Online Supplements to "Revenue Superiors Variants of the Second-Price Auction"

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Proof of the necessity of the characterization in Theorem 1 (i):

1. A bidder's weakly undominated strategy must not recommend bidding strictly above his value and (i.1) follows.

2. From 1., any bidder with value $v > c$ has a strictly positive expected payoff and hence c is the minimum of the support of either bid's distribution. Myerson (1981) implies the continuity and monotonicity with respect to his value of any bidder's interim expected payoff.

3. If $b > c$ is in a bidder's bid support, it must be a point of increase to the left of both bidders' bid cumulative functions. Otherwise, there would exist a gap $(b - \varepsilon, b]$ where no bidder bids and any bidder who is supposed to bid close to b would increase his payoff strictly if he bid $b - \varepsilon$ instead. As a consequence, the supports of the bid distributions are equal to the same interval $[c, \eta(k)]$.

4. There does not exist a bid $b \geq c$ that is a mass point of both bidders' bid distributions. Because the value distributions are atomless, if there existed such a bid, a bidder would submit b with a strictly positive probability for some values strictly smaller than b. This bidder would increase strictly his payoff if he bid slightly above b instead.

5. The bid distributions are atomless strictly above c. In fact, from 4. above, there could only exist an atom $b > c$ of the bid distribution of a single bidder, say bidder j. From 3., bidder $i \neq j$ bids at or below and arbitrarily close to b. For a deviation slightly above b by bidder i not to be strictly profitable, his value must approach b when his bid approaches b. From the

continuity in 2., his payoff when his value is b must therefore be:

$$(1-k)\int_c^b (b-w)\, dG_j(w;k),$$

where $G_j(.;k)$ is bidder j's bid cumulative distribution function. While if bidder i submits bids close to and above bid $b-\varepsilon$, with $\varepsilon > 0$, his payoff would tend towards:

$$\begin{aligned}
& k\varepsilon G_j(b-\varepsilon;k) + (1-k)\int_c^{b-\varepsilon} (b-w)\, dG_j(w;k) \\
= \; & (1-k)\int_c^b (b-w)\, dG_j(w;k) \\
& + \left\{ k\varepsilon G_j(b-\varepsilon;k) - (1-k)\int_{(b-\varepsilon,b)} (b-w)\, dG_j(w;k) \right\} \\
\geq \; & (1-k)\int_c^b (b-w)\, dG_j(w;k) \\
& + \varepsilon \left\{ kG_j(b-\varepsilon;k) - (1-k)\left(G_j\left(b^-;k\right) - G_j(b-\varepsilon;k)\right) \right\} \\
> \; & (1-k)\int_c^b (b-w)\, dG_j(w;k),
\end{aligned}$$

where the last inequality holds for all $\varepsilon > 0$ sufficiently small. Bidder i would then have a strictly profitable deviation, which is impossible at an equilibrium.

This also proves that the bid cumulative functions $G_1(b;k), G_2(b;k)$ and hence the bidder's expected payoffs are continuous in $b > c$.

6. Because a bidder's expected payoff when he bids strictly above c has strictly increasing differences in his bid and value, equilibrium bidding strategies must be nondecreasing and consequently, from 5., strictly increasing (when taking values strictly above c) and, from 3., continuous. Thus, for all i and $b > c$, $G_i(b;k) = F_i(\alpha_i(b;k))$.

7. Any bidder's probability of winning is differentiable with respect to his own bid strictly above c. The main idea of the proof is to express that

an optimal b is better for bidder i with value v than a bid b' as the inequality below;

$$\frac{G_j(b;k) - G_j(b';k)}{b - b'} \geq \frac{G_j(b';k)}{v - b} - \frac{1-k}{v-b} \frac{\int_{b'}^{b} G_j(w;k)\, dw}{b - b'},$$

if $b' \leq b$ and the reverse inequality if $b' \geq b$. Making $b - b'$ tend towards zero in such inequalities and appealing to the continuity of G_j above c (from 5.) gives the result.

8. From 7., the inverse bidding functions are differentiable strictly above c and satisfy the system of differential equations (1). Moreover, from 3., we must have $\alpha_1(\eta(k);k) = \alpha_2(\eta(k);k) = d$ and $\alpha_i(c;k) = c$, for at least one i. That actually $\alpha_i(c;k) = c$ for both $i = 1, 2$ is a property of the differential system and follows from Corollary 6 in Lebrun (1999).

