REVENUE COMPARISON BETWEEN THE FIRST AND SECOND PRICE AUCTIONS IN A CLASS OF ASYMMETRIC EXAMPLES.

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An item is sold at auction to one of two bidders whose valuations are independently distributed over $[0,\ 1]$ according to $F_1(v) = v^\gamma$ and $F_2(v) = v^\delta$, with $0 \leq \gamma < \delta$ and $\gamma\delta \geq 1/2$. Under these assumptions, we prove that the first price auction gives the seller higher expected revenues than the second price auction. Such distributions can follow from collusion among bidders with identical and independent uniform valuation distributions.

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1.Introduction.

The first price auction is the selling procedure where the highest bidder is awarded the item and pays the price equal to his bid. In the second price auction, the highest bidder is still awarded the item but pays the second highest bid, that is, the highest rejected bid. We work in the "independent private value model" where the bidders' valuations of the item are privately known and independently distributed.

In the second price auction, bidding his true valuation is the unique weakly dominant strategy. There thus exists a unique equilibrium in weakly dominant strategies. In the first price auction, no weakly dominant strategy exists. In the symmetric case where the bidders' valuations are identically distributed, there exists a general mathematical expression for the unique Bayesian equilibrium. In this case, the celebrated "Revenue Equivalence Theorem" (see Riley and Samuelson, 1981) states that these equilibria of these auction procedures give the same expected revenues to the seller.

When the valuations are not identically distributed, the Revenue Equivalence Theorem does not hold true, there is no general ranking of the expected revenues (see Vickrey, 1961 and Maskin and Riley, 1995) and no general expression seems to exist for the equilibrium of the first price auction. For one of the rare natural asymmetric cases where it is possible to find closed mathematical formulas, see Griesmer, Levitan and Shubik (1967). Even the existence of the equilibrium is not always guaranteed. However if, for example, the valuations are absolutely distributed over the same interval according to at most two different valuation distributions with a relation of stochastic dominance between them, it is possible to prove the existence and uniqueness of the equilibrium (see Lebrun, 1996). Such a set of valuation distributions can follow from collusion into cartels by bidders with identical valuation distributions. A cartel which maximizes the sum of its members' payoffs, has total control over its members' bids and perfect information about his members' valuations, can be considered as one bidder whose valuation is the maximum of its members'. In fact, if it wins the cartel will allocate the item to its member with the highest valuation. The valuation distributions of the cartels will thus be powers of the same distribution. For example, if the bidders' valuations were initially distributed according to a uniform distribution $F(v) = v$, a cartel's valuation distribution will be $F^{\gamma}(v) = v^{\gamma}$, where γ is the number of its members.

Marshall, Meurer, Richard and Stromquist (1994) approximate numerically the equilibria of several examples where the two different distributions F_1 and F_2 are powers of the uniform distribution, that is, $F_1(v) = v^{\gamma}$ and $F_2(v) = v^{\delta}$, with $0 < \gamma < \delta$. In their examples, they found that the seller's revenue at the first price auction is strictly higher than at the second price auction. As simple examples by Maskin and Riley (1985) show, we cannot expect the same ranking to hold for all situations where $F_1 = F^\gamma$ and $F_2 = F^\delta$. Analytical and numerical results are also obtained by Maskin and Riley (1995).

At the equilibrium in weakly dominant strategies of the second price auction, the winner of the auction is the bidder with the highest valuation. In the case with two bidders, bidder 1 and bidder 2, and valuation distributions $F_1 = F^\gamma$ and $F_2 = F^\delta$, with $0 < \gamma < \delta$, the optimal auction procedure described by Myerson (1981) favors bidder 1, in the sense that a strictly higher valuation for bidder 2 is not a guarantee to win the auction. This valuation must be sufficiently higher.

At the equilibrium of the first price auction, bidder 1 bids more aggressively then bidder 2 does (see Maskin and Riley, 1995, and Lebrun, 1995 and 1996). For the same valuation, bidder 1's bid will be strictly larger than bidder 2's. Consequently, at the equilibrium of the first price auction bidder 1 and only he can win with a smaller valuation. The optimal auction procedure and the first price auction thus favor the bidder with the dominated valuation distribution. However the two procedures are not equivalent and the seller's revenue would be increased if, at the equilibrium of the first price auction and for large valuations of bidder 1, bidder 2's valuation needed not be as high above bidder 1's in order to win.

In this paper, we prove that the first price auction with two bidders still gives higher revenues than the second price auction when $F_1(v) = v^\gamma$ and $F_2(v) = v^\delta$, $0 < \gamma < \delta$, and when $\gamma\delta \geq 1/2$. In particular, the first price auction outperforms the second price auction when the bidders are cartels whose members valuations are distributed uniformly, that is, $\delta > \gamma \geq 1$. The uniform distribution may reflect the lack by cartel's members of precise information about the others' valuations.

The proof proceeds by bounding the bias at the equilibrium of the first price auction towards bidder 1 with high valuations, and showing that it is not substantial enough to offset the beneficial effect on the revenue of favoring bidder 1. If the first price auction equilibrium strategies are (β_1, β_2), bidder 2's v_2 has to be larger than $\phi(v_1) = \beta_2^{-1}\beta_1(v_1)$, where v_1 is bidder 1's valuation, in order to win. The bound on the bias is obtained by showing that the "elasticity" $\frac{d\ln\phi(v)}{d\ln v}$ is nonincreasing and not larger than 1. It turns out that for our choice of distributions, these properties are sufficient to establish the superiority of the first price auction.

Our proof is elementary. The properties of the function ϕ are derived by inspection of the differential equations the equilibrium strategies satisfy. The analytical integration of a mathematical expression then provides the ranking of the first and second price auctions.

2.The First and Second Price Auctions.

An item is sold at auction. Bidders 1 and 2 are the only two bidders. We denote bidder 1's valuation by v_1 and bidder 2's valuation by v_2. The bidders' valuations v_1, v_2 are independently distributed over $[0, 1]$ according to the couple of distributions $F_1(v) = v^\gamma$ and $F_2(v) = v^\delta$, with $0 < \gamma < \delta$. These distributions as well as the rules of the auction procedures are common knowledge. Only bidder i knows v_i. In both auction procedures every bidder has to submit a bid. The item goes to the highest bidder who pays a price equal to his bid in the first price auction and equal to his opponent's bid in the second price auction.

A strategy[1] β_i of bidder i defines a bid probability distribution $\beta_i(\text{v}, .)$, for every v in $[0, 1]$. A strategy β_i is pure if all the probability distributions $\beta_i(\text{v}, .)$ it defined are degenerate distributions. We identify a pure strategy with the bid function it determines. A Bayesian equilibrium is defined as usual. A pure Bayesian equilibrium is a couple of pure strategies which form a Bayesian equilibrium.

In the second price auction, although there exist many equilibria it is well known that the pure strategy Bayesian equilibrium (β, β) where $\beta(\text{v}) = \text{v}$, for all v in $[0, 1]$, is the only equilibrium in weakly dominant strategies. In this paper we always assume that the bidders follow the weakly dominant strategy in the second price auction. In Lemma 0 below, we state useful results pertaining to the first price auction and which follow from Lebrun (1995, 1996).

Lemma 0: Under our assumptions, there exists one and only one Bayesian equilibrium (β_1, β_2) of the first price auction. This unique equilibrium is pure and its strategies β_1 and β_2 are continuous and strictly increasing over $[0, 1]$ and such that

$$(1) \qquad \beta_1(0) = \beta_2(0) = 0, \beta_1(1) = \beta_2(1) < 1$$

and $\beta_1(v) > \beta_2(v)$, for all v in $(0, 1)$. Moreover the functions $\phi = \beta_2^{-1}\beta_1$ and β_1 form a solution over $(0, 1]$ of the following system of differential equations,

$$(2) \qquad \frac{d\phi}{dv}(v) = \frac{\gamma}{v} \frac{\phi(v)}{\delta} \frac{\phi(v) - \beta_1(v)}{v - \beta_1(v)}$$

$$(3) \qquad \frac{d\beta}{dv}(v) = \frac{\gamma}{v} (v - \beta_1(v)).$$

We also have the following bounds for the function ϕ,

$$(4) \qquad v < \phi(v) < v^{\gamma/\delta},$$

for all v in $(0, 1)$.

Proof: see Lebrun (1995, 1996).

