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A SIMPLE MODEL OF FIRST PRICE AUCTION WITH RESALE

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Abstract: If there is asymmetry among the bidders taking part in a first price auction, the equilibrium is not expost efficient. We consider a simple model of first price auction among two bidders where resale always occurs in case of inefficiency. We obtain mathematical formulas for the equilibrium strategies. We study properties of the equilibrium, derive some results of comparative statics and compare the revenue accruing to the auctioneer with his revenue in other auction procedures.

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1 INTRODUCTION

The first price auction is the auction procedure where the item being sold is awarded to the highest bidder who pays the price equal to his bid. Much attention has been devoted to this auction in the symmetric setting where the bidders' valuations are distributed identically. The symmetric equilibrium can be explicitly characterized and it displays the property of expost efficiency, that is, there is no possibility of profitable resale between the bidders after the auction.

The analysis of the first price auction where the bidders valuations are not distributed identically is plagued by the impossibility, even in the two bidder case, of obtaining closed mathematical formulas for the equilibrium strategies. Nevertheless it is possible to show that in asymmetric settings the equilibria are no longer efficient. However, the study of the first price auction is done under the hypothesis that resale is impossible or forbidden. In this paper, we adopt the "opposite" assumption that resale always takes place when there is inefficiency.

A more realistic model should allow for a partial resolution of the inefficiency through resale, as it is likely with bargaining in the presence of imperfect information (see Myerson and Satterthwaite, 1983). Our model can be considered as the first step towards the study of resale in first price auctions. As the description of our model will show, it allows us to focus on the consequences of the possibility of resale on the bidding behaviors without having to consider strategic misrepresentation of the valuations.

Two bidders take part in the first price auction. After the auction, the valuations become common knowledge. If the winner of the auction has a smaller valuation, he sells the item to the other bidder at a resale price which depends on the valuations in an exogenously given manner. It turns out that under this assumption, there exist general mathematical formulas for the equilibrium strategies. It is then simple to study the equilibrium and, for example, to obtain results of comparative statics. It is also possible to compare the revenue accruing to the auctioneer with what he obtains in other auction procedures, like the second price auction.

Riley and Samuelson (1981) give an explicit expression for the equilibrium strategy in the symmetric n bidder case. In the asymmetric case without resale, Griesmer, Levitan and Shubik (1967) consider the first price auction with two bidders whose valuations are uniformly distributed over possibly different intervals. Vickrey (1961) analyzes the asymmetric two bidder case without resale where one bidder knows the other bidder's valuation with certainty. Asymmetric n bidder examples without resale where all bidders except one have the same valuation probability distribution are examined numerically by Marshall, Meurer, Richard and Stromquist (1994). Maskin and Riley(October 1994) consider the existence of an equilibrium in the asymmetric n bidder case without resale by relying on discrete approximations and passing to the limit. Maskin and Riley (July 1994) then study properties of the equilibria when they exist. The first price auction in asymmetric settings and without resale is studied in Maskin and Riley (Nov 1994, Dec 1994 and June 1995) and Lebrun (1996 a and b).

Milgrom (1987) considers resale in auctions with perfect information throughout. Campos e Cunha and Santos (1995) introduces resale in an example due to Krishna (1995) where two units are sequentially auctioned and information is perfect.

Johnson (1979) reports that when they were allowed, resales took place in the auctioning of timber rights by the forest service (see McAfee and McMillan, 1987, Section XII). The Federal Communication Commission allows resale after the airwave auctions it recently held (see McMillan, 1994, Cramton, 1995, and McAfee and McMillan 1996). There exists a very active secondary market in auctions like the auction of Treasury Securities (see Bikhchandani and Huang, 1993). Other examples can be found in Campos e Cunha and Santos (1995).

In our model, the valuations are revealed publicly at the outset of the auction. For example, leaks, the public release of connected information or the realization of some relevant events enable a bidder to deduce his opponent's valuation. Or, we can also envision the resale as the result of collusion occuring with probability one after the auction. Other authors have made similar assumptions in other situations. McAfee, Vincent, Williams and Havens (1994) assume that the two bidders competing for a corporate takeover collude with an exogenous probability α after the auction has started. When collusion takes place, the bidders' valuations are common knowledge.

As mentioned above, resale can be considered as the result of collusion. McAfee and McMillan (1992) examine the allocation of the item and the payoffs in a whole encompassing cartel which would have direct control over the members' bids or could commit to effectively punish the deviator. Collusion in second price auctions is studied in Graham and Marshall (1987), Graham Marshall and Richard (1990) and Mailath and Zemsky (1991).

Although resale introduces a link between the bidders' net expected utilities of acquiring the item, a standard model of affiliation (see Milgrom and Weber, 1982) could not explain the bidders' behaviors at the auction.[1] In fact, in such a model a bidder would have a utility zero of losing the auction while actually he can obtain a strictly positive payoff if he is a buyer at the resale stage. Furthermore, the expectation of payoff conditional on losing depends on the strategies used in the auction. In this sense, we can say that the net expected utilities (or "valuations") are endogenous. Contrary to Krishna (1993), where it was due to the multiplicity of the objects auctioned sequentially, this endogeneity comes from our assumption of imperfect information.

In Section 2 we present the model, derive the equilibrium and study some of its properties. We obtain results of comparative statics in Section 3 and compare the first price auction with other auction procedures in Section 4.

Section 5 is the conclusion where we discuss the generalization to n bidders.

2 THE MODEL AND THE EQUILIBRIUM

An indivisible item is being sold at a first price auction. There are two bidders, bidder 1 and bidder 2. Bidder i's valuation is drawn from the interval $[\underline{c}, \overline{c}]$ with $\underline{c} < \overline{c}$, according to the cumulative distribution function F_i . During the auction, v_i is known only to bidder i. The distribution functions F_1 and F_2 are independent and are strictly increasing and differentiable over $[\underline{c}, \overline{c}]$.