Proof of Lemma A2:

Proof of (i): For all v in $(c, d]$ and all $l > 0$, we obviously have:

$$\begin{aligned} & l \int_c^v \left(\frac{G(w)}{G(v)}\right)^l dw \\ = & \int_c^v \frac{G(w)}{g(w)} d\left(\frac{G(w)}{G(v)}\right)^l, \end{aligned}$$

and consequently $\frac{G(w)}{g(w)}$ is integrable for $\left(\frac{G(w)}{G(v)}\right)^l$ over $[c, v]$. The equality in (i) then follows from the weak convergence of $\left(\frac{G(w)}{G(v)}\right)^l$ towards the degenerate distribution δ_v concentrated at v when l tends towards $+\infty$.

Proof of (ii): The first statement is an immediate consequence of (i). Let ε be an arbitrary strictly positive number. From the convergence at d, there exists l' such that $\left| l \int_c^d G(w)^l\, dw - \frac{G(d)}{g(d)} \right| < \varepsilon$, for all $l > l'$. Let M be the maximum over the interval $[c, d]$ of $\left| \frac{d}{dv} \left(\frac{G(v)}{g(v)}\right)^2 \right| /2$. We have $M < +\infty$. Indeed, as $G(v)/g(v) = 1/\frac{d}{dv} \ln G(v)$ is nondecreasing in an interval $[c, c + \eta]$

and tends towards 0 when v tends towards c, we have $\frac{d}{dv}\left(\frac{G(v)}{g(v)}\right)^2 \geq 0$ in this interval. Furthermore, we have:

$$\begin{aligned}
& \left|\frac{d}{dv}\left(\frac{G(v)}{g(v)}\right)^2\right| \\
= \; & 2\left(\frac{G(v)}{g(v)}\right)^2 \left|1-\frac{d\ln g(v)}{d\ln G(v)}\right| \\
\leq \; & 2(1+B)\left(\frac{G(v)}{g(v)}\right)^2,
\end{aligned}$$

with B is a bound of $\left|\varepsilon_g(v) = \frac{d\ln g(v)}{d\ln G(v)}\right|$[1].

Consider then any l such that $l > \max(l', M/\varepsilon)$. Then,

$$\max_{v\in[c,d]}\left|l\int_c^v \left(\frac{G(w)}{G(v)}\right)^l dw - \frac{G(v)}{g(v)}\right| < \varepsilon.$$

From the definition of l', the inequality holds true if the maximum is reached at d. The inequality is obviously satisfied if the maximum on the LHS is zero. We may thus that it is strictly positive and, hence, that it is not reached at $v = c$.

Assume then that the maximum is different from zero and reached at v^* in the interior of the interval. In this case, the FOC is:

$$l - l\frac{\int_c^{v^*} G(w)^l dw}{G(v^*)^l}\frac{g(v^*)\,l}{G(v^*)} - \frac{d}{dv}\frac{G(v^*)}{g(v^*)} = 0,$$

or, equivalently:

$$l\frac{\int_c^{v^*} G(w)^l dw}{G(v^*)^l} = \frac{G(v^*)}{g(v^*)}\left(1-\frac{1}{l}\frac{d}{dv}\frac{G(v^*)}{g(v^*)}\right).$$

[1] Log-concavity at c and the existence of a lower bound on ε_g imply that ε_g is bounded.

Consequently:

$$
\begin{aligned}
& \max_{v\in[c,d]} \left| l \int_c^v \left(\frac{G(w)}{G(v)} \right)^l dw - \frac{G(v)}{g(v)} \right| \\
= & \left| l \int_c^{v^*} \left(\frac{G(w)}{G(v)} \right)^l dw - \frac{G(v^*)}{g(v^*)} \right| \\
= & \left| \frac{1}{l} \frac{G(v^*)}{g(v^*)} \frac{d}{dv} \frac{G(v^*)}{g(v^*)} \right| \\
\leq & \frac{M}{l} \\
< & \varepsilon.
\end{aligned}
$$

||

Proof of Lemma A3:

(i) follows from $\rho_j(b) \to +\infty$ if $b \to c$.

Extend ρ_1, ρ_2 (for example, linearly) as continuously differentiable and strictly positive functions over $(c, d + 2\mu)$, with $\mu > 0$. Finally, use the same formula in the definition to extend $\gamma_i(.;.)$ to $(c, d+2\mu) \times (-2\zeta', 2\zeta')$, where $\zeta' > 0$. As the partial derivatives will be continuous over $(c, d+2\mu) \times (-2\zeta', 2\zeta')$, the extension will be continuously differentiable over this product.