The first inequality in (4) is equivalent to $\beta_1(\text{v}) > \beta_2(\text{v})$, for all v in $(0, 1)$. The second inequality in (4) is equivalent to $F_2(\beta_2^{-1}(\text{v})) < F_1(\beta_1^{-1}(\text{v}))$, for all v in $(0, 1)$. The probability distribution of the bid from bidder 2 thus stochastically dominates the distribution of the bid from bidder 1.

Let r(v) be the ratio $\frac{\phi(v)-\beta_1(v)}{v-\beta_1(v)}$, for v in $(0, 1]$. From (2), we see that an equivalent definition of r(v) is $r(v) = \frac{d\ln\phi^{\delta}(v)}{d\ln v^{\gamma}}$. In the next section, we will need the following properties of r that we derive in Appendix 1 from Lemma 0.

Lemma 6: Under our assumptions, r is nonincreasing and not larger than δ/γ over $(0, 1]$, or, equivalently, $\frac{d^2\ln\phi(v)}{d(\ln v)^2} \leq 0$ and $\frac{d\ln\phi(v)}{d\ln v} \leq 1$, for all v in $(0, 1]$.

Proof: see Appendix 1.

From Lemma 6 we see that $\ln\phi$ is a concave function of lnv and that the elasticity $\frac{d\ln\phi(v)}{d\ln v}$ is not larger than 1. The ranking between the first and second price auctions we find in the next section will follow from the properties of ϕ stated in Lemma 6.

3.Revenue Comparison.

We now state and prove our main result, Theorem 1 below.

Theorem 1: If $\gamma, \delta > 0$, $\gamma \neq \delta$ and $\gamma\delta \geq 1/2$, then we have $R_f > R_s$, where R_f is the seller's expected revenue at the unique Bayesian equilibrium of the first price auction and R_s is his expected revenue at the unique equilibrium in weakly dominant strategies of the second price auction when $F_1(v) = v^{\gamma}$ and $F_2(v) = v^{\delta}$.

The first price and second price auctions are particular examples of auction games described by Myerson (1981). From equation (4.12) in Myerson (1981), we see that, at the equilibrium of such a game with two players, the contribution of a couple of valuations (v_1, v_2) to the seller's expected revenue is the winner's "transformed valuation". Under our assumptions, bidder 1's transformed valuation is $t_1(v_1) = v_1 - \frac{(1-F_1(v_1))}{\frac{d}{dv}F_1(v_1)} = v_1 - \frac{(1-v_1^{\gamma})}{\gamma\, v_1^{\gamma-1}}$ and similarly bidder 2's is $t_2(v_2) = v_2 - \frac{(1-v_2^{\delta})}{\delta\, v_2^{\delta-1}}$. For an interpretation of these transformed valuations in terms of marginal revenues, see Bulow and Roberts (1989).

Neglecting ties, which occur with probability zero, the choices of the winner in the equilibria of the first and second price auctions differ only for the couples of valuations (v_1, v_2) such that $v_1 < v_2 < \phi(v_1)$. For these couples, the first price auction chooses bidder 1 as the winner while the second price auction chooses bidder 2. Consequently,

(5) $$R_f - R_s = \int_0^1\int_0^1 \{ t_1(v_1) - t_2(v_2) \} \, I\{v_1 < v_2 < \phi(v_1)\} \, dv_1^{\gamma} \, dv_2^{\delta},$$

where $I\{v_1 < v_2 < \phi(v_1)\}$ is equal to one if $v_1 < v_2 < \phi(v_1)$ and zero otherwise.

The difference $t_1(v_1) - t_2(v_2)$ may be positive or negative for (v_1, v_2) in $(0, 1]^2$. Consider the subsets B and C of $(0, 1]^2$ defined as follows, $B = \{ (v_1, v_2) \in (0, 1]^2 \mid t_1(v_1) - t_2(v_2) \leq 0 \}$ and $C = \{ (v_1, v_2) \in (0, 1]^2 \mid t_1(v_1) - t_2(v_2) \geq 0 \}$. The part of the boundary of the subsets B and C which is above the main diagonal is always the graph of a function. Let ψ be this function. It is thus the function defined over $(0, 1]$ such that

(6) $t_2(\psi(v_1)) = t_1(v_1)$ and $\psi(v_1) \geq v_1$,

for all $v_1 \in (0, 1]$. Actually, ψ is equal to $h_2^{-1} t_1$ where $h_2 = t_2$ if $\delta \geq 1$ and $h_2 = t_2 \mid_{(v(\delta),+\infty)}$, where $v(\delta) = \left(\frac{1-\delta}{1+\delta}\right)^{1/\delta}$, if $\delta < 1$ (see the properties of the functions t_i in Lemma 7 in Appendix 1). Notice that $\frac{d}{dv} h_2(v) > 0$, for all v in the definition domain of h_2. We thus see that ψ is differentiable over $(0, 1]$. Above the graph of ψ, that is, for $v_2 > \psi(v_1)$, we are in area B and we have $t_2(v_2) > t_1(v_1)$. Between the graph of ψ and the main diagonal, that is, for $v_1 < v_2 < \psi(v_1)$, we are in area C and we have $t_2(v_2) < t_1(v_1)$.

At $v_i = 1$, the derivative of t_i is equal to 2, for all i (see Lemma 7 in Appendix 1). It follows that the function ψ is tangent to the main diagonal at $v_1 = 1$. However, at this point the derivative of the function ϕ is equal to $\gamma/\delta < 1$ (see equation (2)) and consequently $\phi > \psi$ in a left neighborhood of the largest possible valuation 1. As we mentioned in the introduction, the seller's revenue would thus increase if bidder 1 did not win for high valuations of the bidders as often as he does at the eqilibrium of the first price auction. Nevertheless, as we will see below the reverse inequality $\phi < \psi$ holds true for low valuations of the bidders and the contributions of these low valuations to the revenues will result in higher revenues at the first price auction than at the second price auction.

In Appendix 1, we use the properties of the function ϕ proved in Lemma 6 (Section 2) to establish Lemma 11 below which implies that the functions ϕ and ψ cross at most once and are as in Figure 1.

Lemma 11: There exists w^* in $(0, 1)$ such that $\psi(v) \geq \phi(v)$, for all v in $[0, w^*]$, and $\psi(v) \leq \phi(v)$, for all v in $[w^*, 1]$.

Proof: see Appendix 1.

FIGURE[2] 1.

From Lemma 6 (Section 2), $\frac{d\ln\phi}{d\ln v} \leq 1$ and thus ϕ/v is nonincreasing. Also, from Lemma 0 (Section 2), $\phi(v) \leq v^{\gamma/\delta}$. Consequently, the function $\lambda(v) = \min\left(v^{\gamma/\delta}, \frac{\phi(w^*)}{w^*} v\right)$ (see Figure 1) crosses only once and from below the function ϕ. Moreover, if we replace ϕ by λ in (5), we will integrate the difference between the transformed valuations over a smaller area where this difference is nonnegative and over a larger area where it is nonpositive. We will thus obtain a lower bound of the difference $R_f - R_s$.

The function λ introduced in the previous paragraph is a particular case of a function of the form (7) below,

(7) $\mu(\mathrm{v}) \ = \ \min(\mathrm{v}^{\gamma/\delta}, \zeta\mathrm{v}),$

where $\zeta > 1$. Theorem 1 is thus an immediate consequence of Lemma 12 below.

Lemma 12: Under our assumptions, if $\gamma, \delta > 0$, $\gamma \neq \delta$ and $\gamma\delta \geq 1/2$, then

(8) $\int_0^1\int_0^1\{t_1(v_1) - t_2(v_2)\}I\{v_1 < v_2 < \mu(v_1)\}dv_1^{\gamma}dv_2^{\delta} \ > \ 0,$

for all functions μ of the from (8) where $\zeta > 1$.

Proof: See Appendix 2.

The expression (7) of the function μ is simple enough to enable us to obtain an analytical expression of the integral (8). As we show in Appendix 2, there exist $1 < \xi_1 < \xi_2$ such that this expression is strictly positive and strictly increasing in ζ over $(1, \xi_1]$, strictly decreasing over $[\xi_1, \xi_2]$ and strictly increasing over $[\xi_2, +\infty)$. We then complete the proof by showing that the value of the integral (8) at $\zeta = \xi_2$ is strictly positive when $\gamma\delta \ \geq \ 1/2$.

4.Conclusion.