Every bidder submits a bid. The higher bidder is awarded the item and pays a price equal to his bid. Ties are broken by a fair lottery. If bidder i is the highest bidder and if b_i is his bid, his payoff at the outset of the auction is $(v_i - b_i)$. The lower bidder's payoff is equal to zero.

After the auction, every bidder knows both valuations. Resale occurs between the bidders if the current owner of the object has a strictly smaller valuation. The resale price is equal to the value of a commonly known function r at the couple of valuations (v_1, v_2) . For example, the resale price can be a convex combination of v_1, v_2 with fixed weights k and $(1-k)$, that is, $r\,(v_1, v_2) = kv_1 + (1-k)v_2$. Or, it can be equal to the seller's valuation, that is, $r\,(v_1, v_2) = min\,(v_1, v_2)$, or to the buyer's valuation, that is, $r\,(v_1, v_2)$ $= max\,(v_1, v_2)$. If bidder 1, for example, acquires the item at the resale stage, his payoff is equal to $v_1 - r\,(v_1, v_2)$ and bidder 2's payoff is equal to $r\,(v_1, v_2) - v_2$.

The bidders do not discount their payoffs. Their total payoffs are thus equal to the sum of the payoffs at the auction stage and at the second stage. If bidder 1, for example, with valuation v_1 and bid b wins the auction and sells the object to bidder 2 with valuation v_2, bidder 1's total payoff is equal to $(v_1 - b) + (r(v_1, v_2) - v_1) = r(v_1, v_2) - b$ and bidder 2's total payoff is equal to $v_2 - r(v_1, v_2)$. If bidder 1 wins the auction and resale does not take place, then the payoffs of bidders 1 and 2 are $v_1 - b$ and 0 respectively.

A regular strategy β_i of bidder i is a continuous bid function from $[\underline{c}, \overline{c}]$ to R which is differentiable with a strictly positive derivative over $(\underline{c}, \overline{c}]$ and such that $\beta_i(\underline{c}) = \underline{c}$. The value $\beta_i(v)$ of β_i at v is the bid that strategy β_i prescribes to bidder i when his valuation is equal to v.

Both bidders are risk neutral. A regular equilibrium (β_1, β_2) is a couple of regular strategies which is a Bayesian equilibrium of the game. That is, if (β_1, β_2) is a regular equilibrium, then $\beta_i(v)$ is the bid that maximizes bidder i's expected total payoff if his valuation is equal to v and if his opponent bidder j, $j \neq i$, follows β_j for all v and i.

It turns out that there exists a unique regular equilibrium and that there are closed mathematical formulas for the equilibrium strategies. As we will see, although resale does not occur with probability one at the equilibrium, the resale price function determines the marginal benefit of a change of the bid and thus, through the equations they satisfy, the equilibrium strategies. We have Theorem 1 below.

Theorem 1. *If r is continuous, $r\left(F_1^{-1}(q), F_2^{-1}(q)\right)$ is strictly increasing in q and $r(\underline{c}, \underline{c}) = \underline{c}$ then there exists a unique regular equilibrium (β_1, β_2) of the first price auction with resale. The equilibrium strategies are as follows,*

$$\beta_i(v) = \frac{\int_0^{F_i(v)} r\left(F_1^{-1}(q), F_2^{-1}(q)\right) dq}{F_i(v)}, \qquad (1)$$

for all v in $[\underline{c}, \overline{c}]$. At the unique regular equilibrium, we have

$$F_1(\alpha_1(b)) = F_2(\alpha_2(b)), \qquad (2)$$

for all b in $[\underline{c}, \eta]$, where α_1 and α_2 are the inverses of β_1 and β_2, respectively, and $\eta = \beta_1(\overline{c}) = \beta_2(\overline{c})$.

Despite the asymmetry in our model, the equations the equilibrium strategies have to satisfy are easy to solve. Consider, for example, a bid b in $[\underline{c}, \eta]$ such that $\alpha_1(b) < \alpha_2(b)$. It must not be bidder i's best interest to deviate from b when his valuation is $\alpha_i(b)$. The cost of increasing his bid to $b + db$ is the increase in his payment when he wins the auction. His expected marginal cost of an increase of his bid is thus $F_j(\alpha_j(b))\, db$, with $j \neq i$.

By increasing his bid to $b + db$, bidder 1 now wins the auction when bidder 2's valuation lies in $[\alpha_2(b), \alpha_2(b + db)]$. For these valuations of bidder 2, bidder 1 sells the item to bidder 2 at the resale stage. Bidder 1's payoff in this case is thus the difference between the price $r(\alpha_1(b), \alpha_2(b))$ he will receive and the price b that he pays. Since bidder 2's valuation belongs to $[\alpha_2(b), \alpha_2(b + db)]$ with probability $f_2(\alpha_2(b))\, \alpha_2'(b)\, db$, we see that bidder

1's expected marginal benefit is $(r(\alpha_1(b), \alpha_2(b)) - b) f_2(\alpha_2(b))\alpha_2^{/}(b)\, db$. This marginal benefit of a change in b must equal its marginal cost and we find $(r(\alpha_1(b), \alpha_2(b)) - b) f_2(\alpha_2(b))\, \alpha_2^{/}(b) = F_2(\alpha_2(b))$, that we can rewrite as follows,

$$\frac{d}{db} \ln F_i(\alpha_i(b)) = \frac{1}{r(\alpha_1(b), \alpha_2(b)) - b} \qquad (3)$$

with $i = 1$.[2]

An increase in bidder 2's bid to $b + db$ when his valuation is equal to $\alpha_2(b)$ will give him the object at the auction in those cases where he would have gotten it at the resale stage. His expected net marginal benefit is thus the product of the probability $f_1(\alpha_1(b))\, \alpha_1^{/}(b)\, db$ that v_1 belongs to $[\alpha_1(b), \alpha_1(b + db)]$, with the difference between the price $r(\alpha_1(b), \alpha_2(b))$ he would have paid and the price b he will actually pay in this case. Equalizing the marginal cost with the marginal benefit, we find (3) with i=2.