From the definition, $\gamma_i'(b;k)$ is equal to $1 - k\rho_j(b)^{-2} \frac{d}{db}\rho_j(b)$. From our assumptions (in particular of local log-concavity at c), $\frac{d}{db}\rho_j(b)$ is bounded from above and consequently there exists $0 < \zeta < \zeta'$ such that $\gamma_i'(b;k)$ is strictly above $\frac{d-c}{d+\mu-c}$, which belongs to $(0,1)$, over $(c, d+\mu) \times (-\zeta, \zeta)$.

From the definition also, $\frac{\partial}{\partial k}\gamma_i(b;k)$ is equal to $\rho_j(b)^{-1}$, which is bounded from above.

As $\gamma_i(c;k) = c$ and $\gamma_i'(b;k)$ is strictly above $\frac{d-c}{d+\mu-c}$ over $(c, d+\mu) \times (-\zeta, \zeta)$, $\gamma_i(.;k)$ is a strictly increasing function over $(c, d+\mu)$ such that $\gamma_i(d+\mu;k) > d$, for all k in $(-\zeta, \zeta)$. (ii) is proved.

We then have $\gamma_i\left(\gamma_i^{-1}(v;k); k\right) = v$, for all (v,k) in $(c,d] \times (-\zeta, \zeta)$. For all such (v,k), the implicit function theorem implies that $\frac{\partial}{\partial k}\gamma_i^{-1}(v;k)$ exists and is equal to $-\frac{\partial}{\partial k}\gamma_i\left(\gamma_i^{-1}(v;k);k\right)/\gamma_i'\left(\gamma_i^{-1}(v;k);k\right)$. (iii) follows. ||

Proof of Lemma A5:

Proof of (i): Follows directly from the definition of $x(k)$.

Proof of (ii): From Lemma A3 (ii), for all $i=1,2$, $\gamma_i'(b;k)$ tends towards one uniformly for b in any compact subinterval of $(c,d]$. As, from Lemma A3 (ii), $\gamma_i(b;k)$ tends towards b for all b, $\left(\gamma_i^{-1}\right)'(b;k)=1/\gamma_i'\left(\gamma_i^{-1}(b;k);k\right)$ tends towards one and $\gamma_i^{-1}(b;k)$ tends towards b uniformly in $b\in[\underline{b},\gamma_i(d;k)]$, for all $\underline{b}>c$[2]. As $\gamma_2([\underline{b},x(k)];k)\subseteq[\underline{b},\gamma_1(d;k)]$, $\gamma_1^{-1}(\gamma_2(b;k);k)$ tends towards b and its derivative tends towards one uniformly over any interval $[\underline{b},x(k)]$. (ii) then follows from the definition of $\Psi(.;k)$.

Proof of (iii): That $\Phi(.;k)$ tends to Λ uniformly over any compact subinterval of $(-\infty,0)$ follows directly from Theorem 2 and the definitions of $\Phi(.;k)$ and Λ.

Let K be an arbitrary compact subinterval of $(-\infty,0)$. We prove first the uniform convergence over K of the derivative $\Phi'(.;k)$ towards the derivative l of Λ. From the compactness of K, it suffices to prove:

$$\lim_{(s,k)\to(u,0)}\Phi'(s;k)=l,$$

for all u in K. Let $M>0$ be such that $-M<\min K$.

Suppose there exists u in K such that $\lim_{(s,k)\to(u,0)}\Phi'(s;k)\neq l$. Then, there exists $\varepsilon>0$ and a sequence $(s_t;k_t)_{t\geq 1}$ converging towards $(u;0)$ such that:

$$|\Phi'(s_t;k_t)-l|>\varepsilon. \tag{1}$$

From (i), (ii) above and because the left-hand derivative $\Psi_l'(u;k)$ is zero to the right of $\ln F_1(x(k))$, there exists $t'>0$ and m such that $\max K<-m<0$

[2] That is, for all $\underline{b}$ and for all $\varepsilon>0$, there exists $k'>0$ such that $\left|\left(\gamma_i^{-1}\right)'(b;k)-1\right|,\left|\gamma_i^{-1}(b;k)-b\right|\leq\varepsilon$, for all $0<k<k'$ and b in $[\underline{b},\gamma_i(d;k)]$.

and for all $t > t'$:

$$|\Psi'(s;k_t) - l| < \varepsilon/2, \tag{2}$$

for all s in $[-M,-m]$, and

$$\Psi_l'(u;k_t) < l + \varepsilon/2,$$

for all u in $[-m,0]$. As the limit u of $(s_t)_{t\geq 1}$ belongs to K and hence to the interior of $[-M,-m]$, we may assume that $(s_t)_{t\geq 1}$ is included in $[-M,-m]$. We subdivide the rest of the proof in four parts.

