamma(b)}{\sigma'(b) - b} \right) \right\} \text{(A2-18)}$$

$$\gamma(\eta') = d', \sigma'(\eta') = g(d) \text{(A2-19)}$$

$$\gamma(\underline{c}) = \sigma'(\underline{c}) = \underline{c} \text{(A2-20)}$$

such that

$$\sigma'(b) \leq \gamma(b) \text{(A2-2)}$$

, for all b in $[\underline{c}, \eta']$. In (A2-18) above, H is the cumulative distribution function of the lower valuation, that is, the valuation of the last m' units: $H(u) = F(g^{-1}(u))$, for all u in $[\underline{c}, g(d)]$.

The assumptions we have made in this paper (see Section 2) imply that the cumulative distribution functions H and F are differentiable with derivatives h and f which are locally bounded away from zero over $(c, d]$ and $(c, g(d)]$. As we show in the following lemma, these properties are enough to be able to apply to standard results of the theory of ordinary differential equations.

Lemma A2-3: *Through the change of variables $(b, \psi, \psi') = (b, F(\gamma), H(\sigma'))$, the system (A2-17, A2-18) considered on the domain D' with initial condition (A2-19) is equivalent to the differential system (A2-22, A2-23) considered on the domain $O' = \{(b, \psi, \psi') \mid \psi > F(b), \psi' > H(b), 1 \geq \psi > F(c), 1 \geq \psi' > H(c)\}$ in the unknown functions (ψ, ψ') with initial condition (A2-24)*

$$\frac{d}{db} \psi(b) = \frac{\psi(b)}{H^{-1}(\psi'(b)) - b} \text{(A2-22)}$$

$$\frac{d}{db} \psi'(b) = \frac{\psi'(b)}{F^{-1}(\psi(b)) - b} \left\{ 1 + \frac{\psi(b)}{\psi'(b)} \left(\frac{H^{-1}(\psi'(b)) - F^{-1}(\psi(b))}{H^{-1}(\psi'(b)) - b} \right) \right\} \text{(A2-23)}$$

$$\psi(\eta) = F(d'), \psi_3(\eta) = 1 \text{(A2-24)}$$

Moreover, this system with initial condition satisfies the standard assumptions of the theory of ordinary differential equations.

Proof: The transformations of the system, initial conditions, and domain through the change of variables indicated above are immediate. Our assumptions imply that F^{-1} and H^{-1} are locally Lipschitz over $(c, d]$ and $(c, g(d)]$, respectively. The last statement of the lemma follows. $\|$

Lemma A2-4: *Let d' be such that $g(d) \leq d' \leq d$. If (γ, γ') is a solution over $(\rho, \eta']$ of (A2-3,A2-4,A2-5) considered in D' which satisfies (A2-7), then $\gamma'(b) \geq \gamma(b)$, for all b in $[\rho, \eta]$.*

Proof: From (A2-5), we have $\gamma'(\eta') = d \geq \gamma(\eta') = d'$. From (A2-3,A2-4), we have $\frac{d}{db} \ln F(\gamma'(b)) \leq \frac{d}{db} \ln F(\gamma(b))$, for all b in $(\rho, \eta']$. The result follows. $\|$

Lemma A2-5: *Let (γ, σ') be a solution over $(\rho, \eta']$ of the system (A2-17, A2-18) considered on the domain D'' with initial condition (A2-19), with $g(d) \leq d'$. Then, we have*

$$\sigma'(b) < \gamma(b)$$

for all b in (ρ, η').

Proof: If b' in $(\rho, \eta']$ is such that $\sigma'(b') = \gamma(b')$, (A2-17) and (A2-18) imply

$$\frac{d}{db}\sigma'(b') = \frac{H(\sigma'(b'))}{h(\sigma'(b'))} \frac{1}{\gamma(b') - b'} > \frac{F(\gamma(b'))}{f(\gamma(b'))} \frac{1}{\sigma'(b') - b'} = \frac{d}{db}\gamma(b') \text{ (A2-24)}$$

since $H/h > F/f$. From (A2-19) and $d' \geq g(d)$, we have $\sigma'(\eta') \leq \gamma(\eta')$. If $\sigma'(\eta') = \gamma(\eta')$, the inequality (A2-24) implies that there exists $\varepsilon > 0$ such that $\sigma'(b) < \gamma(b)$, for all b in $(\eta' - \varepsilon, \eta')$. We can thus assume without loss of generality that $\sigma'(\eta') < \gamma(\eta')$.

Assume that there exists b' in (ρ, η') such that $\sigma'(b') = \gamma(b')$. Then, the set $\{b' \in (\rho, \eta') \mid \sigma'(b') = \gamma(b')\}$ is not empty and we can define b^* as follows:

$$b^* = \sup\{b' \in (\rho, \eta') \mid \sigma'(b') = \gamma(b')\}$$

By continuity, we can substitute sup in the definition of b^* by max and we have $\sigma'(b^*) = \gamma(b^*)$ and form our assumption $\sigma'(\eta') < \gamma(\eta')$ we have $b^* < \eta'$. From (A2-24), there exists $\delta > 0$ such that $\sigma'(b) > \gamma(b)$, for all b in $(b^*, b^* + \varepsilon)$. The inequality $\sigma'(\eta') < \gamma(\eta')$ then implies the existence of $\widetilde{b}$ in (b^*, η') such that $\sigma'\left(\widetilde{b}\right) = \gamma\left(\widetilde{b}\right)$. This however contradicts the definition of b^* and we have proved Lemma A2-5. $\|$

Lemma A2-6: *Let d' be such that $g(d) \leq d' \leq d$ and let (γ, σ') be a solution over $(\rho, \eta']$ of the system (A2-17, A2-18) considered on the domain D'' with initial condition (A2-19), where $\eta' > c$ and $\eta' \leq g(d) + (g(d) - d')\left(\frac{m}{m'} - 1\right) F(d')$. Then, $\frac{d}{db}\gamma(b) > 0$ and $\frac{d}{db}\sigma'(b) > 0$, for all b in (ρ, η').*

Proof: The conclusion of the lemma is equivalent to $\frac{d}{db}\ln F(\gamma(b)) > 0$ and $\frac{d}{db}\ln H(\sigma'(b)) > 0$, for all b in (ρ, η'). The first inequality is an immediate consequence of (A2-17). From (A2-18) and (A2-19), we have

$$\begin{aligned} \frac{d}{db}\ln H(\sigma'(\eta')) &= \frac{1}{d' - \eta'}\left\{1 + \left(\frac{m}{m'} - 1\right) F(d') \frac{g(d) - d'}{g(d) - \eta'}\right\} \\ &= \frac{1}{d' - \eta'} \frac{g(d) + \left(\frac{m}{m'} - 1\right)(g(d) - d') F(d') - \eta'}{g(d) - \eta'} \geq 0 \quad \text{(A2-25)} \end{aligned}$$

By carrying out the addition in the factor between braces in (A2-18), we see that $\frac{d}{db}\ln H(\sigma'(b))$ is the same sign as the numerator $H(\sigma'(b))(\sigma'(b) - b) + \left(\frac{m}{m'} - 1\right) F(\gamma(b))(\sigma'(b) - \gamma(b))$. Suppose $\frac{d}{db}\ln H(\sigma'(\eta')) = 0$ or, equivalently, this numerator at η', that is, $g(d) + \left(\frac{m}{m'} - 1\right)(g(d) - d') F(d') - \eta'$ is equal to 0. Then, by using $\frac{d}{db}\ln H(\sigma'(\eta')) = 0$, the equation (A2-17), and by taking the derivative in (A2-18), we see that $\frac{d^2}{db^2}\ln H(\sigma'(\eta'))$ is equal to $-2 - \left(\frac{m}{m'} - 1\right) \frac{F(d')}{f(d')} \frac{1}{g(d) - \eta'}$ and is thus strictly negative. Consequently, in all cases $\frac{d}{db}\ln H(\sigma'(b))$ is strictly positive in an interval $(\eta' - \varepsilon, \eta')$, with $\varepsilon > 0$.

Suppose that there exists $\widetilde{b}$ in (ρ, η') such that $\frac{d}{db}\ln H\left(\sigma'\left(\widetilde{b}\right)\right) = 0$. Then, the set $\left\{\widetilde{b} \in (\rho, \eta'] \mid \frac{d}{db}\ln H\left(\sigma'\left(\widetilde{b}\right)\right) = 0\right\}$ is not empty and we can define b^* as follows

$$b^* = \sup\left\{\widetilde{b} \in (\rho, \eta'] \mid \frac{d}{db}\ln H\left(\sigma'\left(\widetilde{b}\right)\right) = 0\right\}$$

From the previous paragraph, we have $b^* < \eta' - \varepsilon$. From (A2-18) and the continuity of all the functions in the R.H.S. of (A2-18), $\frac{d}{db}\ln H\sigma'$ is continuous and thus $\frac{d}{db}\ln H(\sigma'(b^*)) = 0$ and the factor between braces in (A2-18) is equal to 0 or, equivalently, after some rearranging

$$H(\sigma'(b^*))(\sigma'(b^*) - b^*) + \left(\frac{m}{m'} - 1\right) F(\gamma(b^*))(\sigma'(b^*) - \gamma(b^*)) = 0 \quad \text{(A2-26)}$$

. From the definition of b^*, we have $\frac{d}{db}\ln H(\sigma'(b)) > 0$, for all b in (b^*, η'), and thus from (A2-1)

$$H(\sigma'(b))(\sigma'(b) - b) + \left(\frac{m}{m'} - 1\right) F(\gamma(b))(\sigma'(b) - \gamma(b)) > 0 \quad \text{(A2-27)}$$

for all b in (b^*, η').

Consider the function L of two variables defined as follows:

$$L(s, b) = H(s)(s - b) + \left(\frac{m}{m'} - 1\right) F(\gamma(b))(s - \gamma(b))$$

Since $\frac{\partial}{\partial s}L(s,b) > 0$, for all (s,b) such that $g(d) \geq s > b$ and $b \in (\rho, \eta']$, since $L(g(d), b) = (g(d) - b) + \left(\frac{m}{m'} - 1\right) F(\gamma(b))(g(d) - \gamma(b)) \geq 0$, if $g(d) \geq \gamma(b)$, $L(g(d), b) \geq (g(d) - b) + \left(\frac{m}{m'} - 1\right) F(d')(g(d) - \gamma(b)) \geq g(d) - \eta + \left(\frac{m}{m'} - 1\right) F(d')(g(d) - d') \geq 0$, if $g(d) \leq \gamma(b)$, and since $L(b,b) = \left(\frac{m}{m'} - 1\right) F(\gamma(b))(b - \gamma(b)) \leq 0$, the equation

$$L(\widetilde{\sigma}(b), b) = 0$$

determines a function $\widetilde{\sigma}$ over $(\rho, \eta']$ such that $\widetilde{\sigma}(b) > b$, for all b in $(\rho, \eta']$.

From (A2-27), the definition of $\widetilde{\sigma}$, and $\frac{\partial}{\partial s}L(s,b) > 0$, we have $\sigma'(b) > \widetilde{\sigma}(b)$, for all b in (b^*, η'), and from (A2-26) we have $\sigma'(b^*) = \widetilde{\sigma}(b^*)$. Consequently, $\frac{d}{db}\sigma'(b^*) \geq \frac{d}{db}\widetilde{\sigma}(b^*)$. However, from the definition of b^* we have $\frac{d}{db}\sigma'(b^*) = 0$ and thus

$$\frac{d}{db}\widetilde{\sigma}(b^*) \leq 0 \text{(A2-28)}$$

From the definition of $\widetilde{\sigma}$, we have $\frac{d}{db}\widetilde{\sigma}(b) = -\frac{\partial}{\partial b}L(\widetilde{\sigma}(b), b) / \frac{\partial}{\partial s}L(\widetilde{\sigma}(b), b)$, for all b. We already know that $\frac{\partial}{\partial s}L(\widetilde{\sigma}(b), b) > 0$. From the definition of L, we have

$$\frac{\partial}{\partial b}L(s,b) = -H(s) - \left(\frac{m}{m'} - 1\right) F(\gamma(b)) \frac{d}{db}\gamma(b) + \left(\frac{m}{m'} - 1\right)(s - \gamma(b)) \frac{d}{db}F(\gamma(b))$$

The first term is strictly negative and the second term is nonpositive for all $g(d) \geq s > b$ and $b \in (\rho, \eta']$. At $(s,b) = (\widetilde{\sigma}(b^*), b^*)$, up to the factor $\left(\frac{m}{m'} - 1\right)$, the third term is equal to $(\widetilde{\sigma}(b^*) - \gamma(b^*)) \frac{d}{db}F(\gamma(b^*)) = (\sigma'(b^*) - \gamma(b^*)) \frac{d}{db}F(\gamma(b^*))$ which, from the previous lemma, is nonpositive. Consequently, $\frac{d}{db}\widetilde{\sigma}(b^*) > 0$. However this contradicts (A2-28) and the lemma is proved. ‖

Lemma A2-7: *Let d' be such that $g(d) \leq d' \leq d$ and let (γ, σ') be a solution over $(\rho, \eta']$ of the system (A2-17,A2-18) considered on the domain D'' with initial condition (A2-19), where $\eta' \leq g(d) + (g(d) - d')\left(\frac{m}{m'} - 1\right) F(d')$. Then,*

$$H(\sigma'(b))(\sigma'(b) - b) + \left(\frac{m}{m'} - 1\right) F(\gamma(b))(\sigma'(b) - \gamma(b)) > 0 \text{(A2-29)}$$

for all b in (ρ, η').

Proof: Immediate from the previous lemma and (A2-4). ‖

Lemma A2-8: *Let (γ, σ') and $\left(\widehat{\gamma}, \widehat{\sigma}'\right)$ be two solutions over $(\rho, \eta']$ of the system (A2-17,A2-18) considered on the domain D'' such that $\gamma(\eta') < \widehat{\gamma}(\eta')$ and $\sigma'(\eta') < \widehat{\sigma}'(\eta')$. Then $\gamma(b) < \widehat{\gamma}(b)$ and $\sigma'(b) < \widehat{\sigma}'(b)$, for all b in $(\rho, \eta']$.*

Proof: Define b^* as follows:

$$b^* = \min \left\{ b' \in [\rho, \eta'] \mid \gamma(b) < \widehat{\gamma}(b) \text{ and } \sigma'(b) < \widehat{\sigma}'(b), \text{ for all } b \text{ in } (b', \eta'] \right\}$$

By hypothesis, we know that $b^* < \eta'$. We want to prove that $b^* = \rho$. Assume that $b^* > \rho$. By continuity, we have $\gamma(b^*) = \widehat{\gamma}(b^*)$ or $\sigma'(b^*) = \widehat{\sigma}'(b^*)$ (or both).

The case $\gamma(b^*) = \widehat{\gamma}(b^*)$ and $\sigma'(b^*) = \widehat{\sigma}'(b^*)$ is impossible, since otherwise (γ, σ') and $(\widehat{\gamma}, \widehat{\sigma}')$would be two solutions of the same differential systems which coincide at a point b^* where the system satisfies the standard assumptions (locally Lipschitz with respect to the unknown functions) ensuring the uniqueness of the solution. They would thus coincide everywhere, which is impossible since they differ at η'.

Suppose that $\gamma(b^*) = \widehat{\gamma}(b^*)$ and $\sigma'(b^*) < \widehat{\sigma}'(b^*)$. From (A2-17), we have $\frac{d}{db} \ln F(\gamma(b^*)) > \frac{d}{db} \ln F(\widehat{\gamma}(b^*))$ and there thus exists $\varepsilon > 0$ such that $\gamma(b) > \widehat{\gamma}(b)$, for all b in $(b^*, b^* + \varepsilon)$. However, this is impossible since it contradicts the definition of b^*.

Suppose finally that $\gamma(b^*) < \widehat{\gamma}(b^*)$ and $\sigma'(b^*) = \widehat{\sigma}'(b^*)$. From Lemma A2-5, $\sigma'(b) < \gamma(b)$ and $\widehat{\sigma}'(b) < \widehat{\gamma}(b)$, for all b in $(\rho, \eta']$. The R.H.S. of (A2-18) is thus a strictly decreasing function of γ (or $\widehat{\gamma}$). From (A2-18), we thus have $\frac{d}{db} \ln H(\sigma'(b^*)) > \frac{d}{db} \ln H(\widehat{\sigma}'(b^*))$ and there exists $\varepsilon > 0$ such that $\sigma'(b) > \widehat{\sigma}'(b)$, for all b in $(b^*, b^* + \varepsilon)$, which is impossible since it contradicts the definition of b^*. ‖

Lemma A2-9: *Let d' be such that $g(d) \leq d' \leq d$ and let (γ, σ') be a solution over $(\rho, \eta']$ of the system (A2-17,A2-18) considered on the domain D'' with initial condition (A2-19), where $\eta' \leq g(d) + (g(d) - d') \left(\frac{m}{m'} - 1\right) F(d')$. Then, we have $\sigma'(b) \longrightarrow \rho$ as $b \longrightarrow_> \rho$ if and only if $\gamma(b) \longrightarrow \rho$ as $b \longrightarrow_> \rho$.*

Proof: If $\gamma(b) \longrightarrow \rho$, $\sigma'(b) \longrightarrow \rho$ is an immediate consequence of Lemma A2-5. Assume that $\sigma'(b) \longrightarrow \rho$ and that $\gamma(b) \nrightarrow \rho$ as $b \longrightarrow \rho$. Notice that, from the definition of $\mathcal{D}''$, if $\sigma'(b) \longrightarrow \rho$ as $b \longrightarrow \rho$ then $\rho \geq c$. From the LemmaA2-6 γ is strictly increasing and the limit $\overline{\gamma}$ of $\gamma(b)$ for b tending towards ρ exists. From the definition of $\mathcal{D}''$, if this limit is different from ρ then it must be strictly larger than ρ. From (A2-17,A2-18), we have

$$\frac{d \ln H(\sigma'(b))}{d \ln F(\gamma(b))} = \frac{\sigma'(b) - b}{\gamma(b) - b} + \left(\frac{m}{m'} - 1\right) \frac{F(\gamma(b))}{H(\sigma'(b))} \left(\frac{\sigma'(b) - \gamma(b)}{\gamma(b) - b}\right)$$

This derivative thus tends towards

$$-\left(\frac{m}{m'} - 1\right) \frac{F(\overline{\gamma})}{H(\rho)}$$

as b tends towards ρ. This limit is strictly negative (finite or equal to $-\infty$). However, this is impossible since from LemmaA2-6 $H\sigma'$ and $F\gamma$ are strictly increasing over $(\rho, \eta']$. ‖

Lemma A2-10: *Let d' be such that $g(d) \leq d' \leq d$ and let (γ, σ') be a solution over $(\rho, \eta']$ of the system (A2-17,A2-18) considered on the domain D'' with initial condition (A2-19), where $\eta' < g(d) + (g(d) - d')\left(\frac{m}{m'} - 1\right) F(d')$. Let $(\rho, \eta]$, with $-\infty \leq \rho < \eta$, be the "maximal (to the left) definition interval"of (γ, σ'), that is, (γ, σ') cannot be continued over any interval $(\rho', \eta]$, with $\rho' < \rho$, and still be a solution of (A2-17, A2-18) in the domain D''. Then, either $\rho < c$ and $\gamma(c)$, $\sigma'(c) > c$ or $\gamma(\rho) = \sigma'(\rho) = \rho$.*

Proof: From Lemma A2-6, γ and σ' are strictly increasing, the limits of $\gamma(b)$ and $\sigma'(b)$ for b tending towards ρ exist and are finite since they are smaller than $\gamma(\eta')$ and $\sigma'(\eta')$ respectively. We denote these limits by $\gamma(\rho)$ and $\sigma'(\rho)$, irrespectively.

Since the assumptions of the standard theorem of existence of the solution of a differential system with initial condition are satisfied in $\mathcal{D}''$, $(\rho, \gamma(\rho), \sigma'(\rho))$ must belong to the boundary of $\mathcal{D}''$. Assume $\rho > c$. Then, from the definition of $\mathcal{D}''$, we cannot have $\gamma(\rho) = c$ nor $\sigma'(\rho) = c$. Rather, we must have $\gamma(\rho) = \rho$ or $\sigma'(\rho) = \rho$. In either case, Lemma A2-9 implies that $\gamma(\rho) = \sigma'(\rho) = \rho$. If $r = c$, we must have $\gamma(c) = c$ or $\sigma'(c) = c$ and Lemma A2-9 implies $\gamma(c) = \sigma'(c) = c$. If $\rho < c$, the definition of $\mathcal{D}''$ implies $\gamma(c) > c$ and $\sigma'(c) > c$. $\|$

We say that a "maximal solution" as in the previous lemma, that is, a solution (γ, σ') with its maximal interval of definition $(\rho, \eta']$, is of type I in the first case, that is, when $\rho < c$, and is of type II in the second case, that is, when $\rho \geq c$.