Since the RHS's of the equations (3) with i=1,2 are equal, so are the LHS's and we have $d/db\,(lnF_1\alpha_1) = d/db\,(lnF_2\alpha_2)$. Equation (2) in Theorem 1 follows then from $F_1(\alpha_1(\eta)) = F_2(\alpha_2(\eta))$. At an equilibrium the bid probability distributions are equal. We can thus replace in equation (3), with $i = 1$, α_2 by $F_2^{-1} F_1 \alpha_1$ and we obtain an equation in only one unknown α_1. Rewriting this equation as an equation in β_1 gives (1) with $i = 1$. The equation (1) for $i = 2$ is obtained similarly.

A formal proof of the necessity part of Theorem 1 proceeds by noticing that

$$\pi_i(v_i, b) = (v_i - b)F_j(\alpha_j(b)) + \int_{v_i}^{\alpha_j(b}(r(v_i, v_j) - v_i)dF_j(v_j), \qquad (4)$$

with $j \neq i$, gives bidder i's expected payoff in all cases. Taking the derivative with respect to b and setting it equal to zero at $v_i = \alpha_i(b)$ gives equation (3). As we have seen, solving these equations lead to (1) and (2). For the proof of the sufficiency part of Theorem 1, we first see that $\beta_1$and β_2 are regular strategies and we then observe that the derivatives of $\pi_i(v_i, b)$ with respect to b is nonnegative for $v_i \geq \alpha_i(b)$, that is, for $b \leq \beta_i(v_i)$ and is nonpositive for $v_i \leq \alpha_i(b)$, that is, for $b \geq \beta_i(v_i)$. Because the proof of Theorem 1 is straightforward and relies on arguments standard from the theory of auctions, we have abstained from giving it here.

Equation (1) in Theorem 1 allows us to compute the equilibrium strategies in any given case. For example, if $r(v_1, v_2) = kv_1 + (1-k)v_2$, with k in $[0,1]$, $F_1(v_1) = v_1$ and $F_2(v_2) = v_2^2$ for all v_1 and v_2 in $[0,1]$, an easy computation gives $\beta_1(v) = (k/2)v + (2(1-k)/3)v^{\frac{1}{2}}$ and $\beta_2(v) = (k/2)v^2$ $+ (2(1-k)/3)v$ for all v in $[0,1]$. In the first price auction with resale, bidders will not necessarily bid less than their valuations. An expected loss at the auction stage may be outweighed by an expected benefit at the resale stage. From the equation for β_1 in this example, we see that bidder 1's bid will be strictly larger than his valuation if $0 < v < (4(1-k)/3(2-k))^2$.

Suppose bidder 1, for example, wins the auction and sells the item at the resale stage. If bidder 1's valuation is v_1, his bid is equal to

$$\beta_1(v_1) = \frac{\int_0^{F_1(v_1)}\left(r(F_1^{-1}(q), F_2^{-1}(q))\right)dq}{F_1(v_1)}, \qquad (5)$$

Bidder 2's valuation v_2 must be strictly larger than v_1 and not larger than $\alpha_2(\beta_1(v_1)) = F_2^{-1}F_1(v_1)$ and thus $F_2(v_1) \leq F_1(v_1)$.

The expected resale price conditional on resale occuring is equal to

$$\frac{\int_{F_2(v_1)}^{F_1(v_1)} r(v_1, F_2^{-1}(q))dq}{(F_1(v_1) - F_2(v_1))}$$

Since r is nondecreasing, we thus see that this expected resale price is at least as large as $\beta_1(v_1)$.

We thus have Corollary 1 below.

Corollary 1.*Assume that r satisfies the assumptions of Theorem 1. Let b be the winning bid at the auction. If there is a positive probability of resale, the expectation of the resale price conditional b's being the winning bid is not smaller than b.*

Since the probability distributions are identical at the equilibrium , we obtain the same bid distributions that would have resulted if the bidders' valuations had been identically distributed. In Corollary 2 below, we identify the valuation distributions that would have given the same distribution in the symmetric case.

Corollary 2. *Assume that r satisfies the assumptions of Theorem 1. The bid distributions at the unique regular equilibrium of the first price auction with resale are identical to the bid distributions at the unique regular symmetric equilibrium of the first price auction without resale*[3]*, where both valuations* *are identically distributed over* $[\underline{c}, r(\overline{c}, \overline{c})]$ *according to the strictly increasing cumulative distribution function* G *such that*

$$G^{-1}(q) = r\left(F_1^{-1}(q), F_2^{-1}(q)\right)$$

for all q *in* $[0,1]$. *Moreover,* G *as defined above is the only strictly increasing cumulative function with this property.*

Proof: The only symmetric regular equilibrium of the first price auction without resale when both valuations are distributed according to G is the equilibrium where both bidders follow the strategy β such that $\beta(v) = \int_{\underline{c}}^{v} v dG(v) / G(v)$, or, equivalently, $\beta(v) = \int_0^{G(v)} G^{-1}(q)\, dq / G(v)$, for all v in $[\underline{c}, \eta]$ (see Riley and Samuelson, 1981). We have $F_i \alpha_i = G\alpha$, where $\alpha = \beta^{-1}$ and i =1,2, if and only if we have $\beta_i F_i^{-1} = \beta G^{-1}$ for i =1,2. By substituting $G^{-1}(q)$ to v in the last expression for $\beta(v)$ and $F^{-1}(q)$ to v in (1), we find $\beta_i\left(F_i^{-1}(q)\right)$ $= \int_0^q r\left(F_1^{-1}(q), F_2^{-1}(q)\right) dq/q$ and $\beta\left(G^{-1}(q)\right) = \int_0^q G^{-1}(q)\, dq/q$. Corollary 2 follows.||