(a) In (a) and in (b) below, we suppose that $\underline{\lim}_{t\to+\infty}\Phi'(s_t;k_t) < l - \varepsilon$. Extracting the subsequence if necessary, we may assume $\lim_{t\to+\infty}\Phi'(s_t;k_t) < l-\varepsilon$. There then exists $t^* > 0$, which we may assume larger than t', such that:

$$\Phi'(s_t;k_t) < l - \varepsilon,$$

for all $t > t^*$.

Here in (a), we consider the case where there exists a subsequence $(k_{t_r})_{r\geq 1}$ such that $t_r \geq t^*$ and $\Phi(s_{t_r};k_{t_r}) \leq \Psi(s_{t_r};k_{t_r})$, for all $r \geq 1$. Extracting the subsequence again if necessary, we may assume that this the case of the original sequence. For all $t \geq t^*$, as, from (1) and (2), $\Phi'(s_t;k_t) < \Psi'(s_t;k_t)$, there exists $\theta_t > 0$, such that $\Phi(s;k_t) < \Psi(s;k_t)$, for all s in $(s_t, s_t+\theta_t)$. From Lemma A4 (iv), as $\Phi(s;k_t)$, being below $\Psi(s;k_t)$, is concave over $(s_t, s_t+\theta_t)$, we have $\Phi'(s;k_t) < l-\varepsilon$, for all s in $(s_t, s_t+\theta_t)$. As long as it remains strictly below $\Psi(s;k_t)$, $\Phi(s_t;k_t)$ will remain strictly concave and its derivative will decrease and therefore will remain smaller than $l-\varepsilon$ if s increases. As, from (2), $\Psi'(s;k_t) > l-\varepsilon/2$, $\Phi(s;k_t)$ will never meet again $\Psi(s;k_t)$ to the right of s_t. Consequently, for all $t \geq t^*$, $\Phi(s;k_t)$ is strictly concave over $(s_t,-m)$ and $\Phi'(s;k_t) < l-\varepsilon$ over this interval. As s_t tends towards $u < -m$, for all t large enough $s_t < (u-m)/2$ and hence $\Phi'(s;k_t) < l-\varepsilon$ over $((u-m)/2,-m)$. The different functions are then as in Figure 1 below.