Consider the following function:

$$\overline{\eta}' : [g(d), d] \to [c, g(d)]$$
$$\overline{\eta}'(d') = g(d) + \left(\frac{m}{m'} - 1\right)(g(d) - d') F(d')$$

for all d' in $[c, d]$. The function $\overline{\eta}'$ is continuous and strictly decreasing. We now defined the function δ' as follows:

$$\delta' : [c, g(d)] \to [g(d), d]$$
$$\delta'(\eta') = \min\left(\overline{\eta}'^{-1}(\eta'), d\right)$$

for all η' in $[c, g(d)]$. The function δ' is continuous and nonincreasing (see a possible graph of the function δ' in Figure 1) and such that $\delta'(g(d)) = g(d)$ and $\eta' \leq g(d) + \left(g(d) - \delta'(\eta')\right) F\left(\delta'(\eta')\right)$, for all η' in $[c, g(d)]$.

Lemma A2-11: The lower extremity ρ of the maximal solution of (A2-17, A2-18, A2-19) with d' and η' such that $d' = \delta'(\eta')$ is a nondecreasing function of η.

Proof: It follows immediately from Lemma A2-8. In fact, if $\eta' < \widetilde{\eta}$ let (γ, σ') and $\left(\widetilde{\gamma}, \widetilde{\sigma}'\right)$ be the solutions of the system (A2-17, A2-18) considered in the domain $\mathcal{D}''$ with initial condition (A2-19) with the values $\eta', \delta'(\eta')$, and $\delta'(\widetilde{\eta})$ respectively, and let $(\rho, \eta']$ and $(\widetilde{\rho}, \widetilde{\eta}]$ their respective maximal definition intervals. Since δ' is nondecreasing, we have $\delta'(\widetilde{\eta}) \leq \delta'(\eta')$. Consequently, $\widetilde{\gamma}(\eta') \leq \widetilde{\gamma}(\widetilde{\eta}) = \delta'(\widetilde{\eta}) \leq \delta'(\eta') = \gamma(\eta')$. Moreover, $\widetilde{\sigma}(\widetilde{\eta}) = \sigma(\eta') = g(d)$. From Lemma A2-8, we have $\widetilde{\gamma}(b) < \gamma(b)$ and $\widetilde{\sigma}'(b) < \sigma'(b)$, for all b in $(\max(\rho, \widetilde{\rho}), \eta']$. Assume that $\rho > \widetilde{\rho}$. Since $\left(\rho, \widetilde{\gamma}(\rho), \widetilde{\sigma}'(\rho)\right)$ belongs to the closure of $\mathcal{D}''$we have $\widetilde{\gamma}(\rho)$, $\widetilde{\sigma}'(\rho) \geq \max(\rho, c)$ and thus $\gamma(\rho), \sigma'(\rho) > \max(\rho, c)$ and $(\rho, \gamma(\rho), \sigma'(\rho))$ belongs to the (interior) of $\mathcal{D}''$ and ρ cannot be the right-hand extremity of the maximal definition interval of (γ, σ'). $\|$

Lemma A2-12: *Let be the left-hand extremity of the maximal definition interval $(\rho, \eta']$ of the solution (γ, σ') of (A2-17, A2-18) and the initial condition (A2-19) for the values η' and $\delta'(\eta')$. Then, as a function of η', $\max(\rho, c)$ is continuous from the right over $[c, g(d))$.*

Proof: Let η' in $(c, g(d))$ be such that $\rho < c$. The corresponding solution is thus defined at $(\rho + c)/2$. Then from the continuity of the solution of (A2-17, A2-18, A2-19) with respect to η' and the continuity of δ'(here the standard assumptions-local Lipschitz with respect to the unknown functions and the variable- under which this continuity holds are satisfied) there exists $\delta > 0$ such that for all $\widetilde{\eta}$ such that $\eta' - \delta < \widetilde{\eta} < \eta' + \delta$ the solution of (A2-17, A2-18, A2-19) for the value $\widetilde{\eta}$ of the parameter is defined at $(\rho + c)/2$, that is, the left-hand extremity $\widetilde{\rho}$ of the maximal definition interval is strictly smaller than $(\rho + c)/2$ and is thus strictly smaller than c and $\max(\widetilde{\rho}, c) = c$.

Let η' in $(c, g(d))$ be such that $\rho > c$ and let $\varepsilon > 0$. Let b' be such that $\rho < b' < \min(\rho + \epsilon, \eta')$. From the continuity with respect to η' of the solution of (A2-17, A2-18, A2-19) and the continuity of δ', there exists $\delta > 0$ such that for all $\eta' < \widetilde{\eta} < \eta' + \delta$ the solution of (A2-17, A2-18, A2-19) for the value $\widetilde{\eta}$ of the parameter is defined at b' and thus the left-hand extremity $\widetilde{\rho}$ of the maximal definition interval is strictly smaller than b' and than $\rho + \varepsilon$. From Lemma A2-11 $\rho \leq \widetilde{\rho}$ and thus $\rho \leq \widetilde{\rho} \leq \rho + \varepsilon$ and Lemma A2-12 is proved. $\|$

Lemma A2-13: *Let d' be such that $g(d) \leq d' \leq d$. Let η' be such that $c < \eta' \leq g(d) + (g(d) - d')\left(\frac{m}{m'} - 1\right) F(d')$. Let (γ, σ') be a solution over $(\rho, \eta']$ of the system (A2-17, A2-18) with initial condition (A2-19). Then,*

$$\frac{d}{dv}\left[\left(g(v) - \beta'(v)\right) F\left(\gamma \circ \beta'(v)\right)\right] = \left(\frac{d}{dv} g(v)\right) F\left(\gamma \circ \beta'(v)\right) \text{(A2-30)}$$

for all v in $\left(g^{-1} \circ \sigma'(r), d\right]$, where $\beta' = \sigma'^{-1} \circ g$.

Proof: The difference between the R.H.S. and the L.H.S. of the equation above is equal to $\frac{d}{db}\left[(g(v)-b)F(\gamma(b))\right]_{b=\beta'(v)}\frac{d}{dv}\beta'(v)$ which is equal to $\left(g(v)-\beta'(v)\right)F\left(\gamma\circ\beta'(v)\right)\frac{d}{dv}\beta'(v)\left\{-\frac{1}{g(v)-\beta'(v)}+\left[\frac{d}{db}\ln F(\gamma(b))\right]_{b=\beta'(v)}\right\}$. From (A2-16) the second term in the factor between braces is equal to $\left[\frac{1}{\sigma'(b)-b}\right]_{b=\beta'(v)}=\frac{1}{g(v)-\beta'(v)}$ and this factor and thus the difference between the two sides of (A2-30) are equal to 0. $\|$

Lemma A2-14: *If (γ,σ') is a solution over $(\rho,\eta']$ of (A2-17, A2-18) considered in the domain D' and if $\frac{d}{db}\gamma(b)>0$, for all b in $(\rho,\eta]$, and $\frac{d}{db}\sigma'(b)>0$, for all b in (ρ,η), then the functions $\chi=H\circ\sigma'\circ\gamma^{-1}\circ F^{-1}$ and $\omega=\gamma^{-1}\circ F^{-1}$ are differentiable and solutions over $(F(\gamma(\rho)),F(\gamma(\eta))]$ of the system considered on the domain $D=\{(p,\rho,\chi')\mid\chi'>H(\omega),p>F(\omega),1\geq\chi>H(c),1\geq p>F(c)\}$*

$$\frac{d}{dp}\omega(p)=\frac{H^{-1}(\chi'(p))-\omega(p)}{p}\text{(A2-31)}$$

$$\frac{d}{dp}\chi'(p)=\frac{\chi'(p)}{p}\frac{H^{-1}(\chi'(p))-\omega(p)}{F^{-1}(p)-\omega(p)}\left\{1+\left(\frac{m}{m'}-1\right)\frac{p}{\chi'(p)}\left(\frac{H^{-1}(\chi'(p))-F^{-1}(p)}{H^{-1}(\chi'(p))-\omega(p)}\right)\right\}\text{(A2-32)}$$

Inversely, if (ω,χ') is a solution over $(p,p']$, with $F(c)<p<p'\leq 1$, of (A2-31, A2-32) in the domain D, then $\sigma'=H^{-1}\circ\chi'\circ\omega^{-1}$ and $\gamma=F^{-1}\circ\omega^{-1}$ are differentiable and form s solution over $(\omega(p),\omega(p')]$ of the system (A2-17, A2-18) considered in the domain D'. Moreover, the system (A2-31, A2-32) satisfies in the domain D the standard assumptions of the theory of ordinary differential equations. The initial condition (A2-19) is equivalent to the condition (A2-33) below:

$$\omega(F(d'))=\eta',\chi'(F(d'))=1\text{(A2-33)}$$

and the initial condition (A2-20) is equivalent to (A2-34) below

$$\omega(F(\underline{c}))=\underline{c},\chi'(F(\underline{c}))=H(\underline{c})\quad\text{(A2-34)}$$

Proof: The two first statements of the lemma follow immediately from the previous lemma through the transformation $(p,\rho,\chi')=\left(\psi(b),\psi^{-1},\psi'\left(\psi^{-1}\right)\right)$ and from the observation that, in the domain D, $\frac{d}{dp}\rho(p)$ is strictly positive. The last statement of the lemma follows from the observation that, under our hypothesis, F^{-1} and H^{-1} are locally Lipschitz $\|$

Lemma A2-15: *Let d' be such that $g(d)\leq d'\leq d$. Let η' be such that $\eta'\leq g(d)+(g(d)-d')\left(\frac{m}{m'}-1\right)F(d')$. Let (γ,σ') be a solution over $(\rho,\eta']$ of the system (A2-17, A2-18) with initial condition (A2-19). Then,*

$$\frac{d}{dv}\left[(v-\beta(v))\left(m'F(\gamma'\circ\beta(v))+(m-m')F(v)\right)\right]=m'F(\gamma'\circ\beta(v))+(m-m')F(v)\text{(A2-35)}$$

for all v in $\left(g^{-1}\circ\sigma'(r), d\right]$, where $\gamma' = g^{-1}\circ\sigma'$ and $\beta = \gamma^{-1}$.

Proof: The difference between the R.H.S. and the L.H.S. of the equation above is equal to $\frac{d}{db}\left[(v-b)\left(m'F\left(\gamma'(b)\right)+(m-m')F\left(\gamma(b)\right)\right)\right]_{b=\beta(v)}\frac{d}{dv}\beta(v)$. Substituting to $F(\gamma'(b))$ its value $H(\sigma'(b))$, carrying out the derivation with respect to b and using (A2-17, A2-18), we see that this derivative at $b=\beta(v)$ is equal to 0 and the lemma is proved. Or, simply, one can remember that the differential equations (A2-17, A2-18) were (partly) from the requirement that the derivative of the expression between brackets be equal to 0 at $b=\beta(v)$. $\|$

Lemma A2-16: *Let ρ be the left-hand extremity of the maximal definition interval $(\rho,\eta']$ of the solution (γ,σ') of (A2-17, A2-18) and initial condition (A2-19) with the values $d'=\delta'(\eta')$ and η'. Then, as a function of η',* $\max(\rho, c)$ *is continuous over $[c, g(d))$.*

Proof: Let η' in $[c, g(d))$ and let ρ be the left-hand extremity of the maximal definition interval of the solution of (A2-17, A2-18) with initial condition (A2-19) for the values $d'=\delta'(\eta')$, η'. Assume $\rho<c$. From the first paragraph in the proof of Lemma A2-12, $\max(\rho,c)$ is continuous at η'. Assume next that $\rho\geq c$. From Lemma A2-12, we know that $\max(\rho,c)$ is continuous from the right at η'. We need to prove that it is continuous from the left. When $\rho=c$, it immediately follows from Lemma A2-11. We can thus assume that $r>c$ and thus that $\max(r,c)=r$. Let ε be an arbitrary strictly positive number. Without loss of generality, we can assume that $\rho-\varepsilon>c$. Let $\nu>0$ be defined as follows

$$\nu=\int_{\rho-\varepsilon}^{\rho}F(u)\,du \text{ (A2-36)}$$

From Lemma A2-10, we have $\gamma(\rho)=\rho$ or, equivalently, $\rho=\beta(\rho)$ with $\beta=\gamma^{-1}$. From the continuity of the functions involved, there exists w in (ρ,d') such that

$$|w-\beta(w)|<\frac{\nu}{2} \text{ (A2-37)}$$

Notice that $w>s=g^{-1}(\rho)$ implies

$$g(w)>\rho \text{ (A2-38)}$$

From Lemma A2-14, $\chi'=H\circ\sigma'\circ\gamma^{-1}\circ F^{-1}$and $\omega=\gamma^{-1}\circ F^{-1}$form a solution over $(F(\rho),1]$ of (A2-31, A2-32) and satisfy the initial condition

$$\chi'\left(F\left(\delta'(\eta')\right)\right)=1, \omega\left(F\left(\delta'(\eta')\right)\right)=\eta' \text{ (A2-33)}$$

Since, from Lemma A2-14 and the continuity of δ', the system (A2-31, A-32) considered in the domain D satisfies the standard assumptions of the theory of ordinary differential equations, there exists $\tau>0$ such that the solution $\left(\widetilde{\chi}',\widetilde{\omega}\right)$

of (A2-31, A2-32) considered on the domain D with initial condition (A2-33) with the values $\delta'(\eta'), \widetilde{\eta}$ of the parameters is defined at $F(w)$ and is such that $|\omega(p) - \widetilde{\omega}(p)| < \nu/2$ at $p = F(w)$ and thus

$$\left|\beta(w) - \widetilde{\beta}(w)\right| < \frac{\nu}{2} \quad \text{(A2-39)}$$

for all $\widetilde{\eta}$ such that $\eta - \tau < \widetilde{\eta} \leq \eta$. From Lemma A2-14 again, for all $\widetilde{\eta}$ such that $\eta - \tau < \widetilde{\eta} \leq \eta$, the solution $(\widetilde{\gamma}, \widetilde{\sigma}')$ of (A2-17, A2-18) considered in the domain $\mathcal{D}'$ and of (A2-19) with the values $\delta'(\eta'), \widetilde{\eta}$ of the parameters is such that $\widetilde{\chi}' = H \circ \widetilde{\sigma}' \circ \widetilde{\gamma}^{-1} \circ F^{-1}$and $\widetilde{\omega} = \widetilde{\gamma}^{-1} \circ F^{-1}$ form a solution of (A2-31, A2-32) considered on the domain D and of (A2-33) with the values $\delta'(\eta'), \widetilde{\eta}$ of the parameters and thus $\widetilde{\beta} = \widetilde{\gamma}^{-1}$ is defined at w and is such that (A2-39) holds true.

Let $\widetilde{\eta}$ be such that $\eta - \tau < \widetilde{\eta} \leq \eta$, let $(\widetilde{\gamma}, \widetilde{\sigma}')$ be the solution of (A2-17, A2-18, A2-19) for the values $(\widetilde{\eta}, \delta'(\widetilde{\eta}))$ of the parameters, and let $\widetilde{\rho}$ be the left-hand extremity of the maximal definition interval of $(\widetilde{\gamma}, \widetilde{\sigma}')$. From Lemma A2-14, $H(\sigma'(\widetilde{\rho}))$ is the left-hand extremity of the maximal definition interval of $(\widetilde{\chi}', \widetilde{\rho})$ and since this solution is defined at $H(g(w))$, we have $g(w) \geq \widetilde{\sigma}'(\widetilde{\rho})$. From Lemma A2-15, we have

$$\begin{aligned} & \int_{\widetilde{\gamma}(\widetilde{\rho})}^{w} \left\{ m' F\left(\widetilde{\gamma}' \circ \widetilde{\beta}(u)\right) + (m - m') F(u) \right\} du \\ = \quad & \left(w - \widetilde{\beta}'(w)\right) \left\{ m' F\left(\widetilde{\gamma}' \circ \widetilde{\beta}(w)\right) + (m - m') F(w) \right\} \\ & - (\widetilde{\gamma}(\widetilde{\rho}) - \widetilde{\rho}) \left\{ m' F\left(\widetilde{\gamma}'(\widetilde{\rho})\right) + (m - m') F(\widetilde{\gamma}(\widetilde{\rho})) \right\} \quad \text{(A2-40)} \end{aligned}$$

In the closure of the domain $\mathcal{D}'$, we have $\gamma(b) \geq b$. Consequently we obtain, after changing the variables in the integral,

$$\begin{aligned} & \int_{\widetilde{\gamma}(\widetilde{\rho})}^{w} \left\{ m' F\left(\widetilde{\gamma}' \circ \widetilde{\beta}(u)\right) + (m - m') F(u) \right\} du \\ \leq \quad & \left(w - \widetilde{\beta}'(w)\right) \left\{ m' F\left(\widetilde{\gamma}' \circ \widetilde{\beta}(w)\right) + (m - m') F(w) \right\} \end{aligned}$$

Moreover, we know that $\widetilde{\gamma}' \geq \widetilde{\gamma}$ (see Lemma A2-4) and thus

$$m \int_{\widetilde{\gamma}(\widetilde{r})}^{w} F(u)\, du \leq \left(w - \widetilde{\beta}'(w)\right) \left\{ m' F\left(\widetilde{\gamma}' \circ \widetilde{\beta}(w)\right) + (m - m') F(w) \right\}$$

. From (A2-37) and (A2-39), we then obtain, after dividing both sides by m,

$$\int_{\widetilde{\gamma}(\widetilde{\rho})}^{w} F(u)\, du \leq \nu$$

The definition (A2-36) of ν then implies

$$\rho - \epsilon \leq \widetilde{\gamma}(\widetilde{\rho}) \leq \rho$$

and we have thus proved the following result

$$\lim_{\widetilde{\eta}\to_{<}\eta}\widetilde{\gamma}\left(\widetilde{\rho}\right)=\rho \text{ (A2-41)}$$

We next show that there exists $\widetilde{\eta}<\eta$ such that the corresponding left-hand extremity $\widetilde{\rho}$ is not smaller than c. Otherwise, for all $\widetilde{\eta}<\eta$ we would have $\widetilde{\rho}<c$. From Lemma A2-10, we have

$$\gamma\left(\rho\right)=\rho$$

Consider b in (ρ,η). Applying the standard theorems of the theory of ordinary differential equations to the system (A2-22, A2-23) in Lemma A2-3, we see that there exists $\tau>0$ such that for all $\widetilde{\eta}$ such that $\eta-\tau\leq\widetilde{\eta}\leq\eta$ the solution $\left(\widetilde{\gamma},\widetilde{\sigma}'\right)$ of (A2-17, A2-18, A2-19) with the values $\delta'\left(\widetilde{\eta}'\right),\widetilde{\eta}'$ of the parameters is defined at b. From monotonicity, we have $\widetilde{\gamma}\left(b'\right)\leq\widetilde{\gamma}\left(b\right)$for all b' in $[c,\rho]$. We thus have

$$\widetilde{\gamma}'\left(\widetilde{\rho}\right)\leq\widetilde{\gamma}\left(b'\right)\leq\widetilde{\gamma}\left(b\right) \text{ (A2-42)}$$

for all b' in $[c,\rho]$. From the continuity of the solution of (A2-22, A2-23, A2-24) in Lemma A2-3 with respect to the parameter η, we have $\lim_{\widetilde{\eta}\to_{<}\eta}\widetilde{\gamma}\left(b\right)=\gamma\left(b\right)$ and thus $\limsup_{\widetilde{\eta}\to_{<}\eta}\widetilde{\gamma}\left(b'\right)\leq\gamma\left(b\right)$ for all b' in $[c,\rho]$ and all b in (ρ,η). By making b tend towards ρ we find $\limsup_{\widetilde{\eta}\to_{<}\eta}\widetilde{\gamma}\left(b'\right)\leq\gamma\left(\rho\right)=\rho$. Moreover, (A2-41) and (A2-42) imply $\liminf_{\widetilde{\eta}\to_{<}\eta}\widetilde{\gamma}\left(b'\right)\geq\liminf_{\widetilde{\eta}\to_{<}\eta}\widetilde{\gamma}'\left(\widetilde{\rho}\right)=\rho$, for all b' in $[c,\rho]$. Consequently, $\lim_{\widetilde{\eta}\to_{<}\eta}\widetilde{\gamma}\left(b'\right)$ exists and is equal to ρ, for all b' in $[c,\rho]$, that is

$$\lim_{\widetilde{\eta}\to_{<}\eta}\widetilde{\gamma}\left(b'\right)=r \text{ (A2-43)}$$

for all b' in $[c,\rho]$. In particular, we have

$$\lim_{\widetilde{\eta}\to_{<}\eta}\left[\ln F\left(\widetilde{\gamma}\left(\frac{c+\rho}{2}\right)\right)-\ln F\left(\widetilde{\gamma}\left(c\right)\right)\right]=0 \text{ (A2-44)}$$

However, from (A2-17) we have $\ln F\left(\widetilde{\gamma}\left(\left(c+\rho\right)/2\right)\right)-\ln F\left(\widetilde{\gamma}\left(c\right)\right)=\int_{c}^{(c+\rho)/2}\frac{1}{\widetilde{\sigma}'\left(b'\right)-b'}db'$. Since $\widetilde{\sigma}'\leq\widetilde{\gamma}$, (A2-43) implies that there exists $\theta>0$ such that $\widetilde{\sigma}'\left(\left(c+\rho\right)/2\right)<\rho+\varepsilon$, for all $\eta-\theta\leq\widetilde{\eta}\leq\eta$. We thus find

$$\ln F\left(\widetilde{\gamma}\left(\frac{c+\rho}{2}\right)\right)-\ln F\left(\widetilde{\gamma}\left(c\right)\right)\geq\int_{c}^{(c+\rho)/2}\frac{1}{\rho+\varepsilon-b'}db'>0$$

for all $\eta-\theta\leq\widetilde{\eta}\leq\eta$. However, this result contradicts (A2-44) and we have proved that there exists $\widetilde{\eta}<\eta$ such that the corresponding left-hand extremity $\widetilde{\rho}$ is not smaller than c.