If bidder i's valuation is v, his interim payoff $P_i(v)$ is the expectation of his payoff conditional on his valuation. It is equal to $\pi_i(v, \beta_i(v))$ where π_i is the function given in (4). By substituting the expression for $\beta_i(v)$ given in (1) to b in (4) and rearranging, we find the following expression for $P_i(v)$, i=1,2,

$$P_1(v) = vF_2(v) - \int_0^{F_1(v)} r\left(F_1^{-1}(q), F_2^{-1}(q)\right) dq + \int_{F_2(v)}^{F_1(v)} r(v,, F_2^{-1}(q))dq$$
$$P_2(v) = vF_2(v) - \int_0^{F_2(v)} r\left(F_1^{-1}(q), F_2^{-1}(q)\right) dq + \int_{F_1(v)}^{F_2(v)} r(F_1^{-1}(q), v)dq \quad (6)$$

From these equalities, we obtain the expression (7) below for the difference $P_2(v) - P_1(v)$,

$$P_2(v) - P_1(v) =$$

$$\int_{F_2(v)}^{F_1(v)} [(v + r(F_1^{-1}(q), F_2^{-1}(q)) - (r(F_1^{-1}(q), v) + r(v, F_2^{-1}(q))] dq \qquad (7)$$

Under the assumptions of Corollary 3 below, it is easily seen that (7) can be transformed into (8). The other statements of Corollary 3 are immediate consequences of (8).

Corollary 3. *Assume that r satisfies the assumptions of Theorem 1 and is such that $r(v, v) = v$, for all v. Let v be an element of $[\underline{c}, \eta]$. The equality (8) below holds true if r is twice continuously differentiable over the interior of the domain of the multiple integral in the R.H.S.,*

$$P_2(v) - P_1(v) = -\int_{F_2(v)}^{F_1(v)} \int_q^{F_1(v)} \int_{F_2(v)}^q r_{12}(F_1^{-1}(s), F_2^{-1}(t))\, dF_2^{-1}(s)\, dF_1^{-1}(t)\, dq, \quad (8)$$

Suppose that $F_2(v) \leq F_1(v)$. If $r_{12} \geq 0$ over the integration domain then $P_2(v) \leq P_1(v)$ and if $r_{12} \leq 0$ over this domain then $P_2(v) \geq P_1(v)$.

Remark that the inequality $F_2(v) \leq F_1(v)$ is equivalent to $\beta_2(v) \leq \beta_1(v)$ and thus implies $\alpha_1(b) \leq \alpha_2(b)$ where $b = \beta_1(v)$. Bidder 1 with valuation v can only be a seller at the resale stage. Similarly, bidder 2 with valuation v can only be a buyer.

From corollary 3, we see that if $r_{12} = 0$, as it is the case where $r(v_1, v_2) = kv_1 + (1-k)v_2$, then $P_1(v) = P_2(v)$, for all v. This result is in contrast with the properties of the first price auction without resale. For example, when F_2 stochastically dominates F_1 as it would be the case if bidder 2 had the reputation of being very interested in the item, bidder 2's interim payoffs would be larger than bidder 1's in the first price auction without resale. This follows from the fact that, in this auction, the same relation of stochastic dominance passes from the valuation distributions to the bid distributions (see for example Lebrun 1996b). Notice also that in our first price auction with resale, the bid probability distributions are identical. In particular, it implies in our example where $F_2(v) \leq F_1(v)$, for all v, that $\beta_1(v) \geq \beta_2(v)$, for all v. In such an example, the bidder reputed less interested bids higher[4] and will thus be the only seller at the resale stage.

3 COMPARATIVE STATICS

From (1) in Theorem 1, we immediately see that the equilibrium strategies depend only on the values of r over the "curve" $C = \{(F_1^{-1}(q), F_1^{-1}(q)) : q\epsilon[0,1]\} = \{(v, F_2^{-1}(F_1(v)) : v\epsilon[\underline{c}, \overline{c}]\}$. Remark that since $\beta_1(v) = \beta_2(F_2^{-1}(F_1(v))$, the couples in C are these couples of valuations giving rise to the same bid from both bidders. Furthermore, $\beta_i(v)$ is increasing with r. Thus, regardless of whether bidder i is a seller or a buyer at the resale stage, he will bid higher at the equilibrium if the resale price increases. The intuition is simple and similar to the intuition given by Campos e Cunha and Santos (1995) in their analysis of an example with perfect information where two objects are auctioned sequentially. A higher resale price makes winning at the auction more attractive not only for the seller at the resale stage but also for the buyer. The latter desires to increase his probability of getting the object at the auction rather than to acquire it later at a high price. In fact, we have seen after Theorem 1 (Section 2) that the marginal benefit of an increase of the bid is $(r(\alpha_1, \alpha_2) - b)\, f_j(\alpha_j\,(b)\,\alpha_j^{/}$ and increases thus with r. In Corollary 4 below, an "individually rational" resale price function r is such that $min(v_1, v_2) \leq r(v_1, v_2) \leq \max(v_1, v_2)$, for all v_1, v_2.

Corollary 4. *If r and $\widetilde{r}$ satisfy assumptions of Theorem 1, and if $r \leq \widetilde{r}$ over $C = \{(v, F_2^{-1}(F_1\,(v)) : v\epsilon\,[\underline{c}, \overline{c}]\}$,then*

$$\beta_i \leq \widetilde{\beta_i},$$

for all i=1,2, when β_i, i=1,2, are the equilibrium bid functions corresponding to r and $\widetilde{\beta_i}$, i=1,2, to $\widetilde{r}$. Among the individually rational resale functions verifying the assumptions of Theorem 1, $r\,(v_1, v_2) = \max\,(v_1, v_2)$ gives the maximum equilibrium bid functions and $r\,(v_1, v_2) = min\,(v_1, v_2)$ gives the minimum equilibrium bid functions.