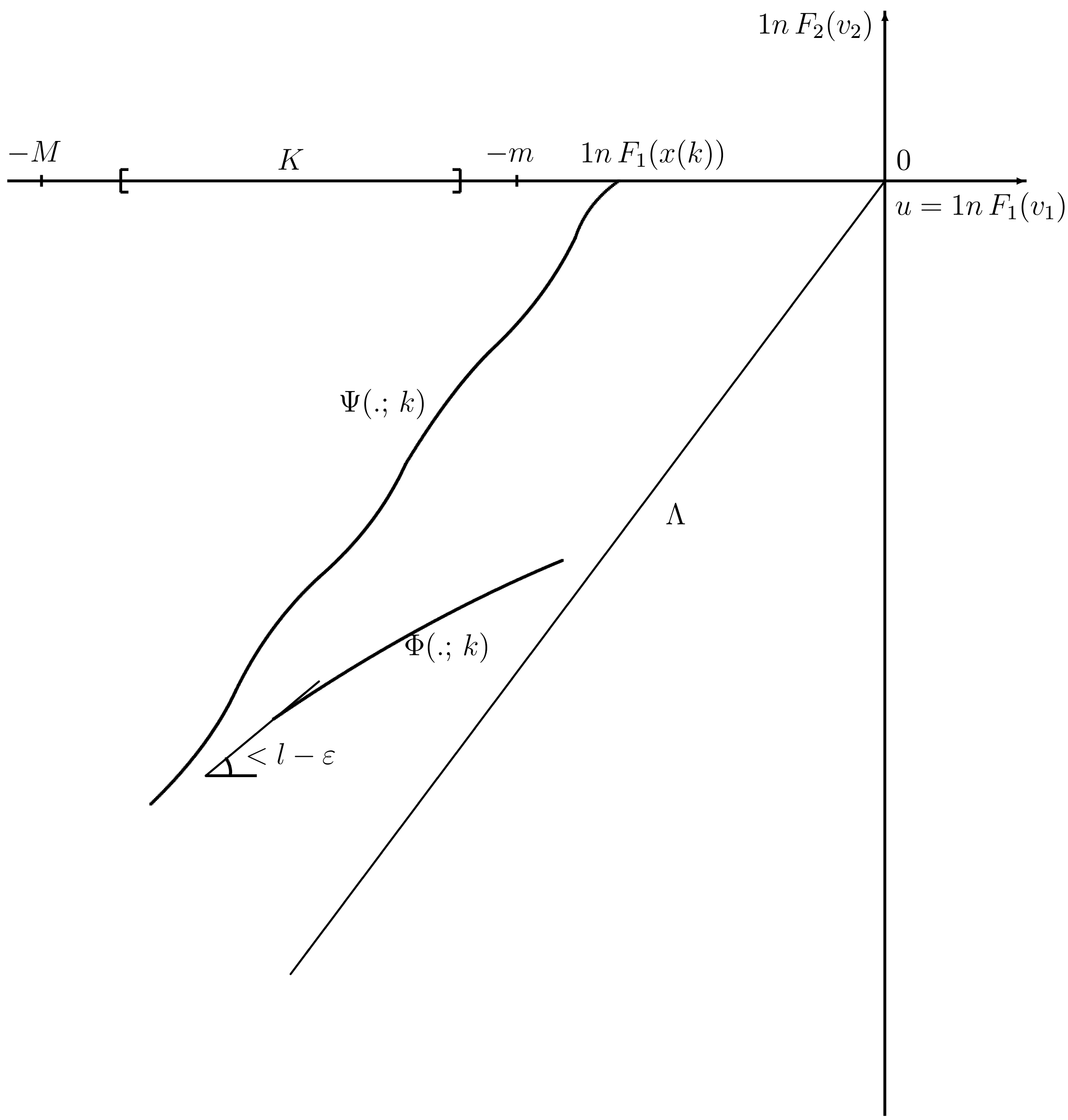

FIGURE 1: Ruling out $\Phi'\,(.; k)$ further below l than $\Psi'\,(.; k)$ is while $\Phi\,(.; k)$ is not larger than $\Psi\,(.; k)$.

For all t large enough, we then have:

$$\int_{(u-m)/2}^{-m} \Phi'(s;k_t)\,ds < (-u-m)(l-\varepsilon)/2.$$

However, $\Phi(.;k)$ tends towards Λ and consequently we also have, from :

$$\begin{aligned} & \lim_{t\to+\infty} \int_{(u-m)/2}^{-m} \Phi'(s;k_t)\,ds \\ = & \lim_{t\to+\infty} \left(\Phi(-m;k_t) - \Phi((u-m)/2;k_t)\right) \\ = & \Lambda(-m) - \Lambda((u-m)/2) \\ = & l(-u-m)/2, \end{aligned}$$

and we obtain a contradiction.

(b) We consider next the case where there exist a subsequence $(k_{t_r})_{r\geq 1}$ such that $t_r \geq t^*$ and $\Phi(s_{t_r};k_{t_r}) > \Psi(s_{r_r};k_{r_r})$, for all $r \geq 1$. Extracting the subsequence if necessary, we may again assume that this holds true for the original sequence. From Lemma A4 (iv), as long as it remains strictly above $\Psi(s;k_t)$, $\Phi(s;k_t)$ will remain strictly convex and its derivative will decrease and therefore will remain smaller than $l-\varepsilon$ if s decreases. As, from (2), $\Psi'(s;k_t) > l-\varepsilon/2$, $\Phi(s;k_t)$ will never meet $\Psi(s;k_t)$ to the left of s_t. Consequently, for all $t \geq t^*$, $\Phi(s;k_t)$ is strictly convex over $(-M, s_t)$ and $\Phi'(s;k_t) < l-\varepsilon$ over this interval. As s_t converges towards u, for all t large enough $\Phi(s;k_t)$ is strictly convex over $\left(-M, \frac{u-M}{2}\right)$ and $\Phi'(s;k_t) < l-\varepsilon$ over this interval. The configuration of the graphs are as in Figure 2 below.