From the monotonicity of the left-hand extremity of the maximal definition interval with respect to η (Lemma A2-11), there then exists $\kappa>0$ such that $\widetilde{\rho}>c$, for all $\widetilde{\eta}$ such that $\eta-\kappa\leq\widetilde{\eta}\leq\eta$. From Lemma A2-10, for all such $\widetilde{\eta}$ we have $\widetilde{\sigma}'\left(\widetilde{\rho}\right)=\widetilde{\rho}$. From (A2-41), we then immediately obtain

$$\lim_{\widetilde{\eta}\to_{<}\eta}\widetilde{\rho}=\rho$$

and we have proved the left-continuity with respect to η of the left-hand extremity ρ. $\|$

Lemma A2-17: *Let ρ be the left-hand extremity ρ of the maximal definition interval of the solution of (A2-17, A2-18, A2-19) with the values $\eta', \delta'(\eta')$ of the parameters with $\eta' < g(d)$. Then, we have $\lim_{\eta' \to_{>} g(d)} \rho = g(d)$.*

Proof: Let ε be a strictly positive number such that $\varepsilon < g(d) - c$. Let $\eta' < g(d)$ such that

$$\left(\delta'(\eta') - \eta'\right)\left(m' + (m - m') F\left(\delta'(\eta')\right)\right) \leq m \int_{g(d)-\varepsilon}^{\delta'(\eta')} F(u)\, du \text{(A2-45)}$$

Because $\delta'(\eta')$ tends towards $g(d)$ when η' tends towards $g(d)$, the L.H.S. of the inequality above tends towards 0 and the R.H.S. towards $m \int_{g(d)-\varepsilon}^{g(d)} F(u)\, du > 0$ and such a η' exists. For the sake of convenience, denote $\delta'(\eta')$ by d'. Let ρ be the left-hand extremity of the maximal definition interval of the solution of (A2-17, A2-18) and initial condition (A2-19) with the values d' and η' of the parameters. Assume that $\rho < g(d) - \varepsilon$. From Lemma A2-15 we have that $\int_{\gamma(\rho)}^{d'} \{m' F(\gamma' \circ \beta(u)) + (m - m') F(u)\}\, du$ is equal to $(d' - \beta(d'))(m' F(\gamma' \circ \beta(d')) + (m - m') F(d'))$ $- (\gamma(\rho) - \rho)(m' F(\gamma'(\rho)) + (m - m') F(\gamma(\rho)))$ and thus

$$\begin{aligned} & \int_{\gamma(\rho)}^{d'} \{m' F(\gamma' \circ \beta(u)) + (m - m') F(u)\}\, du \\ = \quad & (d' - \eta')(m' + (m - m') F(d')) - (\gamma(\rho) - \rho)(m' F(\gamma'(\rho)) + (m - m') F(\gamma(\rho))) \end{aligned}$$

From Lemma A2-4, we know that $\gamma' \geq \gamma$ and thus $(d' - \eta')(m' + (m - m') F(d'))$ $- (\gamma(\rho) - \rho)(m' F(\gamma'(\rho)) + (m - m') F(\gamma(\rho)))$ is not smaller than $m \int_{\gamma(\rho)}^{d'} F(u)\, du$. Consequently, $(d' - \eta')(m' + (m - m') F(d'))$ is not smaller than $m \int_{\gamma(\rho)}^{d'} F(u)\, du$ $+ (\gamma(\rho) - \rho)(m' F(\gamma'(\rho)) + (m - m') F(\gamma(\rho)))$ which, in turn, is not smaller than $m \int_{\gamma(\rho)}^{d'} F(u)\, du + m(\gamma(\rho) - \rho) F(\gamma(\rho))$. Since F is nondecreasing, we obtain

$$(d' - \eta')(m' + (m - m') F(d')) \geq m \int_{\rho}^{d'} F(u)\, du$$

However, we have assumed[18] that $\rho < g(d) - e$ and we thus find

$$(d' - \eta')(m' + (m - m') F(d')) > m \int_{g(d)-\varepsilon}^{d'} F(u)\, du$$

, which contradicts the inequality;(A2-45). Consequently, $\rho \geq g(d) - \epsilon$. Since the left-hand extremity of the maximal definition interval is a nondecreasing function of η', we have $\rho\left(\widetilde{\eta}'\right) \geq g(d) - \varepsilon$, for all $\widetilde{\eta}' \geq \eta'$. Since $\varepsilon > 0$ was arbitrary, the lemma follows. $\|$

Lemma A2-18: *There exists $\eta' < g(d)$ which satisfies the conditions the conditions of Lemma A1-1.*

Proof[19]**:** Consider the set Λ of $\eta' < g(d)$ such that the left-hand extremity ρ of the maximal definition interval of the solution of (A2-17, A2-18, A2-19) with the values $\eta', \delta'(\eta')$ of the parameters is such that $\rho \geq \underline{c}$. From Lemma A2-17 this set is not empty. From Lemma A2-11, this set is an interval. Moreover, the left-hand extremity of this interval is finite since it is strictly larger than $\underline{c}$. In fact, when $\eta' = \underline{c}$, the system (A2-17, A2-18) and initial condition (A2-19) with values $\eta' = \underline{c}$ and $\delta'(\eta') = d$ admits a solution which is defined over a neighborhood of $\eta' = \underline{c}$ and the left-hand extremity of its definition interval is thus strictly smaller than $\underline{c}$. Denote by η'^* the left-hand extremity of Λ, that is, $\eta'^* = \inf \Lambda$.

The left-hand extremity ρ^* of the definition interval of the solution of (A2-17, A2-18, A2-19) for the values η'^* ,$\delta'(\eta'^*)$ of the parameters is equal to $\underline{c}$. In fact, we have seen in the previous paragraph that there exist values η' such that the left-hand extremity of the definition interval is strictly smaller than $\underline{c}$. Consequently, if $\rho^* > \underline{c}$ there would exists a discontinuity with respect to η' of the function $\max(\rho, c)$. From Lemma A2-16, no such discontinuity exists and thus $\rho^* \leq \underline{c}$. Moreover, since $\rho \geq \underline{c}$ for all η in Λ the same lemma implies that $\rho^* \geq \underline{c}$. Consequently, $\rho^* = \underline{c}$. Lemma A2-10 then implies $\gamma^*(\rho^*) = \sigma^{*\prime}(\rho^*) = \rho^*$, where $(\gamma^*, \sigma^{*\prime})$ is the solution of (A2-17, A2-18, A2-19) for the values $\eta'^*, \delta'(\eta'^*)$ of the parameters. From these results and Lemma A2-5, this solution satisfies all the conditions in Lemma A2-1. $\|$

Lemma A2-19: *Let (χ', ω) be a solution over $\left(q, F\left(\delta'(\eta')\right)\right]$ of (A2-31, A2-32) considered on the domain D and of the condition (A2-33) with the values $\eta' < g(d)$ and $\delta'(\eta')$ and let $\left(\widetilde{\chi}', \widetilde{\omega}\right)$ be a solution over $\left(q, F\left(\delta'\left(\widetilde{\eta}'\right)\right)\right]$ of (A2-31, A2-32) considered on the domain D and of the condition (A2-33) with the parameter $\widetilde{\eta}' < \eta'$. Then, $\widetilde{\chi}'(p) > \chi'(p)$ and $\widetilde{\omega}(p) < \omega(p)$, for all p in $\left(q, F\left(\delta'(\eta')\right)\right)$.*

Proof: Denote $\delta'\left(\widetilde{\eta}'\right)$ and $\delta'(\eta')$ by $\widetilde{d}'$ and d', respectively. Because the function δ' is nondecreasing, we have $\widetilde{d}' \geq d'$. From (A2-33), we have $\widetilde{\omega}(F(d')) \leq \widetilde{\omega}\left(F\left(\widetilde{d}'\right)\right) = \widetilde{\eta}' < \eta' = \omega(F(d'))$. If $\widetilde{d}' > d'$, then (A2-33) implies $\widetilde{\chi}'(F(d')) < \widetilde{\chi}'\left(F\left(\widetilde{d}'\right)\right) = \chi'(F(d')) = 1$.

Assume now that $\widetilde{d}' = d'$. Remark that (A2-32) can be equivalently rewritten as $\frac{d}{dp}\chi'(p) = \frac{\chi'(p)}{p}\frac{H^{-1}\left(\chi'(p)\right) - \omega(p)}{F^{-1}(p) - \omega(p)} + \left(\frac{m}{m'} - 1\right)\frac{H^{-1}\left(\chi'(p)\right) - F^{-1}(p)}{F^{-1}(p) - \omega(p)}$ or, equivalently, (A2-46) below

$$\frac{d}{dp}\chi'(p) = \frac{\chi'(p)}{p} + \frac{m'\chi'(p) + (m - m')\,p}{p}\frac{H^{-1}(\chi'(p)) - \omega(p)}{F^{-1}(p) - \omega(p)} \text{(A2-46)}$$

From (A2-46) and (A2-33), we have $\frac{d}{dp}\chi'(F(d')) = \frac{1}{F(d')} + \frac{g(d)-d'}{d'-\eta'} \frac{m'+(m-m')F(d')}{m'F(d')}$.

If $d' > g(d)$, $\frac{d}{dp}\chi(1)$ is a strictly decreasing function of η' and thus $\frac{d}{dp}\chi'(F(d')) < \frac{d}{dp}\widetilde{\chi}'(F(d'))$. Assume $d' = g(d)$. From Lemma A2-1 (1), we have $d' = g(d) = d$. We then have $\frac{d}{dp}\chi(1) = \frac{d}{dp}\widetilde{\chi}(1) = 1$. Calculating $\frac{d^2}{dp^2}\chi'(p)$ from (A2-46) and substituting 1 to p and d to $g(d)$, using (A2-31), (A2-32), and (A2-33), and simplifying, we find:

$$\frac{d^2}{dp^2}\chi'(1) = \frac{m}{d-\eta'}\left(\frac{1}{h(d)} - \frac{1}{f(d)}\right)$$

From the assumption $\frac{d}{dv}\frac{F}{H}(v) > 0$, at $v = d$, we have $f(d) < h(d)$ and thus $\frac{d^2}{dp^2}\chi'(1)$ is a strictly decreasing of η' and $\frac{d^2}{dp^2}\widetilde{\chi}'(1) > \frac{d^2}{dp^2}\chi'(1)$. In all cases, there thus exists $\varepsilon > 0$ such that $\widetilde{\chi}'(p) > \chi(p)$ and $\widetilde{\omega}(p) < \omega(p)$, for all p in $(F(d') - \varepsilon, F(d'))$.

Define p^* as follows:

$$p^* = \min\left\{p' \in [q, F(d')] \mid \widetilde{\chi}'(p) > \chi(p) \text{ and } \widetilde{\omega}(p) < \omega(p), \text{ for all } p \text{ in } (p', 1)\right\}$$

From the previous paragraph, we know that $p^* \leq F(d') - \varepsilon$. We want to prove that $p^* = q$. Assume that $p^* > q$. By continuity, we have $\widetilde{\chi}'(p^*) = \chi(p^*)$ or $\widetilde{\omega}(p^*) = \omega(p^*)$ (or both).

The case $\widetilde{\chi}'(p^*) = \chi'(p^*)$ and $\widetilde{\omega}(p^*) = \omega(p^*)$ is impossible, since otherwise (χ', ω) and $(\widetilde{\chi}', \widetilde{\omega})$ would be two solutions of the same differential systems which coincide at a point b^* where the system satisfies the standard assumptions (locally Lipschitz with respect to the unknown functions) ensuring the uniqueness of the solution. They would thus coincide everywhere, which is impossible since they differ over $(F(d') - \varepsilon, F(d'))$.

Suppose that $\widetilde{\chi}'(p^*) = \chi'(p^*)$ and $\widetilde{\omega}(p^*) < \omega(p^*)$. In the second term of the R.H.S. of (A2-46) the second factor is equal to $\left(1 + \left(H^{-1}(\chi'(p)) - F^{-1}(p)\right) / \left(F^{-1}(p) - \omega(p)\right)\right)$ and is thus a strictly increasing function of $\omega(p)$ since $H^{-1}(p) - F^{-1}(\chi'(p)) < 0$. In fact, from Lemma A2-5 $\sigma' < \gamma$ over the interior of the definition domain and thus $\chi' = H \circ \sigma' \circ \gamma^{-1} \circ F^{-1} < H \circ F^{-1}$ over the interior of the definition domain. Since $\widetilde{\omega}(p^*) < \omega(p^*)$, we thus have $\frac{d}{dp}\widetilde{\chi}(p^*) < \frac{d}{dp}\chi(p^*)$ and there exists $\delta > 0$ such that $\widetilde{\chi}(p) < \chi(p)$, for all p in $(p^*, p^* + \delta)$. However, this is impossible since it contradicts the definition of p^*.

Suppose finally that $\widetilde{\chi}(p^*) > \chi(p^*)$ and $\widetilde{\omega}(p^*) = \omega(p^*)$. From the equation (A2-31), we see that $\frac{d}{dp}\omega(p)$ is a strictly increasing function of $\chi'(p)$. Consequently, $\frac{d}{dp}\widetilde{\omega}(p^*) > \frac{d}{dp}\omega(p^*)$ and there exists $\delta > 0$ such that $\widetilde{\omega}(p) > \omega(p)$, for all p in $(p^*, p^* + \delta)$, which is impossible since it contradicts the definition of p^*. Thus $p^* = q$, $\widetilde{\chi}(p) > \chi(p)$, for all p in $(q, F(d'))$, and the lemma is proved. $\|$

Lemma A2-20: *There exists only one equilibrium.*

Proof: Assume there are two different equilibria and let (γ, σ') and $\left(\widetilde{\gamma}, \widetilde{\sigma}'\right)$ be the couples of their inverse bid functions. Let the corresponding parameters be η' and $\widetilde{\eta}'$, respectively. From the theory of ordinary differential equations (applied to the system (A2-17, A2-18)), we have $\eta' \neq \widetilde{\eta}'$. Without loss of generality, assume that $\widetilde{\eta}' < \eta'$. Because the function δ' is nonincreasing, we have $d' = \delta'(\eta') \leq \widetilde{d}' = \delta'\left(\widetilde{\eta}'\right)$[20].

From Lemma A2-13, we have $\int_{\underline{c}}^{d} F\left(\gamma \circ \beta'(u)\right) dg(u) = (g(d) - \eta') F(d')$ and $\int_{\underline{c}}^{d} F\left(\widetilde{\gamma} \circ \widetilde{\beta}'(u)\right) dg(u) = \left(g(d) - \widetilde{d}'\right) F\left(\widetilde{d}'\right)$. Consequently, we have

$$\int_{\underline{c}}^{d} F(\varphi(u))\, du < \int_{\underline{c}}^{d} F(\widetilde{\varphi}(u))\, du$$

where $\varphi = \gamma \circ \beta'$ and $\widetilde{\varphi} = \widetilde{\gamma} \circ \widetilde{\beta}'$.

From Lemma (A2-19) with $q = F(\underline{c})$, we have $\widetilde{\chi}'(p) > \chi'(p)$, for all p in $(F(\underline{c}), F(d'))$. Since $\chi' = H \circ \sigma' \circ \gamma^{-1} \circ F^{-1} = H \circ g \circ \beta' \circ \gamma^{-1} \circ F^{-1} = H \circ g \circ \varphi^{-1} \circ F^{-1}$ and, similarly, $\widetilde{\chi}' = H \circ g \circ \widetilde{\varphi}^{-1} \circ F^{-1}$, it implies $\widetilde{\varphi}^{-1}(u) > \varphi^{-1}(u)$, for all u in $(\underline{c}, d')$, and thus (if $\widetilde{d}' > d'$ the last inequality would imply $\widetilde{\varphi}^{-1}\left(\widetilde{d}'\right) > \widetilde{\varphi}^{-1}(d') \geq \varphi^{-1}(d') = d$, which is impossible since $\widetilde{\varphi}^{-1}\left(\widetilde{d}'\right) = d'$ and thus) we find $\widetilde{d}' = d'$ and $\widetilde{\varphi}(u) < \varphi(u)$, for all u in $(\underline{c}, d)$. Consequently, $\int_{\underline{c}}^{d} F(\widetilde{\varphi}(u))\, du > \int_{\underline{c}}^{d} F(\varphi(u))\, du$, we obtain a contradiction, and the proof is complete. $\|$

Appendix 3: Comparative Statics I

In Lemma A3-1 below, we show that over the interval $(\underline{c}, \min(g(d), \eta))$ the two parts of the characterization in Theorem 4 (Section 3) of the symmetric regular equilibrium in the case $m > m'$ can be subsumed in a single system of differential equations. We extend the inverses γ, γ', and $\sigma' = g \circ \gamma'$ to $\mathcal{R}$ as in Section 3.

Lemma A3-1: *Let $\left(\beta, \beta'\right)$ be the unique symmetric regular equilibrium and let η' and d' be as in Theorem 4 (Section 3). Then (γ, σ') is a solution over $(\underline{c}, \min(g(d), \eta))$ of the following system of differential equations:*

$$\frac{d}{db} \ln F(\gamma(b)) = \min\left(\frac{1}{\sigma'(b) - b}, \frac{1}{\gamma(b) - b}\left\{1 + \frac{1}{m/m' - 1} \frac{1}{F(\gamma(b))}\right\}\right) \quad \text{(A3-1)}$$

$$\frac{d}{db} \ln H(\sigma'(b)) = \max\left(\frac{1}{\gamma(b) - b}\left\{1 + \left(\frac{m}{m'} - 1\right) \frac{F(\gamma(b))}{H(\sigma'(b))} \left(\frac{\sigma'(b) - \gamma(b)}{\sigma'(b) - b}\right)\right\}, 0\right) \quad \text{(A3-2)}$$

Proof: From Theorem 4 (Section 3), the equation (3-3) holds true over $(\underline{c}, \eta']$ and since $\frac{d}{db}\ln H(\sigma'(b)) \geq 0$ over this interval, the equation (A3-2) holds true over the same interval. From the link (3-5) between η' and d' and the initial condition at η' in the boundary conditions (3-4), we see that $\frac{d_l}{db}\ln H(\sigma'(\eta'))$ is equal to zero. Over $[\eta', +\infty)$ the function σ' is identically equal to $g(d)$. Thus the right-hand derivative of σ' at η' is equal to zero and the (two-sided) derivative of σ' exists and is equal to zero. The function $\ln H(\sigma'(b))$ is thus differentiable over $(\underline{c}, +\infty)$. Over $(\eta', \min(g(d), \eta'))$ the first argument of the maximum operator in the R.H.S. of (A3-2) is nonpositive. In fact, since $\gamma(b) \geq \sigma(b) = g(d)$ over this interval the factor $1 + \left(\frac{m}{m'} - 1\right) F(\gamma(b)) \left(\frac{g(d)-\gamma(b)}{g(d)-b}\right)$ is a nonincreasing function of b which vanishes at η' and which is thus nonpositive for b$\geq$ η'. Consequently, the equation (A3-2) holds true over the interval $(\underline{c}, \min(g(d), \eta'))$.

Over the interval $(\underline{c}, \eta']$ the first argument of the maximum operator in (A3-1) is not larger than the second argument of this operator. In fact, $\frac{1}{\sigma'(b)-b} \leq \frac{1}{\gamma(b)-b}\left\{1 + \frac{1}{m/m'-1}\frac{1}{F(\gamma(b))}\right\}$ is equivalent to

$$(m-m')\,F(\gamma(b))\,(\sigma'(b)-\gamma(b)) + m'(\gamma(b)-b) \geq 0 \quad \text{(A3-3)}$$

. However, over $(\underline{c}, \eta']$ the equation (3-3) holds true and since $\frac{d}{db}\ln H(\sigma'(b)) \geq 0$, we have $1+\left(\frac{m}{m'} - 1\right)\frac{F(\gamma(b))}{H(\sigma'(b))}\left(\frac{\sigma'(b)-\gamma(b)}{\sigma'(b)-b}\right) \geq 0$ or, after rearranging, $(m-m')\,F(\gamma(b))\,(\sigma'(b)-\gamma(b)) \geq -m'H(\sigma'(b))\,(\sigma'(b)-b)$. Consequently, the L.H.S. of (A3-3) is not smaller than $-m'H(\sigma'(b))\,(\sigma'(b)-b) + m'(\gamma(b)-b) \geq -m'(\sigma'(b)-b) + m'(\gamma(b)-b) = m'(\gamma(b)-\sigma'(b)) \geq 0$ and (A3-3) holds true over $(\underline{c}, \eta']$. From (3-2), we then see that (A3-1) is satisfied over $(\underline{c}, \eta')$ and is satisfied at η' when the derivative is a left-hand derivative.