From Corollary 4, $r\,(v_1, v_2) = v_2$ and $\widetilde{r}\,(v_1, v_2) = \max\,(v_1, v_2)$, for example, give the same equilibrium bid functions when $F_2(v) \leq F_1(v)$, for all v, as in the example $F_1(v) = v$ and $F_2(v) = v^2$ over $[0, 1]$. Corollary 4 also implies that the highest bid functions are obtained when the buyer's valuation determines the price, that is, when the seller captures the whole surplus of the resale.

Corollary 5 is a direct consequence of the expression (6) for the interim payoffs.

Corollary 5. *Assume that r and $\tilde{r}$ satisfy the assumptions of Theorem 1. Let v be an element of $[\underline{c}, \overline{c}]$. If $F_j(v) \geq F_i(v)$ and $\tilde{r} \geq r$ over the domains of the integrals in (6) corresponding to $P_i(v)$, then*

$$P_i(v) \geq \widetilde{P_i}(v)$$

where $P_i(v)$ and $\widetilde{P_i}(v)$ are bidder i's interim payoffs when the resale price functions are r and $\tilde{r}$ respectively.

Reasoning as after the statement of Corollary 3 (Section 2), we see that under the assumption of Corollary 5, bidder i with valuation v can only be a buyer at the resale stage. Corollary 5 implies that if the resale price increases, the interim payoff of the potential buyer decreases. If we consider only individually rational resale functions r such that $r_{12} = 0$, Corollaries 5 and 2 (Section 2) imply that an increase of r in this family decreases both bidders' interim payoffs.

Rather than obtaining more results of comparative statics by, for example, focusing on a particular class of resale functions, we will compare in the next section the auctioneer's revenue at the first price auction with resale with his revenue at other auction procedures.

4 REVENUE COMPARISONS

We first compare the first price auction with resale with the second price auction. In the latter auction, the winner is the highest bidder and pays a price equal to the second highest bid, here his opponent's bid. The equilibrium where every bidder submits the bid equal to his valuation is the only Bayesian equilibrium in weakly dominated strategies of this second price auction without resale[5] (see Vickrey, 1961).

We know from Corollary 4 (Section 3) that $r(v_1, v_2) = \max(v_1, v_2)$ is the individually rational resale price function which gives the highest equilibrium bid functions of the first price auction with resale and thus also the highest revenues $R(F_1, F_2)$ to the auctioneer. From Corollary 2 (Section 2), the bid probability distributions are identical to the bid distributions when both bidders' valuations are distributed according to G such that $G^{-1} = max\left(F_1^{-1}, F_2^{-1}\right)$. The auctioneer's expected revenues $R(F_1, F_2)$ and $R(G, G)$ are thus also identical, that is, $R(F_1, F_2) = R(G, G)$. From the revenue equivalence theorem (see Riley and Samuelson, 1981), the expected revenues $R(G, G)$ from the first price auction are equal to the expected revenues $R_s(G, G)$ from the second price auction in the symmetric case, that is, $R(G, G) = R_s(G, G)$ and thus $R(F_1, F_2) = R_s(G, G)$.

Since the valuation distribution functions are strictly increasing, the equality $G^{-1} = max\left(F_1^{-1}, F_2^{-1}\right)$ is equivalent to $G = min(F_1, F_2)$ and G thus dominates F_1 and F_2. The expectation of the second highest valuation (here the minimum) is thus higher when the valuations are both distributed according to G than when they are distributed according to F_1 and F_2, that is, $R_s(G, G) \geq R_s(F_1, F_2)$.

Consequently, $R(F_1, F_2) \geq R_s(F_1, F_2)$, and if $F_1 \neq F_2$ the inequality $R_s(G, G) \geq R_s(F_1, F_2)$ is strict and thus $R(F_1, F_2) > R_s(F_1, F_2)$. When there is asymmetry and when the resale price is the buyer's valuation, the first price auction with resale gives higher revenues than the second price auction does. We can similarly show that if $r(v_1, v_2) = min(v_1, v_2)$ we have $R(F_1, F_2) \leq R_s(F_1, F_2)$, with a strict inequality when $F_1 \neq F_2$. When $r(v_1, v_2) = k \min(v_1, v_2) + (1-k) \max(v_1, v_2)$,there exists an intermediate k where $R = R_s$ and we have Corollary 6 below.

Corollary 6. *Suppose* $F_1 \neq F_2$ *and* $r(v_1, v_2) = k min(v_1, v_2) + (1-k)\, max(v_1, v_2)$, *with* $0 \leq k \leq 1$. *Then* R *is a strictly decreasing function of* k *and*

$$R > R_s,$$

for all k *in* $[0, k^*)$ *and*

$$R < R_s,$$

for all k *in* $(k^*, 1]$, *where*

$$k^* = \frac{\int_{\underline{c}}^{\overline{c}} \max_i (1 - F_i(v))^2 dv - \int_{\underline{c}}^{\overline{c}} (1 - F_1(v)(1 - F_2(v)) dv}{\int_{\underline{c}}^{\overline{c}} \max_i (1 - F_i(v))^2 dv - \int_{\underline{c}}^{\overline{c}} \min_i (1 - F_i(v))^2 dv} \tag{1}$$

where R and R_s are the auctioneer's expected revenues at the first price auction with resale and at the second price auction (respectively).

Proof. See the Appendix.

In the example $F_1(v) = v$ and $F_2(v) = v^2$ over $[0, 1]$, direct computation shows $k^* = 5/12$. From Corollary 6, we thus see that if $k < 5/12$, the auctioneer's revenue is higher in the first price auction with resale and if $k > 5/12$, it is higher in the second price auction.

Further results can be obtained if we restrict the resale price functions. Corollary 7 below is an example.