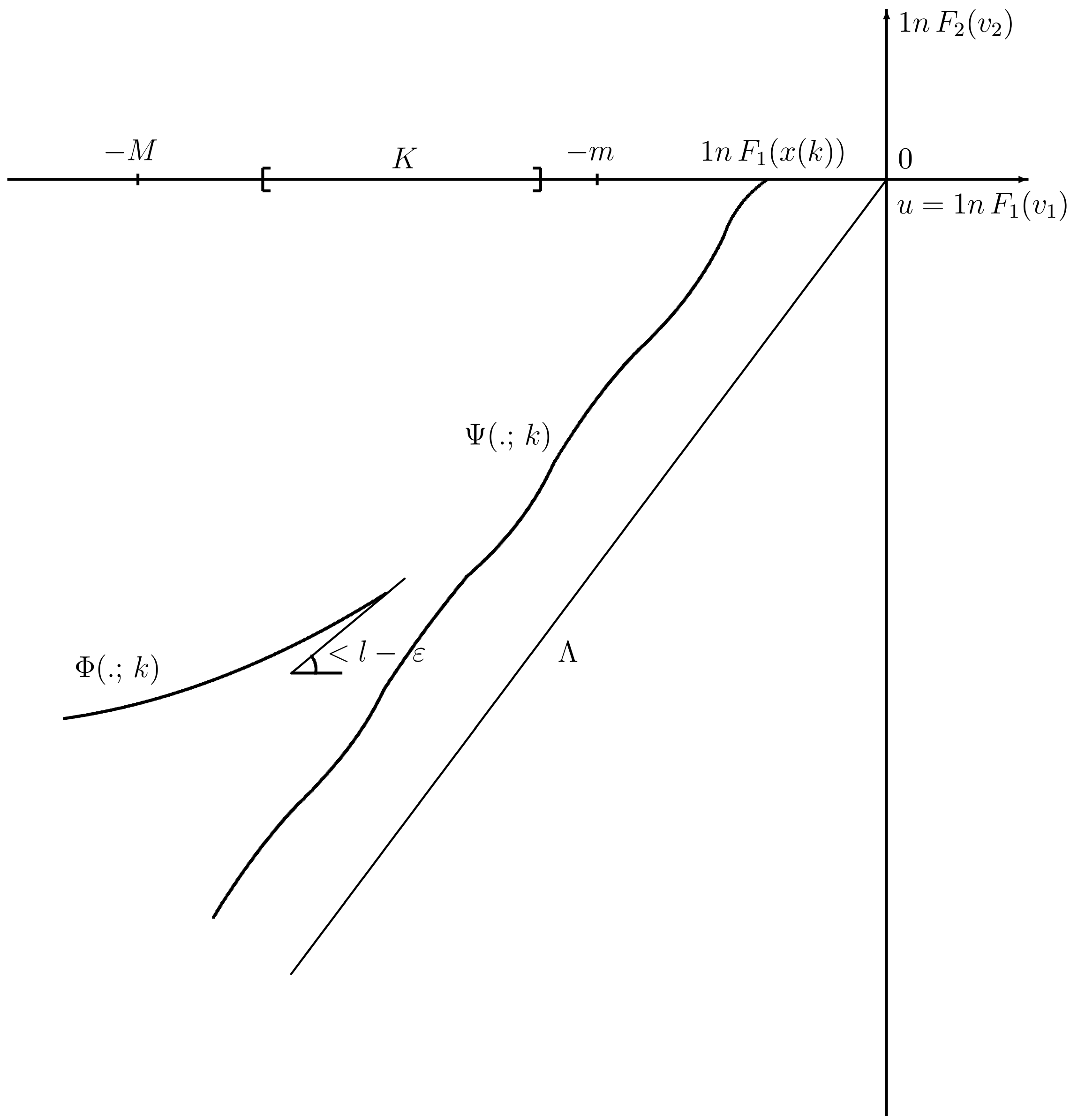

FIGURE 2: Ruling out $\Phi'(.;k)$ further below l than $\Psi'(.;k)$ is while $\Phi(.;k)$ is not smaller than $\Psi(.;k)$.

For all such t, we then have:

$$\int_{-M}^{(u-M)/2} \Phi'(s;k_t)\, ds < (u+M)\,(l-\varepsilon)/2.$$

However, we also have:

$$
\begin{aligned}
& \lim_{t\to+\infty} \int_{-M}^{(u-M)/2} \Phi'(s;k_t)\,ds \\
= & \lim_{t\to+\infty} \left(\Phi\left((u-M)/2;k_t\right) - \Phi\left(-M;k_t\right)\right) \\
= & \Lambda\left((u-M)/2\right) - \Lambda\left(-M\right) \\
= & l\left(u+M\right)/2,
\end{aligned}
$$

and we obtain a contradiction. As (a) and (b) exhaust all possibilities, we have ruled out $\underline{\lim}_{t\to+\infty}\Phi'(s_t;k_t) < l-\varepsilon$.