Over $[\eta', \min(g(d), \eta))$, the second argument of the minimum operator in (A3-1) is smaller than the first argument. In fact, from (3-5) and (3-4) these two arguments are equal at $b = \eta'$. Moreover, their difference $\frac{1}{\gamma(b)-b}\left\{1 + \frac{1}{m/m'-1}\frac{1}{F(\gamma(b))}\right\} - \frac{1}{g(d)-b}$ is equal to $\frac{1}{\gamma(b)-b}\left\{\frac{g(d)-\gamma(b)}{g(d)-b} + \frac{1}{m/m'-1}\frac{1}{F(\gamma(b))}\right\}$ and, since $\gamma \geq \sigma$ and $g(d) > b$ over $[\eta', g(d))$, the factor between braces is a nonincreasing function of b over $[\eta', \min(g(d), \eta))$. The difference is thus nonpositive and the second argument of the minimum operator is the smaller argument over this interval. From (A2-11), the equation (A3-1) holds true over $[\eta', \min(g(d), \eta))$. Consequently, $\frac{d}{db}\ln F(\gamma(b))$ is differentiable over $(\underline{c}, \eta)$ and the equation (A3-1) holds true over $(\underline{c}, \min(g(d), \eta))$. ||

Lemma A3-2: *Let $m, m', \widetilde{m}$, and $\widetilde{m}'$ be such that $\frac{\widetilde{m}}{\widetilde{m}'} > \frac{m}{m'} > 1$ and let (β, β') and $\left(\widetilde{\beta}, \widetilde{\beta}'\right)$ the corresponding equilibria. Let $\overline{b}$ be in $\left[\min\left(\eta', \widetilde{\eta}'\right), \max\left(\eta', \widetilde{\eta}'\right)\right]$.*

If $\gamma\left(\overline{b}\right) \leq \widetilde{\gamma}\left(\overline{b}\right)$ and $\sigma'\left(\overline{b}\right) \leq \widetilde{\sigma}'\left(\overline{b}\right)$, then $\eta' \geq \widetilde{\eta}'$, $\gamma(b) < \widetilde{\gamma}(b)$, and $\sigma'(b) < \widetilde{\sigma}'(b)$ for all b in $\left[\min\left(\eta',\widetilde{\eta}'\right),\overline{b}\right)$. If $\overline{b} > \min\left(\eta',\widetilde{\eta}'\right)$, then $\eta' > \widetilde{\eta}'$.

Proof: The result is immediate if $\overline{b} = \min\left(\eta',\widetilde{\eta}'\right)$. Assume thus that $\overline{b} \in \left(\min\left(\eta',\widetilde{\eta}'\right),\max\left(\eta',\widetilde{\eta}'\right)\right]$ and thus $\min\left(\eta',\widetilde{\eta}'\right) < \max\left(\eta',\widetilde{\eta}'\right)$. We show that in this case $\eta' > \widetilde{\eta}'$.Denote $\max\left(\eta',\widetilde{\eta}'\right)$ by $\overline{\eta}$. The equations (A3-1) and (A3-2) apply over $(c,\overline{\eta}]$ since η' and $\widetilde{\eta}'$ are strictly smaller than $g(d)$.

We rule out $\eta' < \overline{\eta} = \widetilde{\eta}'$. Suppose this is the case. Then, Theorem 4 implies

$$\frac{d}{db}\ln H\left(\sigma'\left(\overline{b}\right)\right) = 0 < \frac{d}{db}\ln H\left(\widetilde{\sigma}'\left(\overline{b}\right)\right) \text{(A3-4)}$$

. However, the first argument $\frac{1}{\gamma(\overline{b})-\overline{b}}\left\{1+\left(\frac{m}{m'}-1\right)\frac{F(\gamma(\overline{b}))}{H(\sigma'(\overline{b}))}\left(\frac{\sigma'(\overline{b})-\gamma(\overline{b})}{\sigma'(\overline{b})-\overline{b}}\right)\right\}$ of the maximum operator in the R.H.S. of (A3-2) at $\overline{b}$ is a strictly decreasing function $\gamma\left(\overline{b}\right)$, a strictly decreasing function of $\sigma'\left(\overline{b}\right)$ (the ratio $\frac{\sigma'(\overline{b})-\gamma(\overline{b})}{\sigma'(\overline{b})-\overline{b}}$ can also be written as $\left(1+\frac{\overline{b}-\gamma(\overline{b})}{\sigma'(\overline{b})-\overline{b}}\right)$), and a strictly decreasing function of the ratio m/m'. Since $\gamma\left(\overline{b}\right) \leq \widetilde{\gamma}\left(\overline{b}\right)$, $\sigma'\left(\overline{b}\right) < \widetilde{\sigma}\left(\overline{b}\right) = g(d)$, and $m/m' \leq \widetilde{m}/\widetilde{m}'$, this first argument at $\overline{b}$ in the equation (A3-2) for $\frac{d}{db}\ln H\left(\widetilde{\sigma}'\left(\overline{b}\right)\right)$ is strictly smaller than in the equation for $\frac{d}{db}\ln H\left(\sigma'\left(\overline{b}\right)\right)$. Since the second arguments are identical (they are equal to the constant 0), we have $\frac{d}{db}\ln H\left(\widetilde{\sigma}'\left(\overline{b}\right)\right) \leq \frac{d}{db}\ln H\left(\sigma'\left(\overline{b}\right)\right)$, which contradicts (A3-4) and we have proved $\overline{\eta} = \eta' \geq \widetilde{\eta}'$ and thus, since $\min\left(\eta',\widetilde{\eta}'\right) < \max\left(\eta',\widetilde{\eta}'\right)$, $\eta' > \widetilde{\eta}'$.

The inequality $\sigma'(b) < \widetilde{\sigma}'(b)$, for all b in $\left[\widetilde{\eta}',\overline{b}\right)$, is immediate since $\sigma'(b) < g(d) = \widetilde{\sigma}'(b)$, for all b in $\left[\widetilde{\eta}',\eta'\right)$. Since $\widetilde{\eta}' < \overline{b}$, the value at b of the R.H.S. of equation (A3-1) $\frac{d}{db}\ln F\left(\widetilde{\gamma}(b)\right)$ is equal to the value of its second argument $\frac{1}{\widetilde{\gamma}(b)-b}\left\{1+\frac{1}{\widetilde{m}/\widetilde{m}'-1}\frac{1}{F(\widetilde{\gamma}b)}\right\}$ which is thus not larger than the value of its first argument $\frac{1}{\widetilde{\sigma}'(b)-b} = \frac{1}{g(d)-b}$, for all b in $\left[\widetilde{\eta}',\overline{b}\right)$. Since $\overline{b} \leq \eta'$, the value at b of the R.H.S. of the equation (A3-1) for $\frac{d}{db}\ln F\left(\gamma(b)\right)$ is equal to the value of its first argument $\frac{1}{\sigma'(b)-b}$ which, since $\sigma'(b) < g(d)$, is strictly larger than $\frac{1}{g(d)-b}$, for all b in $\left[\widetilde{\eta}',\overline{b}\right)$. Consequently, $\frac{d}{db}\ln F\left(\widetilde{\gamma}(b)\right) < \frac{d}{db}\ln F\left(\gamma(b)\right)$, for all b in $\left[\widetilde{\eta}',\overline{b}\right)$, and $F\left(\widetilde{\gamma}(b)\right)/F\left(\gamma(b)\right)$ is strictly decreasing over this interval. Since $F\left(\widetilde{\gamma}\left(\overline{b}\right)\right)/F\left(\gamma\left(\overline{b}\right)\right) \geq 1$, we find $\widetilde{\gamma}(b) > \gamma(b)$, for all b in $\left[\widetilde{\eta}',\overline{b}\right)$. ||

Lemma A3-3: *Let $m, m', \widetilde{m}$, and $\widetilde{m}'$ be such that $\frac{\widetilde{m}}{\widetilde{m}'} > \frac{m}{m'} > 1$ and let*

$\left(\beta,\beta'\right)$ and $\left(\widetilde{\beta},\widetilde{\beta}'\right)$ the corresponding equilibria. Let $\overline{b}$ be in $\left(\underline{c},\min\left(\eta',\widetilde{\eta}'\right)\right]$. If $\gamma\left(\overline{b}\right) \leq \widetilde{\gamma}\left(\overline{b}\right)$ and $\sigma'\left(\overline{b}\right) \leq \widetilde{\sigma}'\left(\overline{b}\right)$, then $\gamma(b) < \widetilde{\gamma}(b)$ and $\sigma'(b) < \widetilde{\sigma}'(b)$ for all b in $\left(\underline{c},\overline{b}\right)$.

Proof: Since $\overline{b} \leq \min\left(\eta', \widetilde{\eta}'\right)$, the equations (3-2) and (3-3) apply over $\left(\underline{c}, \overline{b}\right]$ or, equivalently, in the equations (A3-1) and (A3-2) the values of the max and min operators are equal to the values of their first argument.

We first show that there exists $\varepsilon > 0$ such that such that $\gamma(b) < \widetilde{\gamma}(b)$ and $\sigma(b) < \widetilde{\sigma}(b)$, for all b in $\left(\overline{b} - \varepsilon, \overline{b}\right)$. The existence of such an ε is immediate if $\gamma\left(\overline{b}\right) < \widetilde{\gamma}\left(\overline{b}\right)$ and $\sigma\left(\overline{b}\right) < \widetilde{\sigma}\left(\overline{b}\right)$. Assume that $\sigma\left(\overline{b}\right) = \widetilde{\sigma}\left(\overline{b}\right)$ and $\gamma\left(\overline{b}\right) \leq \widetilde{\gamma}\left(\overline{b}\right)$. Because $\sigma < \gamma$ and the factor between braces is strictly positive over (c, η'), the L.H.S. of (3-3) is a strictly decreasing function of γ and a strictly decreasing function of m/m' over the same interval. Since $\gamma\left(\overline{b}\right) \leq \widetilde{\gamma}\left(\overline{b}\right)$ and $\frac{\widetilde{m}}{\widetilde{m}'} > \frac{m}{m'}$, we thus have $\frac{d}{db} \ln H\left(\widetilde{\sigma}'\left(\overline{b}\right)\right) < \frac{d}{db} \ln H\left(\sigma'\left(\overline{b}\right)\right)$ and an $\varepsilon > 0$ as in the top of this paragraph exists. If $\gamma\left(\overline{b}\right) < \widetilde{\gamma}\left(\overline{b}\right)$ and $\sigma\left(\overline{b}\right) < \widetilde{\sigma}\left(\overline{b}\right)$, equation (3-2) immediately implies $\frac{d}{db} \ln F\left(\widetilde{\gamma}\left(\overline{b}\right)\right) < \frac{d}{db} \ln F\left(\gamma\left(\overline{b}\right)\right)$ and again such an $\varepsilon > 0$ exists.

Define b^* as follows:

$$b^* = \inf\left\{\widehat{b} \text{ in } \left[\underline{c}, \overline{b} - \varepsilon\right] \mid \gamma(b) < \widetilde{\gamma}(b) \text{ and } \sigma(b) < \widetilde{\sigma}(b), \text{ for all b in } \left(\widehat{b}, \overline{b} - \varepsilon\right]\right\}$$

The set in this definition is not empty since it includes $\overline{b} - \varepsilon$. Suppose that $b^* > c$. Then by continuity $\gamma(b^*) = \widetilde{\gamma}(b^*)$ or $\sigma(b^*) = \widetilde{\sigma}(b^*)$. Suppose that $\sigma(b^*) = \widetilde{\sigma}(b^*)$ and $\gamma(b^*) \leq \widetilde{\gamma}(b^*)$. Reasoning as in the proof of the previous lemma, we see that equation (3-3) implies $\frac{d}{db} \ln H\left(\widetilde{\sigma}'(b^*)\right) < \frac{d}{db} \ln H\left(\sigma'(b^*)\right)$. Consequently, there exists a right-hand neighborhood of b^* where $\widetilde{\sigma}'$ is strictly smaller than σ', that is, there exists $\delta > 0$ such that $\widetilde{\sigma}'(b) > \sigma'(b)$, for all b in $(b^*, b^* + \delta)$. However, this contradicts the definition of b^* and $\sigma(b^*) = \widetilde{\sigma}(b^*)$, $\gamma(b^*) \leq \widetilde{\gamma}(b^*)$ is impossible. Suppose that $\sigma(b^*) < \widetilde{\sigma}(b^*)$ and $\gamma(b^*) = \widetilde{\gamma}(b^*)$. Again reasoning as in the previous paragraph we see that $\frac{d}{db} \ln F\left(\widetilde{\gamma}\left(\overline{b}\right)\right) < \frac{d}{db} \ln F\left(\gamma\left(\overline{b}\right)\right)$. Consequently, $\widetilde{\gamma}$ is strictly smaller than γ over a right-hand neighborhood of $\overline{b}$. Again, this contradicts the definition of $\overline{b}$. We have thus shown that $b^* = \underline{c}$ and the proof of the lemma is complete. ||

Lemma A3-4: *Let $m, m', \widetilde{m}$, and $\widetilde{m}'$ be such that $\frac{\widetilde{m}}{\widetilde{m}'} > \frac{m}{m'} > 1$ and let $\left(\beta, \beta'\right)$ and $\left(\widetilde{\beta}, \widetilde{\beta}'\right)$ the corresponding equilibria. Let $\overline{b}$ be in $\left(\underline{c}, \max\left(\eta', \widetilde{\eta}'\right)\right]$. If $\gamma\left(\overline{b}\right) \leq \widetilde{\gamma}\left(\overline{b}\right)$ and $\sigma'\left(\overline{b}\right) \leq \widetilde{\sigma}'\left(\overline{b}\right)$, then $\gamma(b) \leq \widetilde{\gamma}(b)$ and $\sigma'(b) \leq \widetilde{\sigma}'(b)$, for all b in $\left(\underline{c}, \overline{b}\right]$, and $\sigma'(b) < \widetilde{\sigma}'(b)$, for all b in $\left(\underline{c}, \min\left(\eta', \widetilde{\eta}', \overline{b}\right)\right)$.*

Proof: Immediate from the two previous lemmas. ||

Lemma A3-5: *Assume that $r > c$ or, equivalently, that $\underline{c} > c$ (thus $F(\underline{c}) > 0$). Let $m, m', \widetilde{m}$, and $\widetilde{m}'$ be such that $\frac{\widetilde{m}}{\widetilde{m}'} > \frac{m}{m'} > 1$ and let $\left(\beta, \beta'\right)$ and $\left(\widetilde{\beta}, \widetilde{\beta}'\right)$ the corresponding equilibria. Let $\overline{b}$ be in $\left(\underline{c}, \max\left(\eta', \widetilde{\eta}'\right)\right]$. The the following inequalities cannot hold simultaneously: $\gamma\left(\overline{b}\right) \leq \widetilde{\gamma}\left(\overline{b}\right)$ and $\sigma\left(\overline{b}\right) \leq \widetilde{\sigma}\left(\overline{b}\right)$.*

Proof: Suppose $\gamma\left(\overline{b}\right) \leq \widetilde{\gamma}\left(\overline{b}\right)$ and $\sigma\left(\overline{b}\right) \leq \widetilde{\sigma}\left(\overline{b}\right)$. Then, the previous lemma implies $\sigma(b) \leq \widetilde{\sigma}(b)$, for all b in $\left(c, \overline{b}\right)$, and $\sigma(b) < \widetilde{\sigma}(b)$, for all b in $\left(\underline{c}, \min\left(\eta', \widetilde{\eta}', \overline{b}\right)\right)$. From (2), we thus have $\frac{d}{db}\ln F\left(\gamma\left(b\right)\right) \geq \frac{d}{db}\ln F\left(\widetilde{\gamma}\left(b\right)\right)$ over $\left(\underline{c}, \overline{b}\right)$ and $\frac{d}{db}\ln F\left(\gamma\left(b\right)\right) > \frac{d}{db}\ln F\left(\widetilde{\gamma}\left(b\right)\right)$ over $\left(\underline{c}, \min\left(\eta', \widetilde{\eta}', \overline{b}\right)\right)$. Consequently, $F\left(\widetilde{\gamma}\right)/F\left(\gamma\right)$ is nondecreasing over $\left(\underline{c}, \overline{b}\right)$ and is strictly increasing over $\left(\underline{c}, \min\left(\eta', \widetilde{\eta}', \overline{b}\right)\right)$. Thus $F\left(\widetilde{\gamma}\left(\underline{c}\right)\right)/F\left(\gamma\left(\underline{c}\right)\right) > F\left(\widetilde{\gamma}\left(\overline{b}\right)\right)/F\left(\gamma\left(\overline{b}\right)\right) \geq 1$ and $F\left(\widetilde{\gamma}\left(\underline{c}\right)\right) > F\left(\gamma\left(\underline{c}\right)\right)$. However, (4) implies $F\left(\widetilde{\gamma}\left(\underline{c}\right)\right) = F\left(\gamma\left(\underline{c}\right)\right) = F\left(\underline{c}\right)$ and we obtain a contradiction. ||

Lemma A3-6: *Assume that $r > c$ or, equivalently, that $\underline{c} > c$ (thus $F\left(\underline{c}\right) > 0$). Let $m, m', \widetilde{m}$, and $\widetilde{m}'$ be such that $\frac{\widetilde{m}}{\widetilde{m}'} > \frac{m}{m'} > 1$ and let $\left(\beta, \beta'\right)$ and $\left(\widetilde{\beta}, \widetilde{\beta}'\right)$ the corresponding equilibria. Then $\widetilde{\gamma}\left(b\right) < \gamma\left(b\right)$, for all b in $\left(\underline{c}, \max\left(\eta', \widetilde{\eta}'\right)\right]$.*

Proof: Since $\widetilde{\sigma}'\left(\max\left(\eta', \widetilde{\eta}'\right)\right) = \sigma'\left(\max\left(\eta', \widetilde{\eta}'\right)\right) = g\left(d\right)$, the previous lemma implies $\widetilde{\gamma}\left(\max\left(\eta', \widetilde{\eta}'\right)\right) < \gamma\left(\max\left(\eta', \widetilde{\eta}'\right)\right)$ and there thus exists $\varepsilon > 0$ such that $\widetilde{\gamma}\left(b\right) < \gamma\left(b\right)$, for all b in $\left[\max\left(\eta', \widetilde{\eta}'\right) - \varepsilon, \max\left(\eta', \widetilde{\eta}'\right)\right]$.

Let b^* be defined as follows:

$$b^* = \inf\left\{\widehat{b} \text{ in } \left(c, \max\left(\eta', \widetilde{\eta}'\right)\right] \mid \widetilde{\gamma}\left(b\right) < \gamma\left(b\right), \text{ for all b in } \left(\widehat{b}, \max\left(\eta', \widetilde{\eta}'\right)\right]\right\}$$

The set in the definition is not empty since belongs $\max\left(\eta', \widetilde{\eta}'\right) - \varepsilon$ to it. Suppose that $b^* >$ Then by continuity again, we have $\widetilde{\gamma}\left(b^*\right) = \gamma\left(b^*\right)$. The previous lemma implies $\widetilde{\sigma}'\left(b^*\right) < \sigma\left(b^*\right)$ and thus $b^* < \widetilde{\eta}'$. From equation (A3-1), we thus have $\frac{d}{db}\ln F\left(\gamma\left(b^*\right)\right) \leq \frac{1}{\sigma'\left(b^*\right) - b^*} < \frac{1}{\widetilde{\sigma}'\left(b^*\right) - b^*} = \frac{d}{db}\ln F\left(\widetilde{\gamma}\left(b^*\right)\right)$. Consequently, there exists $\tau > 0$ such that $\widetilde{\gamma}\left(b\right) > \gamma\left(b\right)$, for all b in $\left(b^*, b^* + \tau\right)$. However, this contradicts the definition of b^* and consequently we have proved that $b^* = \underline{c}$ and thus $\widetilde{\gamma}\left(b\right) < \gamma\left(b\right)$, for all b in $\left(\underline{c}, \max\left(\eta', \widetilde{\eta}'\right)\right]$. ||

Lemma A3-7: *Let $m, m', \widetilde{m}$, and $\widetilde{m}'$ be such that $\frac{\widetilde{m}}{\widetilde{m}'} > \frac{m}{m'} > 1$ and let $\left(\beta, \beta'\right)$ and $\left(\widetilde{\beta}, \widetilde{\beta}'\right)$ the corresponding equilibria. Then $\widetilde{\gamma}\left(b\right) < \gamma\left(b\right)$, for all b in $\left(\underline{c}, \max\left(\eta', \widetilde{\eta}'\right)\right]$.*

Proof: From the previous lemma, we can assume $\underline{c} = c$ and thus $F\left(\underline{c}\right) = 0$. From the proofs in Appendix 2, we see that the equilibrium inverse bid functions $\left(\gamma, \sigma'\right)$ are limits of equilibrium inverse bid functions $\left(\gamma_\rho, \sigma'_\rho\right)$ for $\rho \rightarrow_> \underline{c}$, where ρ is a "reserve price" strictly larger than $\underline{c}$. In fact, for all $\rho > \underline{c}$, if we consider the case where the new reserve price is equal to ρ the differential equations (2) and (3) in the characterization of the equilibrium will be identical to the equations in the initial environment. The link (5) will remain unchanged. However, in the boundary conditions (4) the initial condition $\gamma\left(\underline{c}\right) = \sigma'\left(\underline{c}\right) = \underline{c}$ will be replaced by $\gamma_\rho\left(\rho\right) = \sigma'_\rho\left(\rho\right) = \rho$. Actually, the couple $\left(\gamma_\rho, \sigma'_\rho\right)$ forms a type II-solution of the differential system (2-3), as we called it in the proofs in Appendix 2. In Appendix 2, we showed that the only equilibrium in the initial environment for the distribution F corresponds to a value of the parameter η' which is the limit (the infimum η'^*) of the values of the parameter corresponding to such type II-solutions. From the continuity, under our hypotheses, of the solution of a differential system with respect to the initial conditions, we obtain $(\gamma(b), \sigma'(b)) = \lim_{\rho \to > \underline{c}} \left(\gamma_\rho(b), \sigma'_\rho(b)\right)$, for all b in $(\underline{c}, \eta']$.