Corollary 7. *Suppose $F_1 \neq F_2$ and $r(v_1, v_2) = kv_1 + (1 - k)v_2$, with $0 \leq k \leq 1$. Without loss of generality, assume that*

$$\int_{\underline{c}}^{\overline{c}} (1 - F_2(v))^2 dv > \int_{\underline{c}}^{\overline{c}} (1 - F_1(v))^2 dv.$$

If

$$\int_{\underline{c}}^{\overline{c}} (1 - F_1(v)) (1 - F_2(v)) \, dv < \int_{\underline{c}}^{\overline{c}} (1 - F_1(v))^2 \, dv,$$

then

$$R > R_s,$$

for all k in $[0,1]$*. If*

$$\int_{\underline{c}}^{\overline{c}}(1-F_1(v))(1-F_2(v))dv \geq \int_{\underline{c}}^{\overline{c}}(1-F_1(v))^2dv,$$

then

$$R > R_s,$$

for all k in $\left[0, k^{/}\right)$ *and*

$$R < R_s,$$

for all k in $\left(k^{/}, 1\right]$*, where*

$$k^{/} = \frac{\int_{\underline{c}}^{\overline{c}}(1-F_2(v))^2dv - \int_{\underline{c}}^{\overline{c}}(1-F_1(v)(1-F_2(v))dv}{\int_{\underline{c}}^{\overline{c}}(1-F_2(v))^2dv - \int_{\underline{c}}^{\overline{c}}(1-F_1(v))^2dv}$$

and where R *and* R_s *are the auctioneer's expected revenues at the first price auction without resale and the second price auction, respectively.*

Proof. See the Appendix.

Our final corollary compares the first price auction with resale with the standard first price auction where resale is not allowed.[6]

Corollary 8. *Assume that* $d/dv[F_2/F_1(v)] > 0$*, for all* $v \epsilon [\underline{c}, \overline{c}]$*. Then there exists one and only one Bayesian equilibrium of the first price auction without resale. Let* R_f *be the auctioneer's expected revenue at this equilibrium. Suppose that* $r(v_1, v_2) = kmin(v_1, v_2) + (1-k)max(v_1, v_2)$ *, for all* v_1 *and* v_2*. Then there exists* k^{**} *in* $(0,1)$ *such that*

$$R > R_f,$$

for all k *in* $[0, k^{**})$ *and*

$$R < R_{f,}$$

for all k *in* $(k^{**}, 1]$.

Proof. See the Appendix.

The assumption $d/dv[F_2/F_1(v)] > 0$, for all v in $[\underline{c}, \overline{c}]$ in Corollary 8 is an assumption of stochastic dominance verified, for example, by $F_1(v) = v$ and $F_2(v) = v^2$ over $[0, 1]$.

5 CONCLUSION

We studied a simple model of first price auction with two bidders where resale always takes place in the case of inefficiency. Unlike the model where resale never takes place, there exist explicit mathematical formulas for the equilibrium strategies. We proved that at the equilibrium, the expectation of the resale price if there is resale and conditional on the winning bid is higher than this bid. Despite the possible asymmetry, the bid probability distributions are equal at the equilibrium. It is this property that allows us to solve the system of equations the equilibrium strategies are solution of. The bidders' bids are distributed as if the valuations are identically distributed. We identified the common valuation distribution that would have given rise to the same bid distributions. We showed that an increase of the resale price as a function of the valuations increases the bid functions and is thus beneficial to the auctioneer. We studied the effect of such a change on the bidders' interim payoffs. If, for example, there is no cross valuation effects on the resale price, a lower resale price function is beneficial to both bidders.

We showed that there is no general ranking between the revenues to the auctioneer at the second price auction and the first price auction with resale. When there is asymmetry, the first price auction where the bidder who sells captures the whole surplus of the resale gives strictly higher revenues than the second price auction. The ranking is reversed when it is the buyer who captures the resale surplus. When an asymmetric Nash bargaining solution determines the resale price, we obtain a formula for the share of the surplus the seller needs to have in order for the first price auction to outperform the second price auction. Allowing resale in the first price auction may or may not be in the best interest of the auctioneer. For high resale prices, allowing resale is profitable and for low resale prices, it is detrimental.

The analysis does not generalize straightforwardly to all n bidder cases. For example, if there are three bidders with three different valuation distributions and if the resale price is the buyer's valuation, we show in the Appendix that there is no regular equilibrium where the bids are distributed identically. However, some cases with two different valuation distributions are simple to treat. If, for example, one bidder's valuation distribution stochastically dominates the valuation distribution common to all other bidders and if the resale price is the seller's valuation, then there exists a regular equilibrium where the bids are identically distributed (see the Appendix). Starting from a symmetric setting, collusion by some bidders into one cartel can give rise to such a case.

APPENDIX

Proof of Corollary 6: From Corollary 2 (Section 2), R is equal to the revenue in the first price auction where both bidders' valuations are distributed according to G as defined in (5). In this case, the equilibrium bid function is $\beta(v) = \int_{\underline{c}}^{v} v dG(v) / G(v)$ and the expected payment from a bidder with valuation v is $\int_{\underline{c}}^{v} v dG(v)$. By integrating by parts this expression over $[\underline{c}, \overline{c}]$ according to G, changing the variable and multiplying by 2, we find

$$R = 2 \int_0^1 (1-q) \left(k \min\left(F_1^{-1}(q), F_2^{-1}(q)\right) + (1-k) \max\left(F_1^{-1}(q), F_2^{-1}(q)\right)\right) dq.$$

Decomposing this expression into a sum of two integrals, using the identities $\min\left(F_1^{-1}, F_2^{-1}\right) = (\max(F_1, F_2))^{-1}$ and $\max\left(F_1^{-1}, F_2^{-1}\right) = (\min(F_1, F_2))^{-1}$, changing variables, we find

$$R = \left[k(-\int_{\underline{c}}^{\overline{c}} v d(\min_i (1 - F_i(v))^2) + (1-k)(-\int_{\underline{c}}^{\overline{c}} v d(\max_i (1 - F_i(v))^2)\right]$$

Finally, if we integrate by parts, we obtain

$$R = \underline{c} + k \int_{\underline{c}}^{\overline{c}} \min_i (1 - F_i(v))^2 dv + (1-k) \int_{\underline{c}}^{\overline{c}} (\max_i (1 - F_i(v))^2) dv.$$

We thus see immediately that R is a strictly decreasing function of k. Before the statement of Corollary 6, we had already proved that $R > R_s$ if $k = 0$ and $R < R_s$ if $k = 1$. The existence of k^* is immediate.