(c) Here and in (d) below, we suppose that $\overline{\lim}_{t\to+\infty}\Phi'(s_t;k_t) > l+\varepsilon$. As above, we may assume that $\lim_{t\to+\infty}\Phi'(s_t;k_t) > l+\varepsilon$. Consequently, there exists $t^* > 0$, which we may assume larger than t' such that:

$$
\Phi'(s_t;k_t) > l+\varepsilon,
$$

for all $t > t^*$.

We show that, for all $t > t^*$, $\Phi(s_t;k_t) < \Psi(s_t;k_t)$. Suppose there exists $t > t^*$ such that $\Phi(s_t;k_t) \geq \Psi(s_t;k_t)$. As $\Phi'(s_t;k_t) > \Psi'(s_t;k_t)$, there exists $\theta_t > 0$ such that $\Phi(s;k_t) > \Psi(s;k_t)$, for all s in $(s_t, s_t+\theta_t)$. From Lemma A4 (iv), $\Phi(s;k_t)$ is convex over $(s_t, s_t+\theta_t)$, and we have $\Phi'(s;k_t) > l+\varepsilon$, for all s in $(s_t, s_t+\theta_t)$. As long as it remains strictly above $\Psi(s;k_t)$, $\Phi(s;k_t)$ will remain strictly convex and its derivative will increase and therefore will remain larger than $l+\varepsilon$ if s increases. As $\Psi'_l(s;k_t) < l+\varepsilon/2$, $\Phi(s;k_t)$ will never meet again $\Psi(s;k_t)$ and therefore will stay strictly above it to the right of s_t. See Figure 4 in the paper. However, this contradicts $\Phi(\ln F_1(x(k_t));k_t) < 0 = \Psi(\ln F_1(x(k_t));k_t)$.

(d) From (c), $\Phi(s_t;k_t) < \Psi(s_t;k_t)$, for all $t > t^*$. As long as it remains strictly below $\Psi(s;k_t)$, $\Phi(s;k_t)$ will remain strictly concave and its derivative will increase and therefore will remain larger than $l+\varepsilon$ if s decreases. As, from (2), $\Psi'(s;k_t) < l+\varepsilon/2$, $\Phi(s;k_t)$ will never meet $\Psi(s;k_t)$

to the left of s_t. Consequently, for all $t \geq t^*$, $\Phi(s; k_t)$ is strictly concave over $(-M, s_t)$ and $\Phi'(s; k_t) > l + \varepsilon$ over this interval. As s_t converges towards u, for all t large enough $\Phi(s; k_t)$ is strictly concave over $(-M, (u - M)/2)$ and $\Phi'(s; k_t) > l + \varepsilon$ over this interval. See Figure 5 in the paper for an illustration of this case.

For all t large enough, we then have:

$$\int_{-M}^{(u-M)/2} \Phi'(s; k_t)\, ds > (u + M)(l + \varepsilon)/2.$$

However, we also have:

$$\begin{aligned}
& \lim_{t \to +\infty} \int_{-M}^{(u-M)/2} \Phi'(s; k_t)\, ds \\
= \; & \lim_{t \to +\infty} \left(\Phi((u - M)/2; k_t) - \Phi(-M; k_t)\right) \\
= \; & \Lambda((u - M)/2) - \Lambda(-M) \\
= \; & l(u + M)/2,
\end{aligned}$$

and we obtain a contradiction. We have ruled out $\overline{\lim}_{t \to +\infty} \Phi'(s_t; k_t) > l + \varepsilon$ and completed the proof of (iii).

Proof of (iv): Suppose $\overline{\lim}_{(s;k) \to (0;0)} \Phi'(s; k) > l$. There then exist $\varepsilon > 0$ and a sequence $(s_t; k_t)_{t \geq 1}$ tending towards $(0; 0)$ and such that $\Phi'(s_t; k_t) > l + \varepsilon$, for all t. We may assume that $(s_t)_{t \geq 1}$ is included in a compact interval $[-M, 0]$. As in the proof of (iii) above, from (i) and (ii) there exists t^*, such that $\Psi_l'(s; k_t) < l + \varepsilon/2$,for all $t > t^*$ and s in $[-M, 0]$.