Carrying out the same truncations for $\left(\widetilde{\gamma}, \widetilde{\sigma}'\right)$, we see from the previous lemma that $\widetilde{\gamma}_\rho(b) < \gamma_\rho(b)$, for all b in $\left(\rho, \max\left(\eta'_\rho, \widetilde{\eta}'_\rho\right)\right]$. Taking the limit for $\rho \to \underline{c}$ and using $\lim_{\rho \to > \underline{c}} \eta'_\rho = \eta'$ and $\lim_{\rho \to > \underline{c}} \widetilde{\eta}'_\rho = \eta'_\rho$, we find

$$\widetilde{\gamma}(b) \leq \gamma(b) \text{ (A3-5)}$$

, for all b in $\left(\rho, \max\left(\eta', \widetilde{\eta}'\right)\right]$.

Assume that there exists b in $\left(\rho, \max\left(\eta', \widetilde{\eta}'\right)\right]$ such that $\widetilde{\gamma}(b) = \gamma(b)$. Suppose that $\widetilde{\sigma}'(b) \geq \sigma'(b)$. Lemma A3-4 then implies that $\gamma(b') < \widetilde{\gamma}(b')$, for all b in $(\underline{c}, b]$, which contradicts (A3-5). Consequently, $\widetilde{\sigma}'(b) < \sigma'(b)$ and the conclusion of Lemma A3-4 also holds true here. The rest of the proof then proceeds as in the proof of the previous lemma. ||

Lemma A3-8: *Let $m, m', \widetilde{m}$, and $\widetilde{m}'$ be such that $\frac{\widetilde{m}}{\widetilde{m}'} > \frac{m}{m'} > 1$ and let*

$\left(\beta, \beta'\right)$ and $\left(\widetilde{\beta}, \widetilde{\beta}'\right)$ the corresponding equilibria. Then $\widetilde{\eta} > \eta$ and $\widetilde{\gamma}(b) < \gamma(b)$, for all b in $(\underline{c}, \eta]$.

Proof: From the previous lemma, we have $\widetilde{\gamma}\left(\max\left(\eta', \widetilde{\eta}'\right)\right) < \gamma\left(\max\left(\eta', \widetilde{\eta}'\right)\right)$. We first prove $\widetilde{\gamma}(b) < \gamma(b)$, for all b in $\left[\max\left(\eta', \widetilde{\eta}'\right), \min(\eta, \widetilde{\eta})\right]$. It is immediate if $\min(\eta, \widetilde{\eta}) < \max\left(\eta', \widetilde{\eta}'\right)$. Assume that $\max\left(\eta', \widetilde{\eta}'\right) < \min(\eta, \widetilde{\eta})$. Let b^* be defined as follows:

$$b^* = \sup\left\{\widehat{b} \in \left[\max\left(\eta', \widetilde{\eta}'\right), \min(\eta, \widetilde{\eta})\right] \mid \widetilde{\gamma}(b) < \gamma(b), \text{ for all b in } \left[\max\left(\eta', \widetilde{\eta}'\right), \widehat{b}\right)\right\}$$

The set in the definition of b^* since (by continuity) it includes an interval of the kind $\left[\max\left(\eta', \widetilde{\eta}'\right), \max\left(\eta', \widetilde{\eta}'\right) + \varepsilon\right]$, where $\varepsilon > 0$. Suppose that $b^* < \min(\eta, \widetilde{\eta})$. From the continuity of γ and $\widetilde{\gamma}$, we have $\widetilde{\gamma}(b^*) = \gamma(b^*)$. The R.H.S. of the differential equation (A1-8) γ satisfies is a strictly decreasing function of the ratio m/m'. Since $\widetilde{m}/\widetilde{m}' > m/m'$, this differential equation thus implies $\frac{d}{db} \ln F(\widetilde{\gamma}(b^*)) < \frac{d}{db} \ln F(\gamma(b^*))$ and there exists $\delta > 0$ such that $\widetilde{\gamma}(b) > \gamma(b)$, for all b in $(b^* - \delta, b^*)$. However, this contradicts the definition of b^* and we have proved that $b^* = \min(\eta, \widetilde{\eta})$ and that $\widetilde{\gamma}(b) < \gamma(b)$, for all b in $\left[\max\left(\eta', \widetilde{\eta}'\right), \min(\eta, \widetilde{\eta})\right]$.

If $\eta \geq \widetilde{\eta}$, then $\min(\eta, \widetilde{\eta}) = \widetilde{\eta}$. From the conclusion of the previous paragraph, we would thus obtain $d = \widetilde{\gamma}(\widetilde{\eta}) < \gamma(\widetilde{\eta})$ which is impossible since $\gamma(\widetilde{\eta}) \leq \gamma(\eta) = d$. The proof of Lemma A3-8 is complete. ||

Lemma A3-9: *If $\left(\beta, \beta'\right)$ is a regular equilibrium, then*

$$\frac{d}{db} \ln F(\gamma'(b)) + (m/m' - 1) F(\gamma(b)) = \frac{1}{\gamma(b) - b}$$

for all b in $(\underline{c}, \eta]$.

Proof: It is an immediate consequence of Theorem 4 (Section 3) (see (A2-10) and how we arrived at (A2-11)). ||

Lemma A3-10: *Let* $m, m', \widetilde{m}$, *and* $\widetilde{m}'$ *be such that* $\frac{\widetilde{m}}{\widetilde{m}'} > \frac{m}{m'} > 1$ *and let* (β, β') *and* $\left(\widetilde{\beta}, \widetilde{\beta}'\right)$ *the corresponding equilibria. Then* $F\left(\widetilde{\gamma}'(b)\right) + (\widetilde{m}/\widetilde{m}' - 1) F(\widetilde{\gamma}(b)) / F(\gamma'(b)) + (m/m' - 1) F(\gamma(b))$ *is strictly increasing over* $(\underline{c}, \eta]$.

Proof: From Lemma A3-9 and Lemma A3-8, we have $\frac{d}{db} \ln \widetilde{m}' F\left(\widetilde{\gamma}'(b)\right) + (\widetilde{m} - \widetilde{m}') F(\widetilde{\gamma}(b)) > \frac{d}{db} \ln m' F\left(\widetilde{\gamma}'(b)\right) + (m - m') F(\gamma(b))$, for all b in $(\underline{c}, \eta]$, and $F\left(\widetilde{\gamma}'(b)\right) + (\widetilde{m}/\widetilde{m}' - 1) F(\widetilde{\gamma}(b)) / F(\gamma'(b)) + (m/m' - 1) F(\gamma(b))$ is strictly increasing over this interval. ||

Appendix 4: Comparative Statics II

The differential equations (A2-31) and (A2-32) in Lemma A2-14 in Appendix 2 can be rewritten equivalently as follows:

$$\frac{d}{dv}\beta(v) = \frac{\varphi(v) - \beta(v)}{F(v)} \text{(A4-1)}$$

$$\frac{d}{dv}\varphi(v) = \frac{H(\varphi(v))}{F(v)} \frac{\varphi(v) - \beta(v)}{v - \beta(v)} \left\{1 + \left(\frac{m}{m'} - 1\right) \frac{F(v)}{H(\varphi(v))} \left(\frac{\varphi(v) - v}{\varphi(v) - \beta(v)}\right)\right\} \text{(A4-2)}$$

, where $\varphi = \sigma' \circ \beta$. From Lemma A2-14, we know that the couple (β, φ) is a solution of (A4-1,A4-2) over $(\underline{c}, \gamma(\eta')]$ which satisfies the following boundary conditions:

$$\beta(\underline{c}) = \varphi(\underline{c}) = \underline{c}, \beta(\gamma(\eta')) = \eta', \varphi(\gamma(\eta')) = g(d) \text{(A4-3)}$$

Lemma A4-1: *The function* δ' *is nonincreasing in* m/m'.

Proof: It follows from the definition of δ' and the observation that since in its definition $g(d) \leq d'$ the function $\overline{\eta}'$ is a nonincreasing function of the ratio m/m'. ||

Lemma A4-2: *Let* $m, m', \widetilde{m}$, *and* $\widetilde{m}'$ *be such that* $\frac{\widetilde{m}}{\widetilde{m}'} > \frac{m}{m'} > 1$ *and let* (β, β') *and* $\left(\widetilde{\beta}, \widetilde{\beta}'\right)$ *the corresponding equilibria. Then,* $\widetilde{\gamma}\left(\widetilde{\eta}'\right) \leq \gamma(\eta')$.

Proof: From Appendix 3 we know that $\widetilde{\eta} > \eta$ and $\widetilde{\gamma}(b) < \gamma(b)$, for all b in $(\underline{c}, \eta]$. Moreover, from the previous lemma the function $\widetilde{\delta}'$ is not larger than δ'. The functions δ' and $\widetilde{\delta}'$ are nonincreasing and the functions γ and $\widetilde{\gamma}$ are strictly increasing. Consequently, the value $\gamma(\eta')$ which is the second component of the intersection of the graphs of δ and γ (or, equivalently, the value $\gamma(x)$ where x is the only solution of the equation $\gamma(x) = \delta(x)$) is not smaller than $\widetilde{\gamma}\left(\widetilde{\eta}'\right)$ (see Figure 1). ||

Lemma A4-3: *Let $m, m', \widetilde{m}$, and $\widetilde{m}'$ be such that $\frac{\widetilde{m}}{\widetilde{m}'} > \frac{m}{m'} > 1$ and let (β, β') and $\left(\widetilde{\beta}, \widetilde{\beta}'\right)$ the corresponding equilibria. Then, there exists $\varepsilon > 0$ such that $\widetilde{\varphi}(v) > \varphi(v)$, for all v in $\left(\widetilde{\gamma}\left(\widetilde{\eta}'\right) - \varepsilon, \widetilde{\gamma}\left(\widetilde{\eta}'\right)\right)$.*

Proof: From the previous lemma, $\widetilde{\gamma}\left(\widetilde{\eta}'\right) \leq \gamma(\eta')$. If $\widetilde{\gamma}\left(\widetilde{\eta}'\right) < \gamma(\eta')$, $\widetilde{\varphi}\left(\widetilde{\gamma}\left(\widetilde{\eta}'\right)\right) = g(d) = \varphi(\gamma(\eta')) > \varphi\left(\widetilde{\gamma}\left(\widetilde{\eta}'\right)\right)$ and the existence of such an $\varepsilon > 0$ is immediate.

Rearranging (A4-2), we find

$$\frac{d}{dv}\varphi(v) = \frac{H(\varphi(v))}{F(v)}\left\{1 + \frac{\varphi(v) - v}{v - \beta(v)}\right\}\left\{1 + \left(\frac{m}{m'} - 1\right)\frac{F(v)}{H(\varphi(v))}\left(\frac{\varphi(v) - v}{\varphi(v) - \beta(v)}\right)\right\} \quad \text{(A4-4)}$$

for all v in $(\underline{c}, \gamma(\eta')]$. Since $\varphi(v) \leq v$ (since $\sigma \leq \beta$) and $\beta(v) < v$ (since $\gamma(v) > v$) over this interval, we see that $\frac{d}{dv}\varphi(v)$ is a nonincreasing function of $\beta(v)$ and m/m', over $(\underline{c}, \gamma(\eta')]$. Moreover, since $\varphi(v) < v$ over $(\underline{c}, \gamma(\eta'))$ we see that $\frac{d}{dv}\varphi(v)$ is a strictly decreasing function of $\beta(v)$ and m/m', over $(\underline{c}, \gamma(\eta'))$.

Assume $\widetilde{\gamma}\left(\widetilde{\eta}'\right) = \gamma(\eta')$ and thus $\varphi\left(\widetilde{\gamma}\left(\widetilde{\eta}'\right)\right) = \widetilde{\varphi}\left(\widetilde{\gamma}\left(\widetilde{\eta}'\right)\right) = g(d)$. From Appendix 3, $\widetilde{\beta}\left(\widetilde{\gamma}\left(\widetilde{\eta}'\right)\right) > \beta\left(\widetilde{\gamma}\left(\widetilde{\eta}'\right)\right)$. Moreover, $\widetilde{m}/\widetilde{m}' > m/m'$. If we substitute $\widetilde{\gamma}\left(\widetilde{\eta}'\right)$ to v in (A4-2), rearranging, and use (A4-3), we thus obtain $\frac{d}{dv}\widetilde{\varphi}\left(\widetilde{\gamma}\left(\widetilde{\eta}'\right)\right) < \frac{d}{dv}\varphi\left(\widetilde{\gamma}\left(\widetilde{\eta}'\right)\right)$ when $g(d) < d$ and thus $\widetilde{\gamma}\left(\widetilde{\eta}'\right) < d$. An $\varepsilon > 0$ as in the previous paragraph thus also exists in this case.

The remaining case we have to examine is $g(d) = d = \widetilde{\gamma}\left(\widetilde{\eta}'\right) = \gamma(\eta')$ and thus $\eta = \eta'$ and $\widetilde{\eta}' = \widetilde{\eta}$. In this case, $\varphi\left(\widetilde{\gamma}\left(\widetilde{\eta}'\right)\right) = \widetilde{\varphi}\left(\widetilde{\gamma}\left(\widetilde{\eta}'\right)\right) = g(d) = d$, $\frac{d}{dv}\widetilde{\varphi}\left(\widetilde{\gamma}\left(\widetilde{\eta}'\right)\right) = \frac{d}{dv}\varphi\left(\widetilde{\gamma}\left(\widetilde{\eta}'\right)\right) = 1$. From (A4-1, A4-2), φ and β are twice differentiable over $(c, d]$ and calculating the second derivatives, substituting d to v, and using the initial condition in (A4-3), and our assumption $\frac{d}{dv}\frac{H}{F}(v) > 0$, for all v in $(\underline{c}, d]$, we find

$$\frac{d^2}{dv^2}\varphi(d) = \frac{d^2}{dv^2}\widetilde{\varphi}(d) = h(d) - f(d) < 0$$

Consider the expression $\zeta(v) = \frac{H(\varphi(v))}{F(v)}\left\{\frac{F(v)}{H(\varphi(v))}\frac{v-\beta(v)}{\varphi(v)-\beta(v)}\frac{d}{dv}\varphi(v) - 1\right\}$ and the similar expression $\widetilde{\zeta}(v)$ for $\widetilde{\varphi}$ and $\widetilde{\beta}$. Both expressions are equal to 0 at $v = d$. From (A4-2), we see that ζ and $\widetilde{\zeta}$ are twice differentiable over $(c, d]$. The first derivative $\frac{d}{dv}\zeta(v)$ is equal to $\left(\frac{m}{m'} - 1\right)\left(\frac{\frac{d}{dv}\varphi(v)-1}{\varphi(v)-\beta(v)} - \frac{(\varphi(v)-v)\left(\frac{d}{dv}\varphi(v)-\frac{d}{dv}\beta(v)\right)}{(\varphi(v)-\beta(v))^2}\right)$

and we thus see that $\frac{d}{dv}\zeta(d) = \frac{d}{dv}\widetilde{\zeta}(d) = 0$. Calculating the second derivative and cancelling the terms which vanish at d, we find $\frac{d^2}{dv^2}\zeta(d) = \left(\frac{m}{m'} - 1\right)\left(\frac{\frac{d^2}{dv^2}\varphi(d)}{\varphi(d)-\beta(d)}\right) = \left(\frac{m}{m'} - 1\right)\frac{h(d)-f(d)}{d-\eta} > \left(\frac{\widetilde{m}}{\widetilde{m}'} - 1\right)\frac{h(d)-f(d)}{d-\widetilde{\eta}} = \frac{d^2}{dv^2}\widetilde{\zeta}(d)$ since $\widetilde{m}/\widetilde{m}' > m/m'$ and, from Appendix 3, $\widetilde{\eta} > \eta$. Consequently, there exists $\varepsilon > 0$ such that $\zeta(v) > \widetilde{\zeta}(v)$, for all v in $(d-\varepsilon, d)$.

From the definition of ζ we have

$$\frac{d}{dv}\varphi(v) = \frac{H(\varphi(v))}{F(v)}\left\{1 + \frac{\varphi(v) - v}{v - \beta(v)}\right\}\{1 + \zeta(v)\} \text{ (A4-5)}$$

for all v in $(\underline{c}, d]$. Since $\varphi(v) < v$, over $(\underline{c}, d)$, the factor $\frac{H(\varphi(v))}{F(v)}\left\{1 + \frac{\varphi(v)-v}{v-\beta(v)}\right\}$ is a strictly increasing function of $\varphi(v)$ and a strictly decreasing function of $\beta(v)$ over $(\underline{c}, d)$. Since, from Appendix 3, $\beta(v) < \widetilde{\beta}(v)$, for all v in $(\underline{c}, d]$, we see that for all v in $(d-\varepsilon, d)$

$$\text{if } \widetilde{\varphi}(v) \leq \varphi(v)\text{, then } \frac{d}{dv}\widetilde{\varphi}(v) < \frac{d}{dv}\varphi(v) \text{ (A4-6)}$$

This property implies that $\widetilde{\varphi}(v) > \varphi(v)$, for all v in $(d-\varepsilon, d)$. We first prove that this property implies that $\widetilde{\varphi}(v) \geq \varphi(v)$, for all v in $(d-\varepsilon, d)$. Suppose that there exists u in $(d-\varepsilon, d)$ such that $\widetilde{\varphi}(u) < \varphi(u)$ or, equivalently, $\widetilde{\varphi}(u) - \varphi(u) < 0$. Define w^* as follows:

$$w^* = \sup\{w \in [u, d] \mid \widetilde{\varphi}(v) < \varphi(v)\text{, for all v in } [u, w)\}$$

The set in this definition is not empty since it includes u. By continuity, we have $w^* > u$. From $\widetilde{\varphi}(d) - \varphi(d) = d - d = 0$ and from the continuity of the functions involved, we have $\widetilde{\varphi}(w^*) - \varphi(w^*) = 0$. There exists (Mean Value Theorem in elementary real analysis) s in (u, w^*) such that $\frac{d}{dv}\widetilde{\varphi}(s) - \frac{d}{dv}\varphi(s) = \frac{(\widetilde{\varphi}(w^*) - \varphi(w^*)) - (\widetilde{\varphi}(u) - \varphi(u))}{w^* - u} < 0$. However, by definition of w^* we have $\widetilde{\varphi}(s) < \varphi(s)$ and we obtain a contradiction with the property (A4-6). We have thus proved that $\widetilde{\varphi}(v) \geq \varphi(v)$, for all v in $(d-\varepsilon, d)$.

We now show that $\widetilde{\varphi}(v) > \varphi(v)$, for all v in $(d-\varepsilon, d)$. Suppose that there exists u in $(d-\varepsilon, d)$ such that $\widetilde{\varphi}(u) = \varphi(u)$. From (A4-5), we then have $\frac{d}{dv}\widetilde{\varphi}(u) < \frac{d}{dv}\varphi(u)$ and $\widetilde{\varphi}$ is strictly smaller than φ over a right-hand neighborhood of u. This contradicts the conclusion of the previous paragraph and the lemma is proved. ||

Lemma A4-4: *Let $m, m', \widetilde{m}$, and $\widetilde{m}'$ be such that $\frac{\widetilde{m}}{\widetilde{m}'} > \frac{m}{m'} > 1$ and let (β, β') and $\left(\widetilde{\beta}, \widetilde{\beta}'\right)$ the corresponding equilibria. Then, $\widetilde{\varphi}(v) > \varphi(v)$, for all v in $\left(\underline{c}, \widetilde{\gamma}\left(\widetilde{\eta}'\right)\right)$ (see Figure 2).*

Proof: From the previous lemma, there exists $\varepsilon > 0$ such that $\widetilde{\varphi}(v) > \varphi(v)$, for all v in $\left(\widetilde{\gamma}\left(\widetilde{\eta}'\right)-\varepsilon, \widetilde{\gamma}\left(\widetilde{\eta}'\right)\right)$. Define v^* as follows:

$$v^* = \inf\left\{w \in \left[\underline{c}, \widetilde{\gamma}\left(\widetilde{\eta}'\right)\right] \mid \widetilde{\varphi}(v) > \varphi(v), \text{ for all v in } \left(w, \widetilde{\gamma}\left(\widetilde{\eta}'\right)\right)\right\}$$

The set in this definition is not empty and $v^* \leq \widetilde{\gamma}\left(\widetilde{\eta}'\right) - \varepsilon$. We want to prove that $v^* = \underline{c}$. Suppose that $v^* > \underline{c}$. Then, by continuity we have $\widetilde{\varphi}(v^*) = \varphi(v^*)$. In the proof of the previous lemma, we saw that $\frac{d}{dv}\varphi(v)$ is a strictly decreasing function of $\beta(v)$ and m/m', over $(\underline{c}, \gamma(\eta')) \supseteq \left(\underline{c}, \widetilde{\gamma}\left(\widetilde{\eta}'\right)\right)$. Since $\widetilde{m}/\widetilde{m}' > m/m'$ and $\widetilde{\beta}(v^*) > \beta(v^*)$, we obtain $\frac{d}{dv}\widetilde{\varphi}(v^*) < \frac{d}{dv}\varphi(v^*)$ and there thus exists $\delta > 0$ such that $\widetilde{\varphi}(v) < \varphi(v)$, for all v in $(v^*, v^* + \delta)$. However, this contradicts the definition of v^* and the lemma is proved. ||

Because payments by bidders to the seller are only transfers which cancel out, the total surplus is the sum of the units' valuations to the bidders who have been awarded them. Let $\left(\beta, \beta'\right)$ be a symmetric regular equilibrium. The first m units go the bidder who has submitted the highest high bid, that is,mth highest type's bidder. The contribution to the surplus of these first m units is thus $m \max(v_1, v_2)$, where v_1 and v_2 are independently and identically distributed according to F. The last $m' = n-m$ units sold go to the highest type bidder if his low bid $\beta'(\max(v_1, v_2))$ is at least as large as his opponent's high bid $\beta(\min(v_1, v_2))$, that is, if $\max(v_1, v_2) \geq \gamma' \circ \beta(\min(v_1, v_2))$ or, equivalently, if $g(\max(v_1, v_2)) \geq \sigma' \circ \beta(\min(v_1, v_2)) = \varphi(\min(v_1, v_2))$. For the highest type bidder, the total valuation of these last m' units is $m' g(\max(v_1, v_2))$. When the highest type bidder's low bid $\beta'(\max(v_1, v_2))$ is smaller than the smallest type bidder's high bid $\beta(\min(v_1, v_2))$, that is, when $\max(v_1, v_2) < \gamma' \circ \beta(\min(v_1, v_2))$ or, equivalently, $g(\max(v_1, v_2)) < \sigma' \circ \beta(\min(v_1, v_2)) = \varphi(\min(v_1, v_2))$, then $\min(m', m)$ units go to the lowest type bidder giving him a total utility of $\min(m', m)\min(v_1, v_2)$, since these units are the only units he is awarded. In this case, if there are remaining units, that is, if $n - m - \min(m', m) = m' - \min(m', m) > 0$ or, equivalently, $m' > m$, they go to the highest type bidder since he has submitted the highest low bid. These units contributes to the total surplus an amount equal to $(m' - \min(m', m))\, g(\max(v_1, v_2))$. We obtain Lemma A4-5 below.