Among the asymmetric examples we considered, the couples $\mathrm{F}_1(\mathrm{v}) \ = \ \mathrm{v}^{\gamma}$, $\mathrm{F}_2(\mathrm{v}) = \mathrm{v}^{\delta}$, with $1 \ \leq \ \gamma \ < \ \delta$, can be the results of collusion into two cartels by bidders whose valuations are uniformly and independently distributed. This uniform distribution can reflect the lack by bidders of precise information about the other bidders' valuations. For these asymmetric examples and for all $\mathrm{F}_1(\mathrm{v}) \ = \ \mathrm{v}^{\gamma}$, $\mathrm{F}_2(\mathrm{v}) = \mathrm{v}^{\delta}$ with $\gamma, \delta \ > 0$, $\gamma \neq \delta$ and $\gamma\delta \geq 1/2$, we proved the superiority of the first price auction over the second price auction.

Using elementary procedures, we were able to show the properties (stated in Lemma 6, Section 2) which guaranteed this superiority. Obviously, our results imply that, for our choices of distributions with F_1 and $\mathrm{F}_2 = \mathrm{F}_1^{\alpha}$, $\alpha > 1$, the second price auction is outperformed by any auction mechanism which shares these properties, that is, chooses bidder 2 as the winner if and only if $\mathrm{v}_2 > \phi(\mathrm{v}_1)$, where ϕ is a function such that $\mathrm{v} \leq \phi(\mathrm{v}) \leq \mathrm{F}_2^{-1}\mathrm{F}_1(\mathrm{v})$ and $\frac{\mathrm{dlnF_2}(\phi(\mathrm{v}))}{\mathrm{dlnF_1}(\mathrm{v})}$ is nonincreasing and not larger than α. We simply showed that the first price auction was such a mechanism.

Appendix 1.

Lemma 1: Under our assumptions, we have $\liminf_{v \to 0} \frac{\phi(v)}{v} \ \leq \ 1 + (1/\gamma).$

Proof: Since all functions involved are differentiable, we see that the function $v^\gamma(v-\beta_1(v))$ is differentiable over $(0,\, 1]$. Computing its derivative, substituting its value given in (3) to $\frac{d}{dv}\beta_1(v)$ and rearranging, we find

$$\frac{d}{dv}\, v^\gamma(v-\beta_1(v)) \;=\; v^\gamma \left\{ \gamma \left(1 - \frac{\phi(v)}{v}\right) \,+ 1 \right\},$$

for all v in $(0,\, 1]$. Since $v^\gamma(v-\beta_1(v)) \;\geq\; 0$, for all $v \;\geq\; 0$, and $v^\gamma(v-\beta_1(v)) \;=\; 0$, for all $v \;=\; 0$, we cannot have $\limsup\limits_{v \to o} \frac{\frac{d}{dv}\, v^\gamma(v-\beta_1(v))}{v^\gamma} \;<\; 0$. Consequently, $\liminf\limits_{v \to 0} \frac{\phi(v)}{v} \;\leq\; 1+\frac{1}{\gamma}$. ||

Lemma 2: Under our assumptions, the ratio $r(v) = \frac{\phi(v)-\beta_1(v)}{v-\beta_1(v)}$ *is differentiable over* $(0,\, 1]$ *and its derivative is equal to (A1) below,*

$$\textit{(A1)}\;\; \frac{d}{dv}\, r(v) \;=\; \frac{\phi(v)-\beta_1(v)}{(v-\beta_1(v))^2}\, \frac{\gamma}{v} \left\{ \left(1+\frac{1}{\delta}\right) \phi(v) - \;\left(1+\frac{1}{\gamma}\right) v \right\},$$

for all v in $(0,\, 1]$.

Proof: The differentiability follows immediately from the differentiability of all functions involved. By computing the derivative of r and substituting the expressions in (2) and (3) to the derivatives of ϕ and β_1, we find

$$\frac{d}{dv}\, r(v) \;=\; r(v) \left\{ \frac{1}{\phi(v)-\beta_1(v)} \left(\frac{\gamma}{v}\, \frac{\phi(v)}{\delta}\, \frac{\phi(v)-\beta_1(v)}{v-\beta_1(v)} \;-\; \frac{\gamma}{v}\, (\phi(v)-\beta_1(v)) \right) \;- \right.$$

$$\left. \frac{1}{v-\beta_1(v)} \left(1 \;-\; \frac{\gamma}{v}\, (\phi(v)-\beta_1(v)) \right) \right\},$$

for all v in $(0,\, 1]$. It suffices then to rearrange to obtain (A1). ||

Lemma 3: For every interior maximum v' of r, we have $v' \leq \frac{\gamma(\delta+1)}{(\gamma+1)\delta}$ *and* $r(v') \;\leq\; \delta/\gamma$.

Proof: From equation (2), we see that an alternative definition of r is as follows,

$$r(v) \;=\; \frac{d\ln\phi^\delta(v)}{d\ln v^\gamma},$$

for all v in $(0,\, 1]$. Suppose v' is an interior maximum of r. Thus $\frac{d}{dv} r(v') \;=\; 0$ and $\frac{d}{dv} r \;\leq\; 0$ in a right neighborhood of v' and $\frac{d}{dv} r \;\geq\; 0$ in a left neighborhood of v'. From equation (A1) , we see that $\phi(v') \;=\; \frac{\delta(\gamma+1)}{(\delta+1)\gamma} v'$, or, equivalently, $\ln\phi^\delta(v') \;=\; \delta\ln\frac{\delta(\gamma+1)}{(\delta+1)\gamma} \;+\; \delta\ln v'$, that $\phi(v) \;\leq\; \frac{\delta(\gamma+1)}{(\delta+1)\gamma} v'$, or, equivalently, $\ln\phi^\delta(v)) \;\leq\; \delta\ln\frac{\delta(\gamma+1)}{(\delta+1)\gamma} \;+\; \delta\ln v$ in a right neighborhood of v' and that $\phi(v) \;\geq\; \frac{\delta(\gamma+1)}{(\delta+1)\gamma} v'$, or, equivalently, $\ln\phi^\delta(v)) \;\geq\; \delta\ln\frac{\delta(\gamma+1)}{(\delta+1)\gamma} \;+$

δlnv in a left neighborhood of v'. Consequently, $\frac{\mathrm{dln}\phi^{\delta}}{\mathrm{dlnv}^{\gamma}}(\mathrm{v'}) \leq \frac{\delta}{\gamma}$. The inequality on v' follows from $\phi(\mathrm{v'}) \leq 1$. ||

<u>Lemma</u> <u>4</u>: Under our assumptions, $r(v) \leq \delta/\gamma$, for all v in $(0, 1]$.

<u>Proof</u>: Let ρ be the supremum of r. If this supremum is interior, Lemma 3 implies Lemma 4. If the supremum is not interior, it cannot be reached at 1, since from (A1) we have $\frac{\mathrm{d}}{\mathrm{dv}}\mathrm{r}(1) < 0$. Consequently, we have $\limsup_{v\to 0} \mathrm{r(v)} = \rho$. Suppose that $\rho > \delta/\gamma$. There exists u > 0 such that $\mathrm{r(u)} > \delta/\gamma$. Then $\mathrm{r(v)} > \mathrm{r(u)}$, for all v in $(0, \mathrm{u}]$. Otherwise there would exist w $<$ u such that $\mathrm{r(w)} < \mathrm{r(u)}$, and the maximum of r over $[\mathrm{w}, 1]$ would be interior and its value would be strictly larger than δ/γ, a contradiction with Lemma 3.

From equation (2), we see that $\frac{\mathrm{d}}{\mathrm{dv}} \ln\phi^{\delta}(\mathrm{v})) \geq \mathrm{r(u)} \frac{\mathrm{d}}{\mathrm{dv}} \ln\mathrm{v}^{\gamma}$, for all v in $(0, \mathrm{u})$. Integrating this inequality from v in $(0, \mathrm{u})$ to u, rearranging and dividing by v^{δ} give

$$\frac{\phi^{\delta}(\mathrm{u})}{\mathrm{u}^{\gamma r(u)}} \frac{\mathrm{v}^{\gamma r(u)}}{\mathrm{v}^{\delta}} \geq \frac{\phi^{\delta}(\mathrm{v})}{\mathrm{v}^{\delta}},$$

for all v in $(0, \mathrm{u})$. If we make v in this inequality tend towards zero, the first ratio in the left hand side is a constant. The second ratio tends towards zero and thus also the right hand side tend towards zero. However this is impossible since $\phi(\mathrm{v}) \geq \mathrm{v}$ (see (4) in Lemma 0) and thus $\frac{\phi^{\delta}(\mathrm{v})}{\mathrm{v}^{\delta}} \geq 1$, for all v. ||

<u>Lemma</u> <u>5</u>: Under our assumptions, $\limsup_{v\to o} r(v) \geq \delta/\gamma$.