As in part (c) of the proof of (iii) above, we can rule out $\Phi(s_t; k_t) \geq \Psi(s_t; k_t)$ for some $t > t^*$. We may then assume $\Phi(s_t; k_t) < \Psi(s_t; k_t)$, for all $t > t^*$. For any $t > t^*$, as longs as $\Phi(s; k_t)$ does not meet $\Psi(s; k_t)$ it will remain strictly concave and therefore its derivative will increase and hence will be larger than $l + \varepsilon$ if s decreases within $[-M, 0]$. However, from $\Psi_l'(s; k_t) < l + \varepsilon/2$, $\Phi(s; k_t)$ will never meet $\Psi(s; k_t)$ to the left of s_t within

$[-M, 0]$. Consequently, $\Phi'(-M; k_t) > l + \varepsilon$, for all $t > t^*$. This contradicts (iii) and we have proved (iv).

Proof of (v):

(a) Differentiating the definition of Ψ, it is straightforward to find the following expression:

$$\begin{aligned} & \Psi'(s;k) \\ = \; & \frac{f_2\left(F_2^{-1}(\exp \Psi(s;k))\right) \exp s}{f_1\left(F_1^{-1}(\exp s)\right) \exp \Psi(s;k)} \\ & \frac{1 + k - k\varepsilon_1(\exp s)}{1 + k - k\varepsilon_2(\exp \Psi(s;k))}, \end{aligned} \tag{4}$$

where $\varepsilon_i(p)$ is the elasticity of the density $f_i\left(F_i^{-1}(p)\right)$ with respect to the cumulative probability p. From the convergence, from (ii), of $\Psi(s;k)$ towards $\Lambda(s)$, there exists $\overline{s} < 0$ such that $F_2 = F_1^l$ is log-concave and hence $\varepsilon_2 \leq 1$ over $\left[c, F_2^{-1}(\exp \Psi(\overline{s};k))\right]$. From Lemma A4 (i), $F_2^{-1}(\exp \Psi(s;k)) \geq F_1^{-1}(\exp s)$. Consequently, for all $s < \overline{s}$, we have:

$$\begin{aligned} & \frac{f_2\left(F_2^{-1}(\exp \Psi(s;k))\right) \exp s}{f_1\left(F_1^{-1}(\exp s)\right) \exp \Psi(s;k)} \\ \leq \; & \frac{f_2\left(F_1^{-1}(\exp s)\right) \exp s}{f_1\left(F_1^{-1}(\exp s)\right) F_2 F_1^{-1}(\exp s)} \\ = \; & l, \end{aligned}$$

where the equality follows from $\rho_2 = l\rho_1$. From (4) and $\varepsilon_2 \leq 1$, we then find, for all $s < \overline{s}$:

$$\begin{aligned} & \Psi'(s;k) \\ \leq \; & l(1 + k - k\varepsilon_1(\exp s)) \\ \leq \; & l(1 + k + kB), \end{aligned}$$

with $-B$ the lower bound of ε_1. The inequality $\overline{\lim}_{(s;k)\to(-\infty;0)} \Psi'(s;k) \leq l$ follows.

(b) We will prove $\lim_{k\to 0} \sup_{s\in\mathbb{R}_-} \Phi'(s;k) \leq l$. This will obviously imply (v). First, note $\Phi'(0;k) = 1$, for all k. Let ε be an arbitrary strictly positive number. From (ii) and part (a) above of the current proof, there exists k' such that $\Psi_l'(s;k) < l + \varepsilon/2$, for all $0 < k < k'$ and all s in $(-\infty, 0)$.

Suppose there exists u in $(-\infty, 0)$ and $k < k'$ such that $\Phi'(u;k) > l + \varepsilon$. Assume first $\Phi(u;k) \leq \Psi(u;k)$. Then, proceeding as in the proofs above, $\Phi(.;k)$ remains concave and below $\Psi(.;k)$ and $\Phi'(.;k)$ remains above $l + \varepsilon$ everywhere to the left of u. Consequently, there exists w in $(u - (\Phi(u;k) - \Lambda(u))/\varepsilon, u)$ such that $\Phi(w;k) = \Lambda(w)$. This contradicts Lemma A4 (iii).

Suppose next $\Phi(u;k) > \Psi(u;k)$. Then, $\Phi(s;k)$ remains strictly convex and strictly above $\Psi(s;k)$ everywhere to the right of u. However, this contradicts $\Phi(\ln F_1(x(k));k) < 0 = \Psi(\ln F_1(x(k));k)$.

We have proved $\sup_{s\in\mathbb{R}_-} \Phi'(s;k) \leq l + \varepsilon$, for all $k > k'$, and consequently $\lim_{k\to 0} \sup_{s\in\mathbb{R}_-} \Phi'(s;k) \leq l + \varepsilon$. As ε was arbitrary, the result follows. ||