Lemma A4-5: *Let $\left(\beta, \beta'\right)$ be a symmetric regular equilibrium and let α be the proportion of high valuation units, that is, $\alpha = m/n$. Then, we have*

$$\begin{aligned} TS &= mE(\max(v_1, v_2)) + \\ &\quad + \min(m', m)\, E\left(g(\max(v_1, v_2))\, I\{g(\max(v_1, v_2)) \geq \varphi(\min(v_1, v_2))\}\right) \\ &\quad + \min(m', m)\, E\left(\min(v_1, v_2)\, I\{g(\max(v_1, v_2)) \geq \varphi(\min(v_1, v_2))\}\right) \\ &\quad + (m' - \min(m', m))\, E(g(\max(v_1, v_2))) \end{aligned}$$

Thus, if $m \geq m'$ we have

$$\begin{aligned} TS &= mE(\max(v_1, v_2)) + \\ &\quad + m'E\left(g(\max(v_1, v_2))\, I\{g(\max(v_1, v_2)) \geq \varphi(\min(v_1, v_2))\}\right) \\ &\quad + m'E\left(\min(v_1, v_2)\, I\{g(\max(v_1, v_2)) \geq \varphi(\min(v_1, v_2))\}\right) \end{aligned}$$

$$
\begin{aligned}
AS &= \alpha E\left(\max\left(v_1, v_2\right)\right) + \\
&\quad +(1-\alpha)E\left(g\left(\max\left(v_1, v_2\right)\right) I\left\{g\left(\max\left(v_1, v_2\right)\right) \geq \varphi\left(\min\left(v_1, v_2\right)\right)\right\}\right) \\
&\quad +(1-\alpha)E\left(g\left(\max\left(v_1, v_2\right)\right) I\left\{g\left(\max\left(v_1, v_2\right)\right) \geq \varphi\left(\min\left(v_1, v_2\right)\right)\right\}\right)
\end{aligned}
$$

and if $m \leq m'$ *we have*

$$
\begin{aligned}
TS &= mE\left(\max\left(v_1, v_2\right)\right) + \\
&\quad +mE\left(g\left(\max\left(v_1, v_2\right)\right) I\left\{g\left(\max\left(v_1, v_2\right)\right) \geq \varphi\left(\min\left(v_1, v_2\right)\right)\right\}\right) \\
&\quad +mE\left(\min\left(v_1, v_2\right) I\left\{g\left(\max\left(v_1, v_2\right)\right) \geq \varphi\left(\min\left(v_1, v_2\right)\right)\right\}\right) \\
&\quad +\left(m'-m\right)E\left(g\left(\max\left(v_1, v_2\right)\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
AS &= \alpha E\left(\max\left(v_1, v_2\right)\right) + \\
&\quad +\alpha E\left(g\left(\max\left(v_1, v_2\right)\right) I\left\{g\left(\max\left(v_1, v_2\right)\right) \geq \varphi\left(\min\left(v_1, v_2\right)\right)\right\}\right) \\
&\quad +\alpha E\left(\min\left(v_1, v_2\right) I\left\{g\left(\max\left(v_1, v_2\right)\right) \geq \varphi\left(\min\left(v_1, v_2\right)\right)\right\}\right) \\
&\quad +\left(1-2\alpha\right)E\left(g\left(\max\left(v_1, v_2\right)\right)\right)
\end{aligned}
$$

Lemma A4-6: *Let* $m, m', \widetilde{m}$, *and* $\widetilde{m}'$ *be such that* $\frac{\widetilde{m}}{\widetilde{m}'} > \frac{m}{m'} > 1$ *and let* $\left(\beta, \beta'\right)$ *and* $\left(\widetilde{\beta}, \widetilde{\beta}'\right)$ *be the corresponding equilibria. Then,* $\widetilde{AS} > AS$.

Proof: From the previous lemma, we have $AS = \alpha E\left(\max\left(v_1, v_2\right)\right) +$

$(1-\alpha)I$, where $a = \frac{m}{n} = 1 - \frac{m'}{n} = 1 - \frac{1}{m/m'+1}$ and I is equal to the expectation of the following expression:

$$g\left(\max\left(v_1, v_2\right)\right) I\left\{g\left(\max\left(v_1, v_2\right)\right) \geq \varphi\left(\min\left(v_1, v_2\right)\right)\right\} + \min\left(v_1, v_2\right) I\left\{g\left(\max\left(v_1, v_2\right)\right) \leq \varphi\left(\min\left(v_1, v_2\right)\right)\right\}$$

We also have $\widetilde{AS} = \widetilde{\alpha}E\left(\max\left(v_1, v_2\right)\right) + \left(1-\widetilde{\alpha}\right)\widetilde{I}$, where $\widetilde{\alpha}$ and $\widetilde{I}$ are defined similarly. Since obviously $\widetilde{AS} - AS = \widetilde{AS} - \left(1-\widetilde{\alpha}\right)I + \left(1-\widetilde{\alpha}\right)I - AS$, we have

$$\widetilde{AS} - AS = \Delta\alpha\left[E\left(\max\left(v_1, v_2\right)\right) - I\right] + \left(1-\widetilde{\alpha}\right)\Delta I$$

where $\Delta\alpha = \widetilde{\alpha} - \alpha$ and $\Delta I = \widetilde{I} - I$. Since α is a strictly increasing function of m/m', we have $\Delta\alpha > 0$. From Lemma A4-3, $\widetilde{\varphi}(v) > \varphi(v)$, for all v in (c, d), and thus

$$\Delta I = E\left\{\left[\min\left(v_1, v_2\right) - g\left(\max\left(v_1, v_2\right)\right)\right] I\left\{\widetilde{\varphi}\left(\min\left(v_1, v_2\right)\right) \geq g\left(\max\left(v_1, v_2\right)\right) \geq \varphi\left(\min\left(v_1, v_2\right)\right)\right\}\right\}$$

Since $\widetilde{\varphi}(v) < v$, for all v in (c, d), we have $g\left(\max\left(v_1, v_2\right)\right) < \min\left(v_1, v_2\right)$, for all $c < v_1, v_2 < d$ such that $\widetilde{\varphi}\left(\min\left(v_1, v_2\right)\right) \geq g\left(\max\left(v_1, v_2\right)\right)$. Consequently, $\Delta I > 0$. Finally, since $\max\left(v_1, v_2\right) > g\left(\max\left(v_1, v_2\right)\right) > \min\left(v_1, v_2\right)$, for all $c <$

$v_1, v_2 < d$, we also have $\left[E\left(\max\left(v_1, v_2\right)\right) - I\right] > 0$. We thus obtain $\widetilde{AS} - AS > 0$ and the lemma is proved. ||

Assume $m \geq m'$. From Theorem 1 (b) in Section 3 and from the proof in Appendix 2 of Theorem 4 (Section 3), we know that we can break down the maximization problem in Lemma 1 (Section 3) into the two following maximization problems:

$$\max_{\underline{c} \leq b} (v - b)\left(m' F\left(\gamma'(b)\right) + (m - m') F\left(\gamma(b)\right)\right)$$

$$\max_{\underline{c} \leq b} (g(v) - b)\, m' F\left(\gamma(b)\right)$$

The first problem consists in the maximization of the total expected payoff on the first m units and the second problem is the maximization of the total expected payoff on the last m' units. It is thus meaningful to talk about the equilibrium expected payoffs on the first m units and on the last m' units. The average (per unit) interim expected payoff $P_H(v)$on one of the first m units, or high valuation units, and the average interim expected payoff $P_L(v)$ on one of the last m' units, or the low valuation units, are, respectively,

$$P_H(v) = \max_{\underline{c} \leq b} (v - b)\left(\frac{m'}{m} F\left(\gamma'(b)\right) + \left(1 - \frac{m'}{m}\right) F\left(\gamma(b)\right)\right)$$

$$P_L(v) = \max_{\underline{c} \leq b} (g(v) - b) F\left(\gamma(b)\right)$$

$\beta(v)$ is the solution of the first maximization problem above and $\beta'(v)$ is the solution of the second maximization problem. Since $\beta(v) > \underline{c}$ for all $v > \underline{c}$ and $\beta'(w) > \underline{c}$ for all $w > g^{-1}(\underline{c})$, the problems above can be rewritten equivalently as:

$$P_H(v) = \max_{\underline{c} < b} (v - b)\left(\frac{m'}{m} F\left(\gamma'(b)\right) + \left(1 - \frac{m'}{m}\right) F\left(\gamma(b)\right)\right)$$

$$P_L(w) = \max_{\underline{c} < b} (g(w) - b) F\left(\gamma(b)\right)$$

for all $v > \underline{c}$ and $w > g^{-1}(\underline{c})$. From Lemma A3-10 in Appendix 3 and from Theorem 6 (a) (Section 6), which we proved in the previous appendix, we know that $\left(\frac{\widetilde{m}'}{\widetilde{m}} F\left(\widetilde{\gamma}'(b)\right) + \left(1 - \frac{\widetilde{m}'}{\widetilde{m}}\right) F\left(\widetilde{\gamma}(b)\right)\right) < \left(\frac{m'}{m} F\left(\gamma'(b)\right) + \left(1 - \frac{m'}{m}\right) F\left(\gamma(b)\right)\right)$ and $\widetilde{\gamma}(b) < \gamma(b)$, for all $b > \underline{c}$. The first part of Lemma A4-7 below follows Since the exante expected payoffs are the expectations of the interim payoffs, the second part also follows.

Lemma A4-7:

(1) $\widetilde{P_H}(v) < P_H(v)$, *for all* $v > c$, $\widetilde{P_L}(w) < P_L(w)$, *for all* $w > g^{-1}(c)$.

(2) $\widetilde{EP_H} < EP_H$, $\widetilde{EP_L} < EP_L$

Appendix 5: The Case g(c)<c

In this appendix we consider the case where $g(c) < c$. The analysis is identical to the analysis in the main text if $r \geq c$. Assume that $r < c$. In Theorem 1 (Section 3) (c) has to be replaced by (the more general) (c') below which consists in three parts: (c'$_1$) applies to all cases, (c'$_2$) holds true only when $m \geq m'$, and (c'$_3$) holds true when $m < m'$.

(c')
(c'$_1$): $\max(g(c), r) \leq \underline{b} = \beta_1(c) = ... = \beta_m(c) \leq c$
(c'$_2$): If $m \geq m'$, then
$\underline{b} = \beta_{m+1}\left(g^{-1}(\underline{b})\right) = ... = \beta_n\left(g^{-1}(\underline{b})\right) = \max \arg\max_{b \in [\max(r,g(c)),c]} (c-b) H(b)$.
(c'$_3$): If $m < m'$, then
$\underline{b} = \beta_{m+1}\left(g^{-1}(c)\right) = ... = \beta_{n-m}\left(g^{-1}(c)\right) = \lambda'\left(g^{-1}(c)\right)$,
$\beta_{m+1}(v) = ... = \beta_{n-m}(v) = \lambda'(v)$, for all v in $\left[g^{-1}(\max(r, g(c))), g^{-1}(c)\right]$,
where $\lambda'(v) = \frac{\max(r,g(c))F\left(g^{-1}(\max(r,g(c)))\right) + \int_{g^{-1}(\max(r,g(c)))}^{v} g(u)dF(u)}{F(v)}$,
for all v in $\left[g^{-1}(\max(r, g(c))), d\right]$.

The rest of Theorem 1 holds true if we understand the property (a) of lumpy bidding means that bidders use one bid function on all their high valuation units and one different bid function on all their low valuation units when they bid above $\underline{b}$. That is, (a) should be rewritten as (a') below where we $\gamma_1, ..., \gamma_n$ denote the inverse bid functions:

(a') $\gamma_1(b) = ... = \gamma_m(b)$, $\gamma_{m+1}(b) = ... = \gamma_n(b)$, for all $b \geq \underline{b}$.

We see then that although the initial condition (c') is different from (c), as (c) it is completely determined. When there are no more low valuation units than high valuation units ($m \geq m'$), the minimum $\underline{b}$ of the bid function on high valuation units is uniquely determined as the maximum of the solutions of the maximization problem $\max_{b \in [\max(r,g(c)),c]} (c-b) H(b)$, where H is the distribution function of the low valuations, or where $H = F \circ g^{-1}$. In this case, the bidders bid $\underline{b}$ on a low valuation unit if and only if this unit's valuation is also $\underline{b}$, that is, if their types are $g^{-1}(\underline{b})$. When there are more low valuation units than high valuation units ($m < m'$), the explicit formula for $\underline{b} = \lambda'\left(g^{-1}(c)\right)$ is

$$\underline{b} = \frac{\max(r, g(c)) F\left(g^{-1}(\max(r, g(c)))\right) + \int_{g^{-1}(\max(r,g(c)))}^{g^{-1}(c)} g(u) dF(u)}{F\left(g^{-1}(c)\right)}.$$

The bid function λ' is the equilibrium bid function when all units are of low valuations and when the reserve price is r and is thus also the equilibrium bid function in a first price auction with homogeneous bidders with valuations distributed according to H. The formula in (c'$_3$) for λ' is easily obtained from the formula in Riley and Samuelson (1981) for this symmetric case of the first price auction by changing the variables from the valuations to the types. In this case of a lower number of high valuation units, for bids below $\underline{b}$, the bid functions on the $(m+1)$th, ..., $(n-m)$th units are equal to the bid function λ'.

The bid functions above $\underline{b}$ satisfy the same conditions and equations as in the main text, these conditions being the differential equations and the initial condition at the upper extremity of the type interval in Theorem 3 when $m \leq m'$ and the differential equations (3-2) and (3-3), the second part of the boundary condition (3-4), the condition (3-5), and the formula (3-6) when $m > m'$. Where there are no conditions on some of the bid functions on low valuations units, that is, over the domain of types $\left[c, g^{-1}\left(\underline{b}\right)\right]$ for $\beta_{m+1}, ..., \beta_n$ when $m \geq m'$ (see (c'$_2$)) and over $\left[c, g^{-1}\left(c\right)\right]$ for $\beta_{n-m+1}, ..., \beta_n$ when $m < m'$ (see (c'$_3$)), the bid functions must of course satisfy the definition of regular strategies. For example, their values cannot exceed the valuations.

We thus notice that the only difference in the equilibrium characterizations with the case $g\left(c\right) = c$ resides in initial conditions at the lower end of the type interval. Additionally, we find that these initial conditions are always completely known from the outset, which is to say that they do not involve parameters such as the maximum bid η on high valuation units and the maximum bid η' on low valuation units that are first unknown and that have then to be determined[21].

In the rest of this appendix, we prove (c'$_1$) for all values of m and m' and we also prove (c'$_2$) and (a') for $m \geq m'$. We then briefly outline the proofs of (c'$_3$) and (a') when $m < m'$.

To prove (c'$_1$), we first establish (A5-1) below:

$$r \leq \beta_1\left(c\right), ..., \beta_{\min\left(\left[\frac{n+1}{2}\right], m\right)}\left(c\right) \leq c \text{(A5-1)}.$$

The inequalities $\beta_1\left(c\right), ..., \beta_m\left(c\right) \leq c$ are immediate consequences of the definition of regular strategies. Let j be the largest index such that $j \leq \min\left(\left[\frac{n+1}{2}\right], m\right)$ and $\beta_j\left(c\right) \geq r$. Assume thus that $j < \min\left(\left[\frac{n+1}{2}\right], m\right)$. We then have $\beta_j\left(c\right) \geq r > \beta_{j+1}\left(c\right)$. Since $n-j \geq j+1$, a type v bidder with v close to c would do better if, instead of submitting a jth bid equal to $\beta_{j+1}\left(v\right) < r$, he submitted, for example, $\left(\beta_j\left(c\right) + \beta_j\left(v\right)\right)/2$. We have thus proved (A5-1).

We then go on to establish (A5-2) below:

$$r \leq \underline{b} = \beta_1\left(c\right) = ... = \beta_{\min\left(\left[\frac{n+1}{2}\right], m\right)}\left(c\right) \leq c \text{(A5-2)}.$$

Suppose that there exists $2 \leq j \leq \min\left(\left[\frac{n+1}{2}\right], m\right)$, such that $\beta_j\left(c\right) < \beta_{j-1}\left(c\right)$. The bid $\beta_j\left(c\right)$ must be the best jth bid of a type c bidder, and in particular, the best bid in the interval $\left[\beta_j\left(c\right), \beta_{j-1}\left(c\right)\right]$. We thus have

$$\beta_j\left(c\right) \in \arg\max_{b \in \left[\beta_j(c), \beta_{j-1}(c)\right]} \left(c-b\right) F\left(\gamma_{n-j+1}\left(b\right)\right) \text{(A5-3)}.$$

Similarly, $\beta_{j-1}\left(c\right)$ must be the best $(j-1)$th bid of a type c bidder, and we therefore have

$$\beta_{j-1}\left(c\right) \in \arg\max_{b \in \left[\beta_j(c), \beta_{j-1}(c)\right]} \left(c-b\right) F\left(\gamma_{n-j+2}\left(b\right)\right) \text{(A5-4)}.$$

For all type $v > \gamma_{n-j+1}\left(\beta_{j-1}(c)\right)$, a type v bidder submits a $(n-j+2)$th bid strictly larger than $\beta_{j-1}(c)$. In fact, a $(n-j+2)$th bid not larger than $\beta_{j-1}(c)$ would be among the winning bids with probability zero while a bid close to and strictly larger than $\beta_{j-1}(c)$, which from the definition of regular strategy is not larger than $\gamma_{n-j+1}\left(\beta_{j-1}(c)\right)$ and is thus strictly smaller than v, would give a strictly positive payoff. Consequently, $\beta_{n-j+1}(v) \geq \beta_{n-j+2}(v) \geq \beta_{j-1}(c)$, for all $v > \gamma_{n-j+1}\left(\beta_{j-1}(c)\right)$. Making v tend towards $\gamma_{n-j+1}\left(\beta_{j-1}(c)\right)$, we find $\beta_{n-j+2}\left(\gamma_{n-j+1}\left(\beta_{j-1}(c)\right)\right) = \beta_{j-1}(c)$ and thus $\gamma_{n-j+1}\left(\beta_{j-1}(c)\right) = \gamma_{n-j+2}\left(\beta_{j-1}(c)\right)$. Consequently, we obtain

$$\left[(c-b)\,F\left(\gamma_{n-j+2}(b)\right)\right]_{b=\beta_{j-1}(c)} = \left[(c-b)\,F\left(\gamma_{n-j+1}(b)\right)\right]_{b=\beta_{j-1}(c)} \text{(A5-5)}.$$

Moreover, since $\gamma_{n-j+2} \geq \gamma_{n-j+1}$, we immediately see that

$$(c-b)\,F\left(\gamma_{n-j+2}(b)\right) \geq (c-b)\,F\left(\gamma_{n-j+1}(b)\right) \text{(A5-6)},$$

for all b. (A5-3) to (A5-6) then imply (A5-7) below:

$$\beta_{j-1}(c), \beta_j(c) \in \arg\max_{b\in[\beta_j(c),\beta_{j-1}(c)]} (c-b)\,F\left(\gamma_{n-j+1}(b)\right) \text{(A5-7)}.$$

However, this is impossible since $(c-b)\,F\left(\gamma_{n-j+1}(b)\right)$ is strictly decreasing over $[\beta_j(c), \beta_{j-1}(c)]$. In fact, in the interval $[b, \beta_{j-1}(c)]$, the bid b is the best jth bid of a type $\gamma_j(b)$ bidder, for all b in $[\beta_j(c), \beta_{j-1}(c)]$. Consequently, $\frac{d_r}{db}\ln F\left(\gamma_{n-j+1}(b)\right) \leq \frac{1}{\gamma_j(b)-b} < \frac{1}{c-b}$, for all b in $(\beta_j(c), \beta_{j-1}(c)]$, and the right-hand derivative of $\ln\left\{(c-b)\,F\left(\gamma_{n-j+1}(b)\right)\right\}$ is strictly negative over $(\beta_j(c), \beta_{j-1}(c)]$ and the function $(c-b)\,F\left(\gamma_{n-j+1}(b)\right)$ is thus strictly decreasing over this interval. We have proved (A5-2).