<u>Proof</u>: Suppose $\limsup_{v\to 0} \mathrm{r(v)} < \delta/\gamma$. Thus there exists v' in $(0, 1]$ and $\rho' < \delta/\alpha$ such that $\mathrm{r(v)} < \rho'$, for all $\mathrm{v} \in (0, \mathrm{v'}]$. From equation (2), we see that $\frac{\mathrm{d}}{\mathrm{dv}} \ln\phi^{\delta}(\mathrm{v}) \leq \rho' \frac{\mathrm{d}}{\mathrm{dv}} \ln\mathrm{v}^{\gamma}$, for all v in $(0, \mathrm{v'}]$. Integrating this inequality from v to v', taking the exponential and rearranging give

$$\text{(A2)} \qquad \frac{\phi^{\delta}(\mathrm{v'})}{\mathrm{v'}^{\gamma\rho'}} \leq \frac{\phi^{\delta}(\mathrm{v})}{\mathrm{v}^{\delta}} \mathrm{v}^{\delta-\gamma\rho'}.$$

From Lemma 1, there exists a sequence of $\mathrm{v} \to 0$ such that the first factor of the right hand side stays bounded. Since $\delta > \gamma\rho'$, for this sequence the R.H.S. of the inequality (A2) tends towards 0. However this is impossible since the L.H.S. is a strictly positive constant. ||

<u>Lemma</u>[3] <u>6</u>: Under our assumptions, r is nonincreasing and not larger than δ/γ over $(0, 1]$, or, equivalently, $\frac{d^2 ln\phi^{\delta}(v)}{d(lnv^{\gamma})^2} \leq 0$ and $\frac{dln\phi^{\delta}(v)}{dlnv^{\gamma}} \leq 1$, for all v in $(0, 1]$.

<u>Proof</u>: From Lemma 2 we see that r is nonincreasing over $(0, 1]$ if and only if $\phi(\mathrm{v}) \leq \frac{\delta(\gamma+1)}{(\delta+1)\gamma}\mathrm{v}$, for all v in $(0, \overline{\mathrm{w}}\,]$ with $\overline{\mathrm{w}} = \frac{\gamma(\delta+1)}{(\gamma+1)\delta}$. Since $1 > \phi(\overline{\mathrm{w}}\,)$, this inequality holds

true in a neighborhood of $\overline{w}$. Suppose it does not always hold true. Thus there exists w' such that $\ln\phi^{\delta}(w') > \ln\lambda^{\delta}(w')$, that is, $\ln\phi^{\delta}(w') - \ln\lambda^{\delta}(w') > 0$, where $\lambda(v) = \frac{\delta(\gamma+1)}{(\delta+1)\gamma}v$, for all v in $(0, 1]$. By continuity there exists $w'' > w'$ such that $\ln\phi^{\delta}(w'') - \ln\lambda^{\delta}(w'') = 0$. By the mean value theorem, there exists w in (w', w'') such that $\left(\frac{d}{d\ln v^{\gamma}} (\ln\phi^{\delta}(v) - \ln\lambda^{\delta}(v)) \right)_w = \frac{0-(\ln\phi^{\delta}(w')-\ln\lambda^{\delta}(w'))}{\ln w''^{\gamma}-\ln w'^{\gamma}} < 0$, and thus $\left(\frac{d\ln\phi^{\delta}(v)}{d\ln v^{\gamma}}\right)_w < \left(\frac{d\ln\lambda^{\delta}(v)}{d\ln v^{\gamma}}\right)_w$.

Since $\ln\phi^{\delta}$ is convex in $\ln v^{\gamma}$ when $\ln\phi^{\delta} \geq \ln\lambda^{\delta}$ and $\ln\lambda^{\delta}$ is affine in $\ln v^{\gamma}$, we see that $\ln\phi^{\delta} > \ln\lambda^{\gamma}$, over (0, w), and $\frac{d}{d\ln v^{\gamma}} \left(\frac{d\ln\lambda^{\delta}(v)}{d\ln v^{\gamma}} - \frac{d\ln\phi^{\delta}(v)}{d\ln v^{\gamma}} \right) < 0$, for all v in (0, w). Consequently, $\lim_{v\to 0} r(v) = \lim_{v\to 0} \frac{d\ln\phi^{\delta}(v)}{d\ln v^{\gamma}}$ exists and is strictly smaller than $\frac{d\ln\lambda^{\delta}(v)}{d\ln v^{\gamma}} = \delta/\gamma$, a contradiction with Lemma 5. We have thus proved that $\phi \leq \lambda$ over $(0, \overline{w}\,]$ and that r is nonincreasing over $(0, 1]$. Moreover, Lemma 4 implies that $\sup r \leq \delta/\gamma$. ||

From Lemma 6, we thus see that $\ln\phi$ is a concave function of lnv. Lemma 6 also implies that ϕ is a concave function of v. In fact, $r(v)\gamma/\delta = \frac{d\ln\phi(v)}{d\ln v} \leq 1$ implies that $\phi(v)/v$ is nonincreasing. Moreover, $\frac{d\phi(v)}{dv} = \delta/\gamma \, \frac{\phi(v)}{v} r(v)$ and all factors in the R.H.S. are strictly positive nonincreasing functions over $(0, 1]$. Another consequence of Lemmas 5 and 6 is that $\lim_{v\to 0} r(v)$ exists and is equal to δ/γ.

<u>Lemma</u> <u>7</u>: *Let* $t(v, \epsilon)$ *be equal to* $v - \frac{(1-v^{\epsilon})}{\epsilon v^{\epsilon-1}}$, *for all v in* $(0, 1]$ *and* ϵ *in* $(0, +\infty)$. *Then,* $t(v, \epsilon)$ *is strictly decreasing in* ϵ *over* $(0, 1] \times (0, +\infty)$, *and is such that* $\frac{\partial}{\partial v}t(v, \epsilon) > 0$, *for all* (v, ϵ) *in* $\left((0, 1] \times [1, +\infty) \right) \cup \left((\left(\frac{1-\epsilon}{1+\epsilon}\right)^{1/\epsilon}, 1] \times (0, 1) \right)$, *and* $\frac{\partial}{\partial v}t(v, \epsilon) < 0$, *for all* (v, ϵ) *in* $(0, \left(\frac{1-\epsilon}{1+\epsilon}\right)^{1/\epsilon}) \times (0, 1)$. *Moreover,* $\frac{\partial}{\partial v}t(v, \epsilon) = t(v, \epsilon)/v + 1/v^{\epsilon}$, *for all* (v, ϵ) *in* $(0, 1] \times (0, +\infty)$.

<u>Proof</u>: From the definition of $t(v, \epsilon)$, we have $\frac{\partial}{\partial \epsilon}t(v, \epsilon) = \frac{(1+\ln v^{\epsilon})-v^{\epsilon}}{\epsilon^2 v^{\epsilon-1}} < 0$, for (v, ϵ) in $(0, 1) \times (0, +\infty)$, and $\frac{\partial}{\partial v}t(v, \epsilon) = \frac{(\epsilon+1)v^{\epsilon}-(\epsilon-1)}{\epsilon v^{\epsilon}}$, for (v, ϵ) in $(0, 1] \times (0, +\infty)$. Lemma 7 follows. ||

<u>Lemma</u> <u>8</u>: *There exists* $\tilde{v}$ *in* $(0, 1)$ *such that* $\psi(v) \geq v^{\gamma/\delta}$, *for all v in* $(0, \tilde{v}]$, *and* $\psi(v) \leq v^{\gamma/\delta}$, *for all v in* $[\tilde{v}, 1)$.