A simple computation shows that

$$R_s = \underline{c} + \int_{\underline{c}}^{\overline{c}} (1 - F_1(v))(1 - F_2(v))dv.$$

The expression for k^* is then obtained by solving the equation $R = R_s$ for k.‖

Proof of Corollary 7: Proceeding as in the proof of Corollary 6, we can show that

$$R = \underline{c} + k \int_{\underline{c}}^{\overline{c}} (1 - F_1(v))^2 dv + (1 - k) \int_{\underline{c}}^{\overline{c}} (1 - F_2(v))^2 dv.$$

Under assumption of Corollary 7, we see that R is a strictly decreasing function of k. The revenue R_s is equal to $\underline{c} + \int_{\underline{c}}^{\overline{c}} (1 - F_1(v))(1 - F_2(v))dv$.

The first statement of Corollary 7 follows immediately.[7] In order to prove the rest of Corollary 7, we have to show that $R > R_s$, when $k = 0$. However, from the Cauchy-Schwartz inequality, we know that

$$\int_{\underline{c}}^{\overline{c}} (1 - F_1(v))(1 - F_2(v))dv. \leq \left[\int_{\underline{c}}^{\overline{c}} (1 - F_1(v))^2 dv \int_{\underline{c}}^{\overline{c}} (1 - F_2(v))^2 dv \right]^{1/2}$$

The inequality sign is strict unless $(1 - F_1)$ and $(1 - F_2)$ are linearly dependent. Since $(1 - F_1)$ and $(1 - F_2)$ are equal to 1 at $\overline{c}$, if they are linearly dependent they are equal. This is not the case and the inequality sign is strict and it follows that

$$\int_{\underline{c}}^{\overline{c}} (1 - F_1(v))(1 - F_2(v))dv < \int_{\underline{c}}^{\overline{c}} (1 - F_2(v))dv$$

and $R_s < R$ when $k = 0$. The formula for $k^/$ is obtained simply by solving the equation $R = R_s$ for k. ‖

Proof of corollary 8: For the existence and uniqueness of the equilibrium, see Lebrun 1996b. Using the same notations and following the same reasoning as in the proof of Corollary 5 and noticing that $\min\{F_1, F_1\} = F_2$ and $\max\{F_1, F_2\} = F_1$, we see that the rest of Corollary 8 is equivalent to the inequalities $R(F_1, F_1) < R_f(F_1, F_2) < R(F_2, F_2)$. Since $R(F_i, F_i) = R_f(F_i, F_i)$, Corollary 8 follows from results in Lebrun 1995.||

Consider the first price auction with resale among three bidders whose valuations are distributed according to F_1, F_2 and F_3 such that $F_1(v) > F_2(v) > F_3(v)$ for all v in $(\underline{c}, \overline{c})$. Suppose that in case of resale, the item is sold to the bidder with the higher valuation among the losers' at the price equal to the buyer's valuation. Assume that $(\beta_1, \beta_2, \beta_3)$ is a regular equilibrium such that $F_1\alpha_1 = F_2\alpha_2 = F_3\alpha_3$ where $\alpha_i = \beta_i^{-1}$ .The assumption on the distributions implies $\alpha_1(b) < \alpha_2(b) < \alpha_3(b)$, for all b in $(\underline{c}, \eta)$, where $\eta = \beta_i(\underline{c})$.

If bidder 3's valuation is $v_3 > \underline{c}$ and his bid is b in a neighborhood of $\beta_3(v_3)$, his expected payoff is equal to $(v_3 - b) F_1(\alpha_1(b)) F_2(\alpha_2(b))$. Taking the derivative and setting it equal to zero at $v_3 = \alpha_3(b)$ give the following equation,

$$\frac{d}{db} lnB(b)\, 2(\alpha_3(b) - b) = 1, \qquad (A.1)$$

where $B = F_i\alpha_i$.

Bidder 1's payoff when his valuation is v_1 and his bid is b in a neighborhood of $\beta_1(v_1)$ is equal to

$$(v_1 - b) F_2(\alpha_2(b)) F_3(\alpha_3(b)) + \int_{v_1}^{\alpha_2(b)} \int_{\underline{c}}^{v_2} (v_2 - v_1) dF_3(v_3) dF_2(v_2) + \int_{v_1}^{\alpha_3(b)} \int_{\underline{c}}^{v_3} (v_3 - v_1) dF_2(v_2) dF_3(v_3)$$

Taking the derivative with respect to b and setting it equal to zero at $v_1 = \alpha_1(b)$, we obtain

$$\frac{d}{db} lnB(b) [2(\alpha_1(b) - b) + (\alpha_2(b) - \alpha_1(b)) \frac{F_3(\alpha_2(b))}{B(b)} + (\alpha_3(b) - \alpha_1(b)) \frac{F_2(\alpha_3(b))}{B(b)}] = 1. \qquad (A.2)$$

From (A.1) and (A.2), we see that the expressions between brackets in the left hand side are equal. By substituting $\beta_3(v)$ to b and using the equalities $B = F_i\alpha_i$,we obtain:

$$2\left(F_1^{-1}F_3(v) - \beta_3(v)\right) + \left(F_2^{-1}F_3(v) - F_1^{-1}F_3(v)\right) F_3F_2^{-1}F_3(v)/F_3(v)$$
$$+\left(v - F_1^{-1}F_3(v)\right) F_2(v)/F_3(v) = 2(v - \beta_3(v))$$
$$or,\ F_1^{-1}F_3(v)\left[2F_3(v) - F_3F_2^{-1}F_3(v) - F_2(v)\right]$$
$$= 2vF_3(v) - vF_2(v) - \left(F_2^{-1}F_3(v)\right)(F_3F_2^{-1}F_3(v)).$$

In general, this equality will not be satisfied. It suffices to take F_1^{-1} such that:

$$F_1^{-1}(q)\left[2q - F_3F_2^{-1}(q) - F_2F_3^{-1}(q)\right]$$
$$\neq \left[2F_3^{-1}(q)\,q - F_2F_3^{-1}(q)\,F_3^{-1}(q) - F_2^{-1}(q)\,F_3F_2^{-1}(q)\right].$$

Consequently, for such distribution functions there is no regular equilibrium where the bids are distributed identically.