We next establish (A5-8) below:

$$r \leq \underline{b} = \beta_1(c) = ... = \beta_m(c) \leq c\text{(A5-8)}.$$

Assume (A5-8) does not hold true. Let i be the smallest index (not larger than m) such that $\beta_i(c) < \underline{b}$. From (A5-2), we have $i > \frac{n+1}{2}$ and thus $(n-i+1) < i$ and $\underline{b} = \beta_{n-i+1}(c) = \beta_{i-1}(c) > \beta_i(c)$. A type v bidder with v close to c would then do better if he submitted a ith bid equal to, for example, $\left(\beta_{i-1}(c) + \beta_{i-1}(v)\right)/2$ instead of $\beta_i(v) < \underline{b} = \beta_{n-i+1}(c)$, which wins with probability zero. We have proved (A5-8).

We now show that $\underline{b} \geq g(c)$. Suppose that $\underline{b} < g(c)$ and thus $r < g(c)$. Consider first the case $m < \frac{n}{2}$, that is, $m < m'$. In this case, just as we proved (A5-1), we can first prove that $r \leq \beta_{m+1}(c), ..., \beta_{\left[\frac{n+1}{2}\right]}(c)$. Then, just as we proved (A5-2), we can prove $\beta_{m+1}(c) = ... = \beta_{\left[\frac{n+1}{2}\right]}(c) = \underline{b}'$. Assume next that $\underline{b}' < \underline{b}$. In this case, a type c bidder would do better if he increased his $(m+1)$th bid from $\beta_{m+1}(c)$, which wins a probability zero, to any bid in $\left(\beta_{m+1}(c), \underline{b}\right)$. Consequently, $\underline{b} = \beta_1(c) = ... = \beta_{\left[\frac{n+1}{2}\right]}(c)$. Assume there exists i such that $\beta_i(c) < \underline{b}$. Let j be the largest such index. We must have $j > \frac{n+1}{2}$ and thus $(n-j+1) < j$. As a consequence of the definition of j, we then have $\beta_{n-j+1}(c) = \beta_{j-1}(c) = \underline{b} > \beta_j(c)$. A type v bidder, for v close to c, would therefore do better by submitting a jth bid equal to, for example, $\left(\beta_{j-1}(c) + \beta_{j-1}(v)\right)/2$ instead of $\beta_j(v) < \beta_{n-j+1}(c)$ which wins with probability zero. We have thus proved that if $\underline{b} < g(c)$ and $m < m'$ then $\beta_1(c) = ... = \beta_n(c) = \underline{b}$. But then a type c bidder would increase his expected payoff if he increased his first bid $\beta_1(c)$. The inequality $\underline{b} < g(c)$ is thus impossible in the case $m < m'$.

Consider now the case $m \geq m'$ or, equivalently, $m \geq \frac{n}{2}$, and suppose again that $\underline{b} < g(c)$. Assume that there exists i such that $\beta_i(c) < \underline{b}$. Let j be the largest such index. From (A3-8), we then have $j > m$ and thus $j \geq n-j+1$. If $n-j+1 = j$, a type c bidder would increase his payoff if he increased slightly his jth bid. If $n-j+1 < j$, from the definition of j we have $\beta_{n-j+1}(c) = \beta_{j-1}(c) = \underline{b} > \beta_j(c)$. Then a type v bidder, for v close to c, would increase his payoff if he changed his jth bid from $\beta_j(v) < \beta_{n-j+1}(c)$, which wins with probability zero, to, for example, $\left(\beta_{j-1}(c) + \beta_{j-1}(v)\right)/2$. We have thus proved (c'$_1$) above in all cases.

In the rest of the appendix, we consider the case $m \geq m'$. We address briefly the case $m < m'$ at the end. Assume thus that $m \geq n/2$, that is, $m \geq m'$. In this case, we prove (A5-9) below:

$$\gamma_m(\underline{b}) = ... = \gamma_n(\underline{b}) = g^{-1}(\underline{b}) \text{ (A5-9).}$$

From the definition of a regular strategy, we know that $\beta_m\left(g^{-1}(\underline{b})\right), ..., \beta_n\left(g^{-1}(\underline{b})\right) \leq g^{-1}(\underline{b})$. Suppose there exists $i \geq m+1$ such that $\beta_i\left(g^{-1}(\underline{b})\right) < \underline{b}$. Let j be the smallest such index. Since $j \geq m+1$, we have $n-j+1 \leq m$. If $j = m+1$, we have $\beta_j\left(g^{-1}(\underline{b})\right) < \underline{b} = \beta_m(c) = \beta_{n-j+1}(c) < \beta_m\left(g^{-1}(\underline{b})\right)$. Therefore, a type $g^{-1}(\underline{b})$ bidder would increase his payoff if he changed his jth bid from $\beta_j\left(g^{-1}(\underline{b})\right)$, which wins with a probability zero, to, for example, b such that $\beta_m(c) < b < \min\left(\beta_m\left(g^{-1}(\underline{b})\right), g^{-1}(\underline{b})\right)$. If $j > m+1$, the definition of j implies $\underline{b} = \beta_m(c) = \beta_{n-j+1}(c) = \beta_{j-1}\left(g^{-1}(b)\right) > \beta_j\left(g^{-1}(\underline{b})\right)$. A type v bidder for v close to and strictly larger than $g^{-1}(\underline{b})$ would increase his payoff if he changed his jth bid from $\beta_j(v) < \beta_{n-j+1}(c)$, which wins with probability zero, to, for example, $\left(\beta_{j-1}(c) + \beta_{j-1}\left(g^{-1}(\underline{b})\right)\right)/2$. (A5-9) is thus proved.

The proof of (a') proceeds as the proof of Theorem 1 (a) in Appendix 1.

The following lemma now completes the proof of (c'$_2$).

Lemma A5-1[22]**:** *Assume $m \geq m'$, $r < c$, and $g(c) < c$. Let (β, β') be a regular symmetric equilibrium Then, we have $\beta(c) = \beta'\left(g^{-1}(\underline{b})\right) = \underline{b}$ where*

$$\underline{b} = \max \arg \max_{b \in [\max(r, g(c)), c]} (c-b) H(b).$$

Proof: Denote by β the common bid function on the 1st, ..., mth units, and denote by β' the common bid function, over $\left[g^{-1}(\underline{b}), d\right]$, on the (m+1)th, ..., nth units. Consider that a bidder with type c submits the same bid b such that $\max(g(c), r) \leq b \leq c$ for his 1st,...,mth units. From the definition of a regular strategy $\beta_{m+1}(c) \leq g(c)$. Since $\beta(c) = \underline{b}$ must be the best such bid, we have

$$\underline{b} \in \arg \max_{b \in [\max(g(c),r),c]} \left\{ (c-b) \sum_{i=1}^{m'} F\left(\gamma_{n-i+1}(b)\right) + (m-m')(c-b) F(\gamma(b)) \right\} \text{(A5-10)},$$

where $\gamma = \beta^{-1}$ and $\gamma_j = \beta_j^{-1}$, for $j \geq m+1$. Bids are not larger than valuations and thus $(c-b)\sum_{i=1}^{m'} F\left(\gamma_{n-i+1}(b)\right) + (m-m')(c-b)F(\gamma(b))$ is not smaller than $m'(c-b)F\left(g^{-1}(b)\right) + (m-m')(c-b)F(b)$. We thus find

$$(c-b) \sum_{i=1}^{m'} F\left(\gamma_{n-i+1}(b)\right) + (m-m')(c-b) F(\gamma(b)) \geq m'(c-b) H(b),$$

for all b in $[\max(g(c), r), c]$, since $F(c) = 0$.

Obviously, $(c-\underline{b})\sum_{i=1}^{m} F\left(\gamma_{n-i+1}(\underline{b})\right) + (m-m')(c-\underline{b})F(\gamma(\underline{b})) = (c-\underline{b})\sum_{i=1}^{m} F\left(\gamma_{n-i+1}(\underline{b})\right) + (m-m')(c-\underline{b})F(c) = m'(c-\underline{b})F(\gamma'(\underline{b}))$, where $\gamma' = \beta'^{-1}$. Since $\gamma'(\underline{b}) = g^{-1}(\underline{b})$, we obtain

$$(c-\underline{b}) \sum_{i=1}^{m'} F\left(\gamma_{n-i+1}(\underline{b})\right) + (m-m')(c-\underline{b}) F(\gamma(\underline{b})) = m'(c-\underline{b}) H(\underline{b})$$

and (simplifying by m')

$$\underline{b} \in \arg \max_{b \in [\max(g(c),r),c]} (c-b) H(b).$$

Since any bid in the open interval $(\max(g(c), r), c)$ gives a strictly positive value of $(c-b)H(b)$, we know that any maximizer is strictly smaller than c. Suppose there exists $\widehat{b}$ in $\arg\max_{b \in [\max(g(c),r),c]} (c-b)H(b)$ such that $\widehat{b} > \underline{b}$. From (A5-10), we have

$$\begin{aligned} m'(c-\underline{b}) H(\underline{b}) &\geq \left(c-\widehat{b}\right) \sum_{i=1}^{m'} F\left(\gamma_{n-i+1}\left(\widehat{b}\right)\right) + (m-m')\left(c-\widehat{b}\right) F\left(\gamma\left(\widehat{b}\right)\right) \\ &\geq m'\left(c-\widehat{b}\right) H\left(\widehat{b}\right). \end{aligned}$$

Since $m'(c-\underline{b})H(\underline{b}) = m'\left(c-\widehat{b}\right)H\left(\widehat{b}\right)$ and $\gamma_{n-i+1}(b) = \gamma'(b)$, for all $b \geq \underline{b}$ and $i = 1, ..., m'$, we obtain

$$\begin{aligned} m'(c-\underline{b}) H(\underline{b}) &= m'\left(c-\widehat{b}\right) F\left(\gamma'\left(\widehat{b}\right)\right) + (m-m')\left(c-\widehat{b}\right) F\left(\gamma\left(\widehat{b}\right)\right) \\ &= m'\left(c-\widehat{b}\right) H\left(\widehat{b}\right) \text{(A5-11)}. \end{aligned}$$

If $m > m'$, (A5-11) implies $F\left(\gamma\left(\widehat{b}\right)\right) = 0$. From $\widehat{b} > \underline{b}$, we find $F\left(\gamma\left(\widehat{b}\right)\right) > F\left(\gamma\left(\underline{b}\right)\right) = F\left(c\right) = 0$. We have a contradiction and the lemma is proved for $m > m'$. If $m = m'$, (A5-11) and $\widehat{b} < c$ imply $F\left(\gamma'\left(\widehat{b}\right)\right) = H\left(\widehat{b}\right)$ and thus $\gamma'\left(\widehat{b}\right) = g^{-1}\left(\widehat{b}\right)$, which is impossible since $\gamma'\left(b\right) > g^{-1}\left(b\right)$, for all $b > \underline{b}$. ||

When $m < m'$, for the $(m+1)$th bid to the $(n-m)$th bid in $\left[\max\left(g\left(c\right), r\right), \underline{b}\right]$, the trade-offs are the same as in the case where all units are of low valuations. It can be seen that $\beta_{m+1}, ..., \beta_{n-m}$ are equal to the valuation when the type is equal to $g^{-1}\left(\max\left(g\left(c\right), r\right)\right)$, or when the low valuation is equal to $\max\left(g\left(c\right), r\right)$ and then coincide over the type interval $\left[g^{-1}\left(\max\left(g\left(c\right), r\right)\right), \gamma_{m+1}\left(\underline{b}\right)\right]$. Over the corresponding valuation interval $\left[g^{-1}\left(\max\left(g\left(c\right), r\right)\right), g\left(\gamma_{m+1}\left(\underline{b}\right)\right)\right]$, the functions $\beta_{m+1} \circ g^{-1}, ..., \beta_{n-m} \circ g^{-1}$ are thus equal to the equilibrium bid function in the first price auction with reserve price r and with two homogenous bidders whose valuations are distributed according to $H = F \circ g^{-1}$. Changing the variables in the known formula for this symmetric case, we find $\beta_{m+1}\left(v\right) = ... = \beta_{n-m}\left(v\right) = \lambda'\left(v\right) = \frac{\max(r,g(c))F\left(g^{-1}(\max(r,g(c)))\right)+\int_{g^{-1}(\max(r,g(c)))}^{v} g(u)dF(u)}{F(v)}$, for all v in $\left[g^{-1}\left(\max\left(r, g\left(c\right)\right)\right), \gamma_{m+1}\left(\underline{b}\right)\right]$.

The bid functions $\beta_{n-m+1}, ..., \beta_n$ are arbitrary over the interval $\left[g^{-1}\left(\max\left(r, g\left(c\right)\right)\right), \gamma_{m+1}\left(\underline{b}\right)\right]$, since over this interval they are always smaller than the bids they compete against. However, it is easily seen that for valuations strictly larger than $g\left(\gamma_{m+1}\left(\underline{b}\right)\right)$, the bids on the $(n-m+1)$th unit to the nth unit are strictly larger than $\underline{b}$ and are among the winning bids with a strictly positive probability. By continuity, we then have $\beta_{n-m+1}\left(\gamma_{m+1}\left(\underline{b}\right)\right) = ... = \beta_n\left(\gamma_{m+1}\left(\underline{b}\right)\right) = \lambda'\left(\gamma_{m+1}\left(\underline{b}\right)\right) = \underline{b}$ and thus all functions β_{m+1},...,β_n,λ' coincide at $\gamma_{m+1}\left(\underline{b}\right)$. Moreover, it is possible to show that the function λ' is extended beyond $\gamma_{m+1}\left(\underline{b}\right)$ according to the formula in (c'$_3$), coming from the case with all low valuation units, then λ' is not larger than the common bid function on low valuation units over $\left[\gamma_{m+1}\left(\underline{b}\right), d\right]$. Over this interval, bids on low valuation units also compete with bids with on high valuation units. This result is thus similar to the result of comparative statics (Section 6) we obtain in this paper when we change the ratio m/m' (notice that here this ratio is strictly smaller than 1).

The proof then proceeds similarly as in the proof of Lemma A5-1 above. From the previous paragraph, $\left(c-b\right)\sum_{i=1}^{m} F\left(\gamma_{n-i+1}\left(b\right)\right)$ is always at least as large, for bids below and above $\underline{b}$, as $m\left(c-b\right)F\left(\lambda'^{-1}\left(b\right)\right)$ and at $b = \underline{b}$ it is equal to $m\left(c-\underline{b}\right)F\left(\lambda'^{-1}\left(\underline{b}\right)\right)$. Consequently, we have $\underline{b} \in \arg\max_{b\in[\max(r,g(c)),c]}\left(c-b\right)F\left(\lambda'^{-1}\left(b\right)\right)$. However, since $\lambda' \circ g^{-1}$ is the equilibrium bidding function of the first price auction with two homogenous bidders and valuation distribution $H = F \circ g^{-1}$, this last maximization problem is the problem faced by a bidder who takes part in this first price auction and whose valuation is equal to c. The unique solution of this problem is thus $\lambda'\left(g^{-1}\left(c\right)\right)$ and we find $\underline{b} = \lambda'\left(g^{-1}\left(c\right)\right)$. Since $\beta_{m+1}, ..., \beta_{n-m}$ coincide with λ' over $\left[g^{-1}\left(\max\left(r, g\left(c\right)\right)\right), \gamma_{m+1}\left(\underline{b}\right)\right]$, we have $\gamma_{m+1}\left(\underline{b}\right) = \lambda'^{-1}\left(\underline{b}\right) = g^{-1}\left(c\right)$ and we therefore obtain (c'$_3$).

Appendix 6

In this appendix, we first obtains properties of the system of differential equations (A6-4, A6-5) below considered on the domain $\mathcal{D} = \left\{(b, \widehat{\gamma}_1, \widehat{\gamma}_2) \mid c < \widehat{\gamma}_1, g^{-1}(c) < \widehat{\gamma}_2,\ b < \widehat{\gamma}_1, g(\widehat{\gamma}_2)\right\}$

$$\frac{d}{db} \ln F(\widehat{\gamma}_1) = \frac{1}{g(\widehat{\gamma}_2(b)) - b} \text{(A6-1)}$$

$$\frac{d}{db} \ln F(\widehat{\gamma}_2) = \frac{1}{\widehat{\gamma}_1(b) - b} \text{(A6-2)}$$

Through the change of variables $(b, \widehat{\gamma}_1, \widehat{\sigma}_2) = (b, \widehat{\gamma}_1, g(\widehat{\gamma}_2))$ the system (A6-1,A6-2) considered in $\mathcal{D}''$ is equivalent to the system (A6-3,A6-4) considered in the domain $\mathcal{D}' = \{(b, \widehat{\gamma}_1, \widehat{\sigma}_2) \mid c, b < \widehat{\gamma}_1, \widehat{\sigma}_2\}$

$$\frac{d}{db} \ln F(\widehat{\gamma}_1) = \frac{1}{\widehat{\sigma}_2(b) - b} \text{(A6-3)}$$

$$\frac{d}{db} \ln H(\widehat{\sigma}_2) = \frac{1}{\widehat{\gamma}_1(b) - b} \text{(A6-4)}$$

where $H = F \circ g^{-1}$ is the cumulative distribution function of the probability distributions of the valuation of the $(m+1)$th,...,nth units. The system (A6-1,A6-2) is in the “type space” and the system (A6-3,A6-4) is in the “valuation space”.

Lemma A6-1: *Assume that $\frac{d}{dv}\frac{F}{H}(v) > 0$, for all v in $(\underline{c}, g(d)]$, or, equivalently that $\frac{d}{dv}\frac{F \circ g}{F}(v) > 0$, for all v in $\left(g^{-1}(\underline{c}), d\right]$. Let $(\widehat{\gamma}_1, \widehat{\sigma}_2)$ be a solution over $\left(\rho, \overline{b}\right]$ of (A6-3, A6-4) considered in the domain D'. Then, we have $\widehat{\gamma}_1(\rho) = \rho$ if and only if $\widehat{\sigma}_2(\rho)$.*

Proof: Suppose, for example, that $\rho > c$, $\widehat{\gamma}_1(\rho) = \rho$, and $\widehat{\sigma}_2(\rho) > \rho$. Consider then the functions $\widehat{\varphi} = \widehat{\gamma}_1 \circ \widehat{\sigma}\ _2^{-1}$ and $\widehat{\beta} = \widehat{\sigma}_2^{-1}$. From (A6-3,A6-4), these functions form a solution over $\left(\widehat{\sigma}_2(\rho), \widehat{\sigma}_2\left(\overline{b}\right)\right]$ of the system of differential equations (A6-5,A6-6) below considered in the domain $\mathcal{D}^* = \left\{\left(v, \widehat{\varphi}, \widehat{\beta}\right) \mid c, \widehat{\beta} < v \leq d, c < \widehat{\varphi} \leq d\right\}$:

$$\frac{d}{dv}\widehat{\varphi}(v) = \frac{F(\widehat{\varphi}(v))}{f(\widehat{\varphi}(v))} \frac{h(v)}{H(v)} \frac{\widehat{\varphi}(v) - \widehat{\beta}(v)}{v - \widehat{\beta}(v)} \text{(A6-5)}$$

$$\frac{d}{dv}\widehat{\beta}(v) = \frac{h(v)}{H(v)} \left(\widehat{\varphi}(v) - \widehat{\beta}(v)\right) \text{(A6-6)}$$

. Since $\widehat{\beta}$ is the inverse of a strictly increasing function, we have

$$\widehat{\beta} \text{ is striclty increasing over } \left(\widehat{\sigma}_2(\rho), \widehat{\sigma}_2\left(\overline{b}\right)\right] \text{ (A6-7)}$$

Consider the identical constant functions $\widehat{\varphi}', \widehat{\beta}'$ defined as follows:

$$\widehat{\varphi}'(v) = \widehat{\beta}'(v) = \rho$$

, for all v in $(\rho, d]$. As it is easily seen $\left(\widehat{\varphi}', \widehat{\beta}'\right)$ is a solution of (A6-5, A6-6) over the interval $(\rho, d]$. From our assumptions $\left[\left(v, \widehat{\varphi}(v), \widehat{\beta}(v)\right)\right]_{v=\widehat{\sigma}_2(\rho)} = (\widehat{\sigma}_2(\rho), \rho, \rho)$ belongs to the interior of $(\rho, d]$, and thus of $\mathcal{D}^*$. Moreover, the solutions $\left(\widehat{\varphi}, \widehat{\beta}\right)$ and $\left(\widehat{\varphi}', \widehat{\beta}'\right)$ coincide at $v = \widehat{\sigma}_2(\rho)$. Form the standard theorems of the theory of ordinary differential equations, they must coincide everywhere over their common definition interval and we obtain a contradiction with (A6-7). We have thus proved that when $\rho > c$, $\widehat{\gamma}_1(\rho) = \rho$ implies $\widehat{\sigma}_2(\rho) = \rho$. The proof that $\widehat{\sigma}_2(\rho) = \rho$ implies $\widehat{\gamma}_1(\rho) = \rho$ when $\rho > c$ is similar.