<u>Proof</u>: From the definition of ψ, we obtain $\frac{d\psi}{dv}(1) = \frac{dt_2}{dv}(1) \;/\; \frac{dt_1}{dv}(1) = 2 \;/\; 2 = 1$. Consequently, $\psi(v) < v^{\gamma/\delta}$, in an open interval (u, 1) with $u < 1$. We see that Lemma 8 will be proved if we prove that there exists one and only one w in (0, 1) such that $\psi(w) = w^{\gamma/\delta}$. Again from the definition of ψ, we have $\psi(w) - \frac{1-\psi(w)^{\delta}}{\delta\psi(w)^{\delta-1}} = 1 - 1 - w^{\gamma\gamma w^{\gamma-1}}$. Substituting $\psi(w)$ by $w^{\gamma/\delta}$ in this equation and rearranging, we see that w is

the solution of $\frac{\gamma}{\delta}\ \mathrm{w}^{(\gamma/\delta)-1} = \frac{(\gamma+1)\mathrm{w}^{\delta}-1}{(\delta+1)\mathrm{w}^{\gamma}-1}$. The L.H.S. is a strictly decreasing function which tends towards $+\infty$ and γ/δ for w tending towards 0 and 1, respectively. The R.H.S. is a strictly increasing function where it is defined. It tends towards 1 and $+\infty$ when v tends in $(0, \left(\frac{1}{1+\delta}\right)^{1/\gamma})$ towards 0 and $\left(\frac{1}{1+\delta}\right)^{1/\gamma}$, respectively, and it tends towards $-\infty$ and γ/δ when v tends in $(\left(\frac{1}{1+\delta}\right)^{1/\gamma}, 1)$ towards in $\left(\frac{1}{1+\delta}\right)^{1/\gamma}$ and 1, respectively. Consequently, there exists[4] one and only one such w. ||

<u>Lemma</u> <u>9</u>: For all v in the interval where ψ is not larger than $v^{\gamma/\delta}$ and in the interior of the interval where ψ is strictly increasing[5], when $\delta > 1$ then $\frac{d^2 ln\psi^{\delta}(v)}{d(lnv^{\gamma})^2} \geq 0$ if $\frac{dln\psi^{\delta}(v)}{dlnv^{\gamma}} \geq \frac{\delta(\gamma-1)}{\gamma(\delta-1)}$, when $\delta < 1$ then $\frac{d^2 ln\psi^{\delta}(v)}{d(lnv^{\gamma})^2} \geq 0$ if $\frac{dln\psi^{\delta}(v)}{dlnv^{\gamma}} \leq \frac{\delta(\gamma-1)}{\gamma(\delta-1)}$ and when $\delta = 1$ then $\frac{d^2 ln\psi^{\delta}(v)}{d(lnv^{\gamma})^2} \geq 0$.

<u>Proof</u>: Differentiating $\mathrm{t}_1(\mathrm{v}) = \mathrm{t}_2(\psi(\mathrm{v}))$, and using Lemma 7, we see that $\frac{\mathrm{dln}\psi^{\delta}(\mathrm{v})}{\mathrm{dlnv}^{\gamma}} = \frac{\delta}{\gamma} \frac{\mathrm{t}_1(\mathrm{v})+(1/\mathrm{v}^{\gamma-1})}{\mathrm{t}_1(\mathrm{v})+(1/\psi(\mathrm{v})^{\delta-1})}$. By rearranging we obtain

$$\text{(A3)} \quad \frac{\mathrm{dln}\psi^{\delta}(\mathrm{v})}{\mathrm{dlnv}^{\gamma}} = \frac{\delta}{\gamma}\left\{ 1 - \frac{1-\frac{\psi(\mathrm{v})^{\delta-1}}{\mathrm{v}^{\gamma-1}}}{1+\mathrm{t}_2(\psi(\mathrm{v}))\psi(\mathrm{v})^{\delta-1}} \right\}.$$

By assumption, $\psi(\mathrm{v})^{\delta}/\mathrm{v}^{\gamma} \leq 1$. By dividing this inequality by $\psi(\mathrm{v})/\mathrm{v} > 1$ we find $\frac{\psi(\mathrm{v})^{\delta-1}}{\mathrm{v}^{\gamma-1}} < 1$ and the numerator of the fraction in (A3) is nonnegative. The function ψ and thus the function $\mathrm{t}_2 \circ \psi = \mathrm{t}_1$ (see the definition (6) of ψ in Section 3 and see Lemma 7) are strictly increasing over a neighborhood of v and thus the denominator is stcirlty increasing around v. Consequently $\frac{\mathrm{dln}\psi^{\delta}(\mathrm{v})}{\mathrm{dlnv}^{\gamma}}$ will be strictly increasing over a neighborhood of v if $\frac{\mathrm{d}}{\mathrm{dv}} \frac{\psi(\mathrm{v})^{\delta-1}}{\mathrm{v}^{\gamma-1}} \geq 0$.

When $\delta = 1$, the inequality $\frac{\mathrm{d}}{\mathrm{dv}} \frac{\psi(\mathrm{v})^{\delta-1}}{\mathrm{v}^{\gamma-1}} \geq 0$ is immediate. The rest of Lemma 9 then follows by computing the derivative in this inequality and by rearranging. ||

<u>Lemma</u> <u>10</u>: If $\delta \leq 1$ or if $\gamma \leq 1 < \delta$ then $\frac{d^2 ln\psi^{\delta}(v)}{d(lnv^{\gamma})^2} \geq 0$, for all v in the interval where ψ is nondecreasing and not larger than $v^{\gamma/\delta}$.

<u>Proof</u>: When $\delta = 1$, Lemma 10 restates Lemma 9. Suppose $\delta < 1$. Then $\frac{\delta(\gamma-1)}{\gamma(\delta-1)} = \frac{\delta}{\gamma}\frac{1-\gamma}{1-\delta} > \frac{\delta}{\gamma} = \alpha$, since $\gamma < \delta$. From (A3), $\left(\frac{\mathrm{dln}\psi^{\delta}(\mathrm{v})}{\mathrm{dlnv}^{\gamma}}\right)_{v=1} = \delta/\gamma$. It suffices then to apply Lemma 9 to prove Lemma 10 in this case. If $\gamma \leq 1 < \delta$ then $\frac{\delta(\gamma-1)}{\gamma(\delta-1)} \leq 0$. However $\frac{\mathrm{dln}\psi^{\delta}(\mathrm{v})}{\mathrm{dlnv}^{\gamma}} \geq 0$, for all v in $(0, 1]$, and Lemma 9 again implies Lemma 10. ||

<u>Lemma</u> <u>11</u>: There exists w^* in $(0, 1)$ such that $\psi(v) \geq \phi(v)$, for all v in $[0, w^*]$, and $\psi(v) \leq \phi(v)$, for all v in $[w^*, 1]$.

Proof: We know that $\phi > \psi$ over $(1-\epsilon, 1)$, for a $\epsilon > 0$. From Lemma 8 and the second inequality in (4), $\phi < \psi$ over $(0, \epsilon')$, for a $\epsilon' > 0$. There thus exists at least one point in $(0, 1)$ where ϕ is equal to ψ. If $\delta \leq 1$ or if $\gamma \leq 1 < \delta$, Lemma 10 implies that $\ln\psi$, over the interval where it is nondecreasing, is a convex function of lnv. Lemma 11 then follows from the concavity of $\ln\phi$ with respect to lnv (see Lemma 6) and the fact that $\ln\phi$ is strictly increasing.

Let γ and δ be strictly larger than 1. Let w^* be the largest w in $(0, 1)$ such that $\phi(w) = \psi(w)$. From Lemma 7 and the definition of ψ, ψ is striclty increasing.

If $\left(\frac{d\ln\psi^\delta(v)}{d\ln v^\gamma}\right)_{w^*} \leq \frac{\delta(\gamma-1)}{\gamma(\delta-1)}$, then $\frac{d\ln\psi^\delta(v)}{d\ln v^\gamma} \leq \frac{\delta(\gamma-1)}{\gamma(\delta-1)}$, for all v in $(0, w^*)$. Otherwise, from the mean value theorem there would exist u in $(0, 1)$ such that $\left(\frac{d^2\ln\psi^\delta(v)}{d(\ln v^\gamma)^2}\right)_u < 0$ and $\left(\frac{d\ln\psi^\delta(v)}{d\ln v^\gamma}\right)_u \geq \frac{\delta(\gamma-1)}{\gamma(\delta-1)}$, which would contradict Lemma 9. If $\left(\frac{d\ln\psi^\delta(v)}{d\ln v^\gamma}\right)_{w^*} > \frac{\delta(\gamma-1)}{\gamma(\delta-1)}$, Lemma 9 similarly implies that $\frac{d\ln\psi^\delta(v)}{d\ln v^\gamma} \leq \left(\frac{d\ln\psi^\delta(v)}{d\ln v^\gamma}\right)_{w^*}$, for all v in $(0, w^*)$. Thus, in general, $\frac{d\ln\psi^\delta(v)}{d\ln v^\gamma} \leq \max\left\{\left(\frac{d\ln\psi^\delta(v)}{d\ln v^\gamma}\right)_{w^*}, \frac{\delta(\gamma-1)}{\gamma(\delta-1)}\right\}$, for all v in $(0, w^*)$. From the definition of w^*, it follows that $\left(\frac{d\ln\psi^\delta(v)}{d\ln v^\gamma}\right)_{w^*} \leq r(w^*) = \left(\frac{d\ln\phi^\delta(v)}{d\ln v^\gamma}\right)_{w^*}$. Since r is nonincreasing and $r(1) = 1$, we have $r(w^*) \geq 1$. However, $\frac{\delta(\gamma-1)}{\gamma(\delta-1)} \leq 1$ and thus $\max\left\{\left(\frac{d\ln\psi^\delta(v)}{d\ln v^\gamma}\right)_{w^*}, \frac{\delta(\gamma-1)}{\gamma(\delta-1)}\right\} \leq r(w^*)$. Again r is nonincreasing and we find that $\frac{d\ln\psi^\delta(v)}{d\ln v^\gamma} \leq r(w^*) \leq r(v) = \frac{d\ln\phi^\delta(v)}{d\ln v^\gamma}$, for all v in $(0, w^*)$. Consequently, $\ln\psi^\delta(v) \leq \ln\phi^\delta(v)$, for all v in $(0, w^*)$, and Lemma 11 is proved. ‖

Appendix 2.