Take now an integer $n \geq 3$ and two distributions F_1 and F_2 such that $F_2(v) \leq F_1(v)$, for all v. Suppose the valuations of bidders $1, 2, ..., (n-1)$ are independently distributed according to F_1 and the valuation of bidder n is distributed according to F_2. Assume also that in case of resale, the bidder among the losers of the auction who has the highest valuation buys the item and pays the price equal to the seller's valuation.

Let $\left(\widetilde{\beta}_1, \widetilde{\beta}_2, ..., \widetilde{\beta}_n\right)$ be a regular equilibrium such that $\widetilde{\beta}_1 = ... = \widetilde{\beta}_{n-1} = \beta_1$ and $\widetilde{\beta}_n = \beta_2$ and $\beta_1 \geq \beta_2$ or equivalently, $\alpha_1 = \beta_1^{-1} \leq \alpha_2 = \beta_2^{-1}$. If bidder i's valuation is v_i and his bid is b, with $1 \leq i \leq (n-1)$, his expected payoff is

$$\begin{gathered}(v_1 - b) F_1(\alpha_1(b))^{n-2} F_2(\alpha_2(b)) + \\ F_1(\alpha_1(b))^{n-2} \int_{v_1}^{\alpha_2(b)} (min(v_1, \widetilde{v}_2) - v_1) dF_2(\widetilde{v}_2) \\ + \int_{\alpha_1(b)}^{v_1} (v_1 - min(v_1, \widetilde{v}_1)) F_2(\widetilde{v}_1) dF_1(\widetilde{v}_1)^{(n-2)}\end{gathered}$$

Then, taking the derivative of bidder 1's payoff with respect to b and setting it equal to zero at $v_1 = \alpha_1(b)$ gives,

$$\frac{d}{db} lnF_1(\alpha_1(b))^{(n-2)} + \frac{d}{db}[lnF_2(\alpha_2(b))] = \frac{1}{\alpha_1(b) - b}. \quad (A.3)$$

Bidder n's expected payoff for a valuation v and a bid b is

$$\left[(v_2 - b) F_1(\alpha_1(b))^{(n-1)} + \int_{\alpha_1(b)}^{v_2} (v_2 - min(\widetilde{v}_1, v_2)) dF_1(\widetilde{v}_1)^{(n-1)}\right]$$

By taking the derivative with respect to b and setting it equal to zero at $v_2 = \alpha_2(b)$, we find

$$\frac{d}{db} lnF_1 \left(\alpha_1^{/} (b) \right)^{(n-1)} = \frac{1}{\alpha_1 (b) - b} \; . \qquad (A.4)$$

From (A.4), we obtain $\beta_1 (v) = \int_{\underline{c}}^{v} v dF_1 (v)^{(n-1)} / F_1 (v)^{(n-1)}$. From (A.3) and (A.4) we have $F_1 \alpha_1 = F_2 \alpha_2$ and the bids are thus distributed identically. The function β_2 is simply $\beta_1 F_1^{-1} F_2$. Notice that $\beta_1 \geq \beta_2$. Inversely, by using arguments similar to those alluded to after the statement of Theorem 1 (Section 2), we can prove that these β_1 and β_2 determine a regular equilibrium.

FOOTNOTES

1. Contrary to the usual assumptions, this model would of course have to be asymmetrical.

2. The denominator is different from zero since $F_2(\alpha_2(b)) \neq 0$ for $b > \underline{c}$.

3. The definition of a regular equilibrium of a first price auction without resale is identical to its definition in the case with resale, it is symmetric if both bidders' strategies are equal. The only symmetric regular equilibrium of the first price auction without resale is actually its only Bayesian equilibrium. (see Maskin and Riley, Nov 1994 and Lebrun 1996b).

4. This property is shared by the equilibria of the first price auction without resale.

5. When we introduce resale, bidding his true valuation is not weakly dominant any longer. For example, take $r(v_1, v_2) = kv_1 + (1-k)v_2$, with $0 < k < 1$. Choose v^* in (0,1) and consider the strategy β_2 of bidder 2 such that $\beta_2(v) = v^*$, for all $v \geq v^*$. It is not too difficult to show that there exists $v < v^*$ such that if bidder 1's valuation is equal to v, a bid strictly larger then v^* gives him a strictly higher expected payoff than the payoff if he bids v.

6. Here, it is not possible to give a general expression for k^{**}.

7. We should not believe that R is always strictly smaller than R_s when $k = 1$. In fact there are distributions F_1 and F_2 verifying the assumption of Corollary 7 and such that $R > R_s$ when $k = 1$. The condition of the

first statement of the corollary is thus meaningful. For example, consider the discrete distributions $\overline{F}_1$ and $\overline{F}_2$ where $\overline{F}_1$ is concentrated at 1 and the support of $\overline{F}_2$ is $\{0,2\}$. If $0 < \overline{F}_2(\{0\}) < 1/2$, we can see that $\int(1-\overline{F}_1(v))(1-\overline{F}_2(v))dv$ is strictly smaller than $\int(1-\overline{F}_1(v))^2 dv$ and $\int(1-\overline{F}_2(v))^2 dv$. It suffices then to construct absolutely continuous distributions F_1 and F_2 over [0,2] which approximate $\overline{F}_1$ and $\overline{F}_2$.

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