Now assume that $\rho = c$ and, for example, that $\widehat{\gamma}_1(\rho) = \rho$. Since the limit of $\ln F(\widehat{\gamma}_1(b))$, as be tends towards ρ, is equal to $-\infty$, its derivative cannot be bounded as b approaches ρ and thus, from (A6-3), $\widehat{\sigma}_2(\rho) = \rho$. Notice that $\widehat{\sigma}_2(\rho)$, the continuous extension of $\widehat{\sigma}_2$ at ρ exists since it is strictly increasing $((b, \widehat{\gamma}_1(b), \widehat{\sigma}_2(b))$ belongs to $\mathcal{D}'$ for all v in $\left(\rho, \overline{b}\right]$). The proof that $\widehat{\sigma}_2(\rho) = \rho$ implies $\widehat{\gamma}_1(\rho) = \rho$ when $\rho = c$ is similar. $||$

We also consider in the domain $\mathcal{D}'' = \{(b, \widehat{\gamma}) \mid \max(c, b) < \widehat{\gamma}\}$ the differential equation (A6-5) below

$$\frac{d}{db} \ln F(\widehat{\gamma}) = \frac{1}{\widehat{\gamma}(b) - b} \text{(A6-5)}$$

Lemma A6-2: *Assume that* $\frac{d}{dv}\frac{F}{H}(v) > 0$*, for all* v *in* $(\underline{c}, g(d)]$*, or, equivalently that* $\frac{d}{dv}\frac{F \circ g}{F}(v) > 0$*, for all* v *in* $\left(g^{-1}(\underline{c}), d\right]$*. Let* $\widehat{\gamma}$ *be a solution over* $\left(\rho, \overline{b}\right]$ *of (A6-5) and let* $(\widehat{\gamma}_1, \widehat{\sigma}_2)$ *be a solution over* $\left(\rho, \overline{b}\right]$ *of (A6-3,A6-4). If* $\widehat{\gamma}\left(\overline{b}\right) \geq \widehat{\gamma}_1\left(\overline{b}\right)$ *and* $\widehat{\gamma}\left(\overline{b}\right) \geq \widehat{\sigma}_2\left(\overline{b}\right)$*, then* $\widehat{\gamma}(b) > \widehat{\gamma}_1(b)$ *and* $\widehat{\gamma}(b) > \widehat{\sigma}_2(b)$*, for all b in* $\left(\rho, \overline{b}\right)$*.*

Proof: We first show that there exists $\varepsilon > 0$ such that $\widehat{\gamma}(b) > \widehat{\gamma}_1(b)$ and $\widehat{\gamma}(b) > \widehat{\sigma}_2(b)$, for all b in $\left(\overline{b} - \varepsilon, \overline{b}\right)$. If $\widehat{\gamma}\left(\overline{b}\right) > \widehat{\sigma}_2\left(\overline{b}\right)$, there exists $\delta > 0$ such that $\widehat{\gamma}(b) > \widehat{\sigma}_2(b)$, for all b in $\left(\overline{b} - \varepsilon, \overline{b}\right)$. Then, (A6-5) and (A6-3) imply that $\frac{d}{db} \ln F(\widehat{\gamma}_1(b)) < \frac{d}{db} \ln F\left(\widehat{\gamma}_1'(b)\right)$, for all b in $\left(\overline{b} - \varepsilon, \overline{b}\right)$, $F(\widehat{\gamma}) / F(\widehat{\gamma}_1)$ is strictly decreasing over this interval, and thus $\widehat{\gamma} > \widehat{\gamma}_1$ over this interval.

If $\widehat{\gamma}\left(\overline{b}\right) = \widehat{\sigma}_2\left(\overline{b}\right)$, we have from (A6-5) and (A6-4)

$$\frac{d}{db}\widehat{\sigma}_2\left(\overline{b}\right) = \frac{H\left(\widehat{\sigma}_2\left(\overline{b}\right)\right)}{h\left(\widehat{\sigma}_2\left(\overline{b}\right)\right)} \frac{1}{\widehat{\gamma}_1\left(\overline{b}\right) - \overline{b}} > \frac{F\left(\widehat{\gamma}\left(\overline{b}\right)\right)}{f\left(\widehat{\gamma}\left(\overline{b}\right)\right)} \frac{1}{\widehat{\gamma}\left(\overline{b}\right) - \overline{b}} = \frac{d}{db}\widehat{\gamma}\left(\overline{b}\right) \text{(A6-6)}$$

since $H/h > F/f$. Consequently there exists $\varepsilon > 0$ such that $\widehat{\gamma}(b) > \widehat{\sigma}_2(b)$, for all b in $\left(\overline{b} - \varepsilon, \overline{b}\right)$. Reasoning as in the previous paragraph, we also have $\widehat{\gamma} > \widehat{\gamma}_1$ over the same interval and we have proved that there always exists $\varepsilon > 0$ such that $\widehat{\gamma}(b) > \widehat{\gamma}_1(b)$ and $\widehat{\gamma}(b) > \widehat{\sigma}_2(b)$, for all b in $\left(\overline{b} - \varepsilon, \overline{b}\right)$.

Let b^* be defined as follows:

$$b^* = \inf\left\{b' \geq c \mid \widehat{\gamma}(b) > \widehat{\gamma}_1(b) \text{ and } \widehat{\gamma}(b) > \widehat{\sigma}_2(b)\text{, for all b in } \left(b', \overline{b}\right)\right\}$$

We want to prove that $b^* = c$. Suppose that $b^* > c$. From what we have already proved, we know that $b^* \leq \overline{b} - \varepsilon$. By continuity, $\widehat{\gamma}(b^*) = \widehat{\gamma}_1(b^*)$ or $\widehat{\gamma}(b^*) = \widehat{\sigma}_2(b^*)$ (or both). Assume that $\widehat{\gamma}(b^*) = \widehat{\sigma}_2(b^*)$. Then an inequality as in (A6-6) at b^* and $\widehat{\gamma}(b^*) \geq \widehat{\gamma}_1(b^*)$ imply $\frac{d}{db}\widehat{\sigma}_2(b^*) > \frac{d}{db}\widehat{\gamma}(b^*)$. There thus exists $\delta > 0$ such that $\widehat{\sigma}_2(b) > \widehat{\gamma}(b)$, for all b in $(b^*, b^* + \delta)$. However, this contradicts the definition of b^* and $\widehat{\gamma}(b^*) = \widehat{\sigma}_2(b^*)$ is thus impossible. Assume that $\widehat{\gamma}(b^*) = \widehat{\gamma}_1(b^*)$ and $\widehat{\gamma}(b^*) > \widehat{\sigma}_2(b^*)$. From (A6-5) and (A6-3) we then have:

$$\frac{d}{db}\ln F\left(\widehat{\gamma}_1(b^*)\right) = \frac{1}{\widehat{\sigma}_2(b^*) - b^*} > \frac{d}{db}\ln F\left(\widehat{\gamma}(b^*)\right) = \frac{1}{\widehat{\gamma}(b^*) - b^*}$$

Again this inequality implies the existence of $\delta > 0$ such that $\widehat{\gamma}_1(b) > \widehat{\gamma}(b)$, for all b in $(b^*, b^* + \delta)$. It contradicts the definition of b^*. Consequently $b^* = c$ and the lemma is proved. ||

Lemma A6-3: *Assume that $\frac{d}{dv}\frac{F}{H}(v) > 0$, for all v in $(\underline{c}, g(d)]$, or, equivalently that $\frac{d}{dv}\frac{F \circ g}{F}(v) > 0$, for all v in $\left(g^{-1}(\underline{c}), d\right]$. Let $\overline{b}$ be such that $\underline{c} < \overline{b} < d$. Let $\widehat{\gamma}$ be a solution over $\left(\rho, \overline{b}\right]$ of the differential system and let $(\widehat{\gamma}_1, \widehat{\sigma}_2)$ be a solution over $\left(\rho, \overline{b}\right]$ of the differential system such that $\widehat{\gamma}(\rho) = \widehat{\gamma}_1(\rho)$. Then the inequalities $\widehat{\gamma}\left(\overline{b}\right) \geq \widehat{\gamma}_1\left(\overline{b}\right)$ and $\widehat{\gamma}\left(\overline{b}\right) \geq \widehat{\sigma}_2\left(\overline{b}\right)$ cannot simultaneously hold.*

Proof: Assume first that $\widehat{\gamma}_1(\rho) > \rho$ or, equivalently, from Lemma A6-1, that $\widehat{\sigma}_2(\rho) > \rho$. Then $(\rho, \widehat{\gamma}(\rho))$ and $(\rho, \widehat{\gamma}_1(\rho), \widehat{\sigma}_2(\rho))$ belong to the (interiors of the) domains where their respective differential equation and system satisfy the standard assumptions of the theory of ordinary differential equations. The solutions can thus be continued to the left of these points and $\widehat{\gamma}$ and $(\widehat{\gamma}_1, \widehat{\sigma}_2)$ are solutions of (A6-5) and (A6-3, A6-4) respectively over an interval $\left(\rho', \overline{b}\right]$ with $\rho' < \rho$. From the previous lemma, we then have $\widehat{\gamma}(\rho) > \widehat{\gamma}_1(\rho)$ which contradicts the assumption $\widehat{\gamma}(\rho) = \widehat{\gamma}_1(\rho)$.

Suppose now that $\widehat{\gamma}_1(\rho) = \rho$ or, equivalently, from Lemma A6-1, that $\widehat{\sigma}_2(\rho) = \rho$. From our assumption $\widehat{\gamma}(\rho) = \widehat{\gamma}_1(\rho)$, we have $\widehat{\gamma}(\rho) = \rho$. However, it is well known that the solution of (A6-5) with this last initial condition is the inverse of the bid function equilibrium bid function $\widehat{\beta}$ of the first price auction with reserve price ρ and two homogenous bidders with valuation distribution F (see, for example, Riley and Samuelson 1981). More generally, the solution of the differential equation (A6-5) with initial condition (A6-7) below, where ρ' belongs to $[\underline{c}, d]$,

$$\widehat{\gamma}(\rho') = \rho' \text{(A6-7)}$$

is the inverse of the function $\widehat{\beta}$ such that $\widehat{\beta}(v) = \frac{\rho' F\left(\rho'\right) + \int_{\rho'}^{v} w dF(w)}{F(v)}$, for all v in $\left[\rho', \widehat{\gamma}\left(\overline{b}\right)\right]$. It is thus a continuous and strictly decreasing function of ρ'

and we denote it by $\widehat{\gamma}(.;\rho')$. Take b' in $\left(\rho,\overline{b}\right)$. From Lemma A6-2 we have $\widehat{\gamma}_1(b') < \widehat{\gamma}(b') = \widehat{\gamma}(b';\rho)$ and $\widehat{\sigma}_2(b') < \widehat{\gamma}(b') = \widehat{\gamma}(b';\rho)$. There thus exists $\rho' > \rho$ such that $\widehat{\gamma}(;\rho')$ is defined at b' and is such that $\widehat{\gamma}_1(b') < \widehat{\gamma}(b';\rho') < \widehat{\gamma}(b')$ and $\widehat{\sigma}_2(b') < \widehat{\gamma}(b';\rho') < \widehat{\gamma}(b')$. From Lemma A6-2 again we then obtain $\widehat{\gamma}_1(b) < \widehat{\gamma}(b;\rho')$ and $\widehat{\sigma}_2(b') < \widehat{\gamma}(b;\rho')$ for all b in $(\rho', b']$ which is clearly impossible since, for example, $\widehat{\gamma}_1(\rho') > \rho'$ while $\widehat{\gamma}(\rho';\rho') = \rho'$. The proof is complete. ||

Footnotes

[1]: In the case with a non binding reserve price and an equal number of low and high valuation units (see Section 4), for example, we will obtain the same marginal distributions of valuations, the same equilibrium, and thus the same marginal bid distributions if, in the space of couples (v_h, v_l) of valuations for the first m units and for the last $m\prime = m$ units, the support of the distribution of valuation couples lies under the graph of φ with $\varphi = g \circ \beta'^{-1} \circ \beta$, where β is the bid function on the high valuation units and β' is the bid function on the low valuation units (see below). In fact, φ is the boundary between the separating and pooling regions since $\beta'\left(g^{-1}(v_l)\right) = \beta(v_h)$ if and only if $v_l = \varphi(v_h)$. Remark that as we go closer to d, for example, the "thickness" of this support must tend towards zero since $(d, g(d))$ belongs to the support and $\varphi(d) = g(d)$.

Let $H(v_l \mid v_h)$ be the distribution function of the valuation v_l conditional on the valuation v_h. The distribution function of the first unit valuation must be equal to F and the distribution function of the second unit valuation must be equal to $H = F \circ g^{-1}$. We must thus have:

$$F\left(\varphi^{-1}(v_l)\right) + \int_{\varphi^{-1}(v_l)}^{d} H(v_l \mid v_h)\, dF(v_h) = F\left(g^{-1}(v_l)\right)$$

for all v_l in $[c, g(d)]$. Since $H(. \mid v_h)$ is a cumulative distribution function whose support is included in $[c, \varphi(v_h)]$, we must also have $H\left(v_l \mid \varphi^{-1}(v_l)\right) = 1$, for all v_l in $[c, g(d)]$. The function $H(v_l \mid v_h)$ is thus a nondecreasing (in v_l) solution of the following integral equations with boundary conditions:

$$\int_{\varphi^{-1}(v_l)}^{d} H(v_l \mid v_h)\, dF(v_h) = F\left(g^{-1}(v_l)\right) - F\left(\varphi^{-1}(v_l)\right)$$

$$H\left(v_l \mid \varphi^{-1}(v_l)\right) = 1$$

for all v_l in $[c, g(d)]$. Any solution $H(. \mid .)$ of the equations, the distribution F, and the function φ above will determine a probability distribution over couples of valuations such that with probability one no pooling of bids occurs at the equilibrium.

[2]: Here, contrary to Lebrun (1997 and 1999a), the upper extremities of the valuation intervals may be different. However, the proofs apply with only minor changes.

[3]: The equilibria we obtain when the reserve price r is binding are also the equilibria when there is no reserve price, when the probability spread by F over $[c, r]$ is concentrated at r, and when the function g is constant and equal to r over $\left[c, g^{-1}(r)\right]$ and is thus only nondecreasing.

[4]: Otherwise, bidders would only submit serious bids for m units and we would return to a case with flat demand curves.

[5]: As we will show, all equilibrium bid functions are differentiable at the upper extremities $\beta_i(d)$ of the bid ranges. In our paper, we do not assume this property from the start; rather, we obtain it as a result.

[6]: γ and γ' are the extended inverse bid functions (see before Lemma 1).

[7]: The formula for β_b was obtained under the assumption that the reserve price for the package was equal to $nr = (m + m')\,r$.

[8]: Applying the Milgrom (2000) and Jehiel and Moldovanu's (1999) result to a partial bundling would result that the seller expected revenue being smaller at the Vickrey auction with two bidders with types $\left(v_1^1, ..., v_n^1\right)$ and $\left(v_1^2, ..., v_n^2\right)$ than at the Vickrey auction with two bidders with types $\left(w_1^1, ..., w_n^1\right)$ and $\left(w_1^2, ..., w_n^2\right)$ where $\left(w_1^1, ..., w_n^1\right) = \left(\frac{1}{l}\sum_{h=1}^{l} v_h^1, ..., \frac{1}{l}\sum_{h=1}^{l} v_h^1, v_{l+1}^1, ..., v_n^1\right)$ and $\left(w_1^2, ..., w_n^2\right) = \left(v_1^2, ..., v_{n-l}^2, \frac{1}{l}\sum_{h=n-l+1}^{n} v_h^2, ..., \frac{1}{l}\sum_{h=n-l+1}^{n} v_h^2\right)$.

[9]: We consider the equation (4-22) in the domain $\mathcal{D} = \{(b, \widehat{\gamma}) \mid d \geq \widehat{\gamma} \geq c, \widehat{\gamma} > b\}$ and the system (4-24) in the domain $\mathcal{D}' = \{(b, \widehat{\gamma}_1, \widehat{\gamma}_2) \mid d \geq \widehat{\gamma}_1, \widehat{\gamma}_2 \geq c, \widehat{\gamma}_1 > b, g(\widehat{\gamma}_2) > b\}$.

[10]: In the case $g(d) < d$, since δ' is continuous and $\delta'(g(d)) = g(d) < d$, there exists such a solution such that $d' = \delta'(\eta') < d$. Consider the left-hand extremity ρ of the maximal definition interval of this solution. As it is easily seen, this solution defines a regular symmetric equilibrium of the discriminatory auction when the reserve price r is set at ρ. There thus exist regular symmetric equilibria of discriminatory auctions such that $\beta'(d) = \eta' = \beta(d') < \beta(d)$.

[11]: This result of the first price auction is quite general and holds true without the assumption $\frac{d}{dv}\frac{F \circ g}{F}(v) > 0$, for all v in $(c, d]$. Consequently, even without this assumption, any equilibrium of the first price auction gives an equilibrium of the discriminatory auction.

[12]: Engelbrecht-Wiggans and Kahn (1998a) work out an example of discriminatory auction with two units, two bidders, and two-dimensional demands. Trying to satisfy their equilibrium equations in a first price auction, they find (see Figure 6 p. 38 in their paper) that the bid for a second unit is larger than the bid for a first unit of *same valuation*. Notice that, contrary to what is alleged in Section 5.2, this property by itself does not imply that the same equations cannot be satisfied in a first price auction since in our model, a bidder's valuations for the units he buys are not identical. Here, the property of more aggressive bidding on the last m' units is consistent with smaller bids on these units thanks to smaller valuations for these units.

[13]: This assumption of stochastic dominance between F_i and F_j requires that over every interval obtained from the initial valuation interval by truncating it from above, the conditional of F_i over this interval first order stochastically dominates the conditional of F_j .

[14]: The examples here are discrete and thus do not fit exactly in our model.

However, they can be approximated by absolutely continuous functions that satisfy our assumptions. That the ranking between the revenues will still hold follows from the continuity (which itself follows from the upper hemi-continuity of the Nash correspondence and the uniqueness of the equilibrium) for the weak topology of the Nash equilibrium correspondence with respect to the valuation distributions (see Lebrun 1999b) and from the continuity of the expected revenues with respect to the equilibrium.

[15]: Actually, it is also the case if F and H are distributions of maxima of (different numbers of) independent uniform variables. For an extension, see Lebrun (1996). Other numerical estimates can be found, for example, in Li and Riley (1997).

[16]: If $F(v) = v^{\alpha}$, for all v in $[0,1]$, the seller's revenue at the first price auction with couple of distributions (F,F) is $\frac{2\alpha^2}{(\alpha+1)(2\alpha+1)}$.

[17]: The inverses γ, γ', and $\sigma' = g \circ \gamma'$ are extended to the interval $(c, +\infty)$ as in Section 2.

[18]: So far, we have not used this assumption.

[19]: The result follows immediately from Lemmas A2-16 and A2-17 when $r > c$, that is, when $\underline{c} > c$. The proof here is thus more relevant to the case $r \leq c$, that is, $\underline{c} = c$. However, it is general enough t apply to both cases.

[20]: The result then follows immediately from Lemma A2-8 when $r > c$, that is, when $\underline{c} > c$. In fact, form Lemma A2-8, we have $\widetilde{\sigma}'(b) > \sigma(b)$, for all b in $\left(\underline{c}, \widetilde{\eta}'\right)$, and thus, from equation (A2-17), $\frac{d}{db} \ln F(\widetilde{\gamma}(b)) < \frac{d}{db} \ln F(\gamma(b))$ over the same interval and $F(\widetilde{\gamma}(b)) / F(\gamma(b))$ is strictly decreasing over this interval. Consequently, $F(\widetilde{\gamma}(\underline{c})) / F(\gamma(\underline{c})) > F\left(\widetilde{\gamma}\left(\widetilde{\eta}'\right)\right) / F\left(\gamma\left(\widetilde{\eta}'\right)\right) \geq 1$, which is impossible since from the initial condition (A2-6) we have $F(\widetilde{\gamma}(\underline{c})) / F(\gamma(\underline{c})) = F(\underline{c}) / F(\underline{c}) = 1$. The proof here is thus more relevant to the case $r \leq c$, that is, $\underline{c} \leq c$. However, it is general enough to apply to both cases.

[21]: This is the reason why we wrote in Section 2 that assuming $g(c) = c$ did not imply the loss of original equilibrium structures.

[22]: When $m \leq m'$, a general expression for $\underline{b}$ is $\max \arg\max_{b \in [\max(r, g(c)), \max(r,c)]} (c-b) H(b)$.

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