Proof of Lemma 12: We first change the variables of integration in $\int_0^1\int_0^1 \{t_1(v_1) - t_2(v_2)\}\, I\{v_1 < v_2 < \mu(v_1)\}\, dv_1^\gamma dv_2^\delta$ from (v_1, v_2), with $v_1 > 0$ and $v_2 \geq v_1$ to $(v_1, \xi) = (v_1, v_2/v_1)$. The inverse of this change is simply given by $(v_1, v_2) = (v_1, \xi v_1)$, for $v_1 > 0$ and $\xi \geq 1$. Through this change of variables, the integral in (8) becomes,

$$\text{(A4)} \int_1^\zeta \int_0^{h(\xi)} \{t_1(v_1) - t_2(\xi v_1)\}\, \gamma v_1^{\gamma-1}\, \delta(\xi v_1)^{\delta-1} \left| \frac{\partial(v_1, v_2)}{\partial(v_1, \xi)} \right| dv_1\, d\xi,$$

where $\left|\frac{\partial(v_1, v_2)}{\partial(v_1, \xi)}\right|$ is the absolute value of the determinant of the Jacobian of the change of variables and $h(\xi)$ is such that $\xi h(\xi) = h(\xi)^{\gamma/\delta}$, that is,

$$\text{(A5)} \quad h(\xi) = \xi^{\frac{\delta}{\gamma-\delta}}.$$

The absolute value of the determinant $\left|\frac{\partial(v_1, v_2)}{\partial(v_1, \xi)}\right|$ is equal to $\begin{vmatrix} 1 & 0 \\ \xi & v_1 \end{vmatrix} = v_1$. By substituting its value to this determinant and their values to $t_1(v_1)$ and $t_2(\xi v_1)$, the inner integral in (A36) becomes $\int_0^{h(\xi)}$ $\{$ $((\gamma+1)\mathrm{v}_1^{\gamma}-1)$ $\delta\xi^{\delta-1}\mathrm{v}_1^{\delta}$ $-$ $((\delta+1)\xi^{\delta}\mathrm{v}_1^{\delta}-1)$ $\gamma\mathrm{v}_1^{\gamma}$ $\}$ dv_1, or, equivalently, $\int_0^{h(\xi)}$ $(\gamma+1)\delta\xi^{\delta-1}\mathrm{v}_1^{\gamma+\delta} -$ $\delta\xi^{\delta-1}\mathrm{v}_1^{\delta}$ $-$ $(\delta+1)\gamma\xi^{\delta}\mathrm{v}_1^{\gamma+\delta} +$ $\gamma\mathrm{v}_1^{\gamma}$ dv_1. It is thus equal to the difference between the values at $\mathrm{h}(\xi)$ and 0 of the function $\{$ $\frac{(\gamma+1)\delta\xi^{\delta-1}}{\gamma+\delta+1}\mathrm{v}_1^{\gamma+\delta+1}$ $-$ $\frac{\delta\xi^{\delta-1}}{\delta+1}\mathrm{v}_1^{\delta+1}$ $-$ $\frac{(\delta+1)\gamma\xi^{\delta}}{\gamma+\delta+1}\mathrm{v}_1^{\gamma+\delta+1}$ $+$ $\frac{\gamma}{\gamma+1}\mathrm{v}_1^{\gamma+1}$ $\}$. Using the value of $\mathrm{h}(\xi)$ given in (A5), we then find the following expression for the value of the inner integral in (A4),

$$\text{(A6)}\quad \frac{(\gamma+1)\delta}{\gamma+\delta+1}\,\xi^{\delta-1+\frac{(\gamma+\delta+1)\delta}{\gamma-\delta}} \;-\; \frac{\delta}{\delta+1}\,\xi^{\delta-1+\frac{(\delta+1)\delta}{\gamma-\delta}}$$

$$-\;\frac{(\delta+1)\gamma}{\gamma+\delta+1}\,\xi^{\delta+\frac{(\gamma+\delta+1)\delta}{\gamma-\delta}} \;+\; \frac{\gamma}{\gamma+1}\,\xi^{\frac{(\gamma+1)\delta}{\gamma-\delta}}.$$

The exponent $\delta-1+\frac{(\gamma+\delta+1)\delta}{\gamma-\delta}$ of ξ in the first term of the expression (A6) is equal to $\frac{2\gamma\delta+2\delta-\gamma}{\gamma-\delta}$. The exponent $\delta-1+\frac{(\delta+1)\delta}{\gamma-\delta}$ of ξ in the second term is equal to $\frac{\gamma\delta+2\delta-\gamma}{\gamma-\delta}$. The exponent $\delta+\frac{(\gamma+\delta+1)\delta}{\gamma-\delta}$ of ξ in the third term is equal to $\frac{2\gamma\delta+\delta}{\gamma-\delta}$. The exponent of ξ in the last term is simply equal to $\frac{\gamma\delta+\delta}{\gamma-\delta}$. By dividing each term by $\xi^{\frac{\gamma\delta+\delta}{\gamma-\delta}}$, we see that the expression in (A6) is equal to (A7) below,

$$\text{(A7)}\;\xi^{\frac{\gamma\delta+\delta}{\gamma-\delta}}\left\{\;\frac{(\gamma+1)\delta}{\gamma+\delta+1}\,\xi^{\frac{\gamma\delta+\delta-\gamma}{\gamma-\delta}} \;-\; \frac{\delta}{\delta+1}\,\xi^{-1} \;-\; \frac{(\delta+1)\gamma}{\gamma+\delta+1}\,\xi^{\frac{\gamma\delta}{\gamma-\delta}} \;+\; \frac{\gamma}{\gamma+1}\;\right\}.$$

The first and third terms of the expression between braces in (A7) above, if they are grouped together, are equal to $\frac{1}{\gamma+\delta+1}$ $\xi^{\frac{\gamma\delta}{\gamma-\delta}}$ $\{$ $(\gamma+1)\delta\xi^{-1}$ $-$ $(\delta+1)\gamma$ $\}$. The second and fourth terms, grouped together, are equal to $-$ $\frac{1}{(\gamma+1)(\delta+1)}$ $\{$ $(\gamma+1)\delta\xi^{-1}$ $-$ $(\delta+1)\gamma$ $\}$. Consequently, we see that (A7) and thus the inner integral in (A4) are equal to (A8) below,

$$\text{(A8)}\;\frac{\xi^{\frac{\gamma\delta+\delta}{\gamma-\delta}}}{(\gamma+\delta+1)(\gamma+1)(\delta+1)}\;\{(\gamma+1)(\delta+1)\xi^{\frac{\gamma\delta}{\gamma-\delta}}-(\gamma+\delta+1)\}\;\{(\gamma+1)\delta\xi^{-1}-(\delta+1)\gamma\}.$$

We now study the sign of the expression (A8), that is, of the inner integral in (A4). We notice that the first factor in (A8) is strictly positive in the range $\xi \geq 1$. The second factor is equal to zero at $\left(\frac{(\gamma+1)(\delta+1)}{(\gamma+\delta+1)}\right)^{\frac{\delta-\gamma}{\gamma\delta}}$, is strictly positive for all ξ smaller than this value and is strictly negative for all other ξ. The third factor is also strictly positive, then strictly negative, but the change of sign occurs at the root $\frac{(\gamma+1)\delta}{(\delta+1)\gamma}$. Consequently, we see that the set of roots of (A8) in the range $\xi > 1$, is equal to

$$\{\;\xi_1,\xi_2\;\}\;=\;\{\;\left(\frac{(\gamma+1)(\delta+1)}{(\gamma+\delta+1)}\right)^{\frac{\delta-\gamma}{\gamma\delta}},\;\frac{(\gamma+1)\delta}{(\delta+1)\gamma}\;\}.$$

Assume that ξ_1 denotes the smallest of the roots, that is, $\xi_1 \;\leq\; \xi_2$. Then, the expression (A8) is strictly positive over $[1, \xi_1)$ and $(\xi_2, \;+\infty)$, and is strictly negative over (ξ_1, ξ_2). Thus the (outer) integral (A4) considered as a function of $\xi > 1$, is strictly positive and strictly increasing over $(1, \;\xi_1)$, then is strictly decreasing over $(\xi_1, \;\xi_2)$ and is again strictly increasing over $(\xi_2, \;+\infty)$. We will thus have proved Lemma 12 if we prove that the integral (A4) is strictly positive at ξ_2.

Suppose[6] first that the maximal root ξ_2 is equal to the root $\frac{(\gamma+1)\delta}{(\delta+1)\gamma}$ of the third factor in (A8). In this case, the third factor in (A8) is nonnegative over the interval $[0, \xi_2]$. Consider the function f defined as the ratio of this factor and $\xi^{\frac{\gamma\delta+\delta}{\gamma-\delta}}$, that is,

$$\text{(A9)} \qquad \mathrm{f}(\xi) \;=\; \frac{(\gamma+1)\delta\xi^{-1}-(\delta+1)\gamma}{\xi^{\frac{\gamma\delta+\delta}{\gamma-\delta}}},$$

or, equivalently, $\mathrm{f}(\xi) \;=\; (\gamma+1)\delta\xi^{\frac{\gamma\delta+\delta}{\delta-\gamma}-1} - (\delta+1)\gamma\xi^{\frac{\gamma\delta+\delta}{\delta-\gamma}}$, for all $\xi > 1$. Over the interval $[0, \xi_2]$, the function f is nonnegative. Moreover, if we take the derivative of f, we find

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\mathrm{f}(\xi) \;=\; \xi^{\frac{\gamma\delta+\delta}{\delta-\gamma}-2}\left\{(\gamma+1)\delta\left(\frac{\gamma\delta+\delta}{\delta-\gamma}-1\right) - (\delta+1)\gamma\frac{\gamma\delta+\delta}{\delta-\gamma}\xi\right\}.$$

Since $\xi > 1$, the factor between braces in the R.H.S. of the equality above is strictly smaller than $(\delta-\gamma)\left(\frac{\gamma\delta+\delta}{\delta-\gamma}\right) - (\delta+1)\gamma \;=\; 0$, and the derivative of f is strictly negative over the domain $\xi > 1$, and thus f is strictly decreasing over this domain. Consequently, if we show that the integral $\int_1^{\xi_2} \xi^{2\frac{(\gamma\delta+\delta)}{\gamma-\delta}} \{(\gamma+1)(\delta+1)\xi^{\frac{\gamma\delta}{\gamma-\delta}} - (\gamma+\delta+1)\}\,\mathrm{d}\xi$ is strictly positive, we will be able to obtain $\int_1^{\xi_2} \xi^{\frac{\gamma\delta+\delta}{\gamma-\delta}} \{(\gamma+1)(\delta+1)\xi^{\frac{\gamma\delta}{\gamma-\delta}} - (\gamma+\delta+1)\}$ $\{(\gamma+1)\delta\xi^{-1} - (\delta+1)\gamma\}\,\mathrm{d}\xi \;>\; 0$ and thus to prove Lemma 12, by applying Lemma 13 (Appendix 3) to f defined in (A9) and g defined in (A10) below,

$$\text{(A10)} \quad \mathrm{g}(\xi) \;=\; \xi^{2\frac{(\gamma\delta+\delta)}{\gamma-\delta}} \{(\gamma+1)(\delta+1)\xi^{\frac{\gamma\delta}{\gamma-\delta}} - (\gamma+\delta+1)\},$$

for all $\xi > 1$.

In this paragraph, we prove that the integral of g defined in (A10) from 1 to ξ_2 is strictly positive. We actually show that $\int_1^{\xi} \mathrm{g}(\xi)\,\mathrm{d}\xi \;>\; 0$, for all $\xi > 1$. The function g can also be written as

$$(\gamma+1)(\delta+1)\xi^{\frac{3\gamma\delta+2\delta}{\gamma-\delta}} - \;(\gamma+\delta+1)\xi^{\frac{2\gamma\delta+2\delta}{\gamma-\delta}},$$

and the integral $\int_1^{\xi} \mathrm{g}(\xi)\,\mathrm{d}\xi$ is thus equal to (A11) below

$$\text{(A11)} \quad \int_1^{\xi} \mathrm{g}(\xi)\,\mathrm{d}\xi \;=\; \left\{\; \frac{(\gamma+1)(\delta+1)(\gamma-\delta)}{3\gamma\delta+\gamma+\delta}\;\xi^{\frac{3\gamma\delta+\gamma+\delta}{\gamma-\delta}} \;-\; \frac{(\gamma+\delta+1)(\gamma-\delta)}{2\gamma\delta+\gamma+\delta}\;\xi^{\frac{2\gamma\delta+\gamma+\delta}{\gamma-\delta}} \;\right\}$$

$$- \left\{ \tfrac{(\gamma+1)(\delta+1)(\gamma-\delta)}{3\gamma\delta+\gamma+\delta} - \tfrac{(\gamma+\delta+1)(\gamma-\delta)}{2\gamma\delta+\gamma+\delta} \right\}.$$

The function g (see (A10)) is strictly positive for ξ strictly smaller than $\left(\frac{(\gamma+1)(\delta+1)}{(\gamma+\delta+1)}\right)^{\frac{\delta-\gamma}{\gamma\delta}}$ and strictly negative for ξ strictly larger. Thus the integral (A11) is strictly increasing and strictly positive before $\left(\frac{(\gamma+1)(\delta+1)}{(\gamma+\delta+1)}\right)^{\frac{\delta-\gamma}{\gamma\delta}}$ and strictly decreasing after. As a consequence, we will have proved $\int_1^\xi \mathrm{g}(\xi)\ \mathrm{d}\xi \ > \ 0$, for all $\xi > 1$, if we prove that the limit of this integral for $\xi \rightarrow +\infty$ is nonnegative. If ξ tends towards $+\infty$, the first difference between braces in the R.H.S. of (A11) tends towards zero. In fact, $\gamma < \delta$ and thus the exponents of ξ in each term of this difference are strictly negative. Consequently, the limit of the integral $\int_1^\xi \mathrm{g}(\xi)\ \mathrm{d}\xi$ for $\xi \rightarrow +\infty$, is equal to the opposite of the difference in the R.H.S. of (A11). By computing this difference, we find

$$\text{(A12)} \lim_{\xi \rightarrow +\infty} \int_1^\xi \mathrm{g}(\xi)\ \mathrm{d}\xi \ = \ \tfrac{(\delta-\gamma)\gamma\delta(2\gamma\delta-1)}{(3\gamma\delta+\gamma+\delta)(2\gamma\delta+\gamma+\delta)}.$$

Since $\delta > \gamma$ and $\gamma\delta \geq 1/2$, we see from (A12) that the limit of $\int_1^\xi \mathrm{g}(\xi)\ \mathrm{d}\xi$ is nonnegative. We have thus proved that $\int_1^\xi \mathrm{g}(\xi)\ \mathrm{d}\xi \ > \ 0$, for all $\xi > 1$, and from the two previous paragraphs we have also proved Lemma 12 in the case $\xi_2 \ = \ \frac{(\gamma+1)\delta}{(\delta+1)\gamma}$.

Assume now that the maximal root ξ_2 is equal to the root $\left(\frac{(\gamma+1)(\delta+1)}{(\gamma+\delta+1)}\right)^{\frac{\delta-\gamma}{\gamma\delta}}$ of the second factor of (A8). In this case, the second factor of (A8) is strictly positive over $[0, \xi_2)$. Since the exponent of ξ in this factor is strictly negative, this factor is a strictly decreasing function of ξ. Consequently, we will have proved Lemma 12 in this case if we prove that $\int_1^{\xi_2} \tilde{\mathrm{g}}(\xi)\ \mathrm{d}\xi \ > \ 0$ where $\tilde{\mathrm{g}}$ is defined in (A13) below,

$$\text{(A13)} \ \tilde{\mathrm{g}}(\xi) \ = \ \xi^{\frac{\gamma\delta+\delta}{\gamma-\delta}} \ \{(\gamma+1)\delta\xi^{-1} - (\delta+1)\gamma\},$$

for all $\xi > 1$. In fact, it will suffice to apply Lemma 13 (Appendix 3) to $\tilde{\mathrm{g}}$ and to $\tilde{\mathrm{f}}$ defined below,

$$\tilde{\mathrm{f}}(\xi) \ = \ (\gamma+1)(\delta+1)\xi^{\frac{\gamma\delta}{\gamma-\delta}} - (\gamma+\delta+1),$$

for all $\xi > 1$.

In this last paragraph, we prove that $\int_1^{\xi_2} \tilde{\mathrm{g}}(\xi)\ \mathrm{d}\xi \ > \ 0$, where $\tilde{\mathrm{g}}$ is defined in (A13). We actually prove that $\int_1^{\xi} \tilde{\mathrm{g}}(\xi)\ \mathrm{d}\xi \ > \ 0$, for all $\xi > 1$. We can rewrite the equality (A13) as follows,

$$\tilde{\mathrm{g}}(\xi) \ = \ (\gamma+1)\delta\xi^{\frac{\gamma\delta+\delta}{\gamma-\delta}-1} - (\delta+1)\gamma\xi^{\frac{\gamma\delta+\delta}{\gamma-\delta}},$$

for all $\xi > 1$. Consequently, the integral of $\tilde{g}$ from 1 to ξ is given by (A14) below,

(A14) $\int_1^\xi \tilde{g}(\xi)\ d\xi \ = \ \left\{(\gamma-\delta)\xi^{\frac{\gamma\delta+\delta}{\gamma-\delta}} - (\gamma-\delta)\xi^{\frac{\gamma\delta+\gamma}{\gamma-\delta}}\right\}.$

From (A14), we see that $\tilde{g}$ is strictly positive over $[1,\ \frac{(\gamma+1)\delta}{(\delta+1)\gamma})$ and is strictly negative over $(\frac{(\gamma+1)\delta}{(\delta+1)\gamma},\ +\infty)$. Thus, the integral $\int_1^\xi \tilde{g}(\xi)\ d\xi$ is strictly positive and strictly increasing over $[1,\ \frac{(\gamma+1)\delta}{(\delta+1)\gamma})$ and strictly decreasing over $(\frac{(\gamma+1)\delta}{(\delta+1)\gamma},\ +\infty)$. If we prove that $\lim\limits_{\xi\to+\infty}$ $\int_1^\xi \tilde{g}(\xi)\ d\xi \ \geq \ 0$, we will thus have proved $\int_1^\xi \tilde{g}(\xi)\ d\xi \ > \ 0$, for all $\xi > 1$. Since the exponents of ξ are strictly negative, each term in the R.H.S. of (A14) tends towards 0 as ξ tends towards $+\infty$. Thus, we have

$$\lim_{\xi\to+\infty} \int_1^\xi \tilde{g}(\xi)\ d\xi \ = \ 0,$$

and we have proved $\int_1^\xi \tilde{g}(\xi)\ d\xi \ > \ 0$, for all $\xi > 1$, and Lemma 12. ||

Appendix 3.

<u>Lemma</u> <u>13</u>: Let a and b be two real numbers such that $a < b$. Let f and g be two continuous functions defined over $[a,\ b]$. Suppose that f is nonincreasing and strictly positive over $(a,\ b)$. Suppose also that there exists c in $(a,\ b)$ such that g is nonnegative over $[a,\ c]$ and nonpositive over $[c,\ b]$. If $\int_a^b g(x)\ dx \ > \ 0$, then

$$\int_a^b f(x)g(x)\ dx \ > \ 0.$$

<u>Proof</u>: Over [a, c], g is nonnegative and f is not smaller than f(c). Consequently, we have

$$\int_a^c f(x)g(x)\ dx \ \geq \ f(c) \int_a^c g(x)\ dx.$$

Over [c, b], g is nonpositive and f is not larger than f(c). Thus we have

$$\int_c^b f(x)g(x)\ dx \ \geq \ f(c) \int_c^b g(x)\ dx.$$

By adding the two previous inequalities, we find

(A15) $$\int_a^b f(x)g(x)\ dx \ \geq \ f(c) \int_a^b g(x)\ dx.$$

Suppose $\int_a^b g(x)\ dx \ > \ 0$. Since f is strictly positive over (a, b), we have $f(c) > 0$. From (A15), we thus see that $\int_a^b f(x)g(x)\ dx \ > \ 0$ and Lemma 13 is proved. ||

Footnotes.

1. See a more detailed definition of strategies in Lebrun (1996).

2. Figure 1 was drawn assuming that $\gamma = 1 < \delta$.

3. The statement given here is obviously equivalent to the statement given in Section 2.

4. Moreover, this w belongs to $(0, \left(\frac{1}{1+\delta}\right)^{1/\gamma})$.

5. Since both intervals are intervals with nonempty interiors and with upper extremities equal to 1, the two requirements define an interval with the same properties.

6. Either of the analytical expressions for the roots of (A8) in the range $\xi > 1$ can give the maximal root ξ_2 for some values of γ and δ.

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Figure

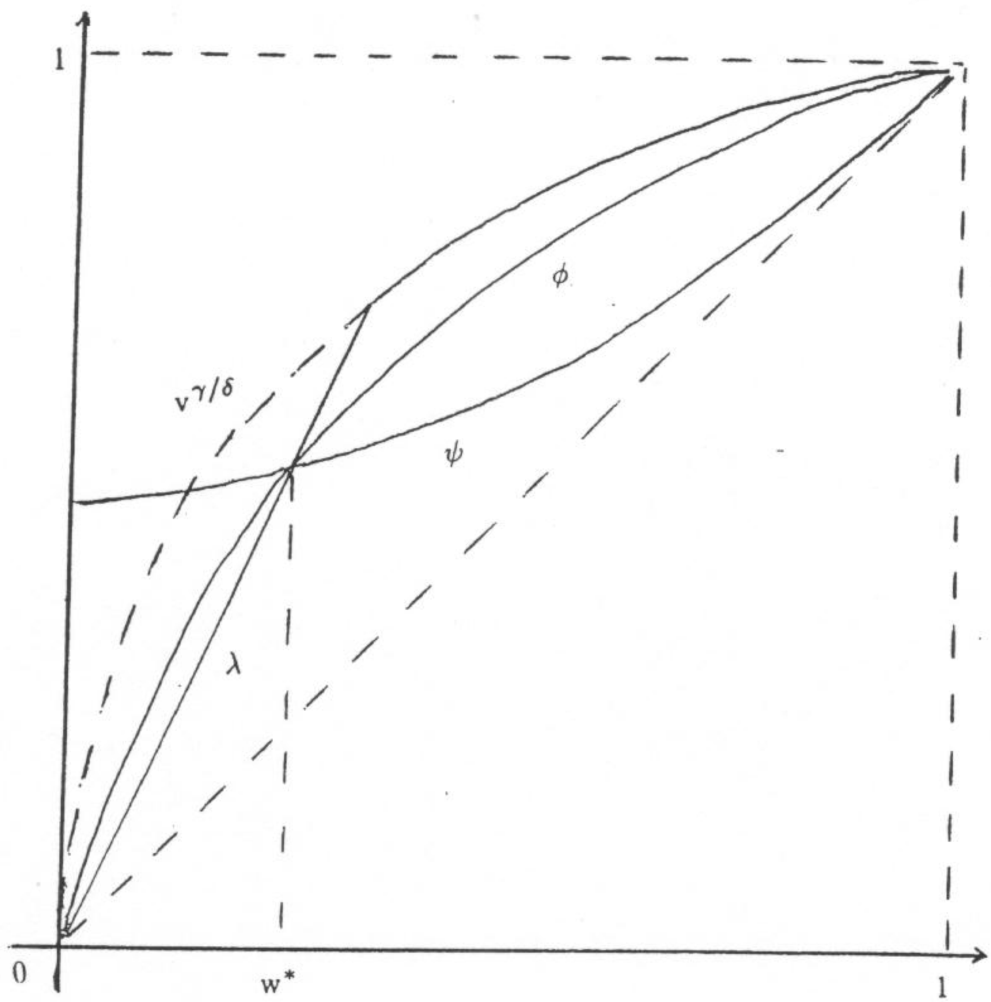

Figure 1