Revenue-Superior Variants of the Second-Price Auction*

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Abstract

Under some assumptions, the introduction of at least a small degree of anonymous "pay-your-bid" in the payment rule of the second-price auction behooves any risk-neutral seller who, while possibly efficiency minded, cares about revenues. In fact, it results in an increase in revenues and a relatively insignificant decrease in efficiency. This can be achieved by adding to the winner's payment a uniform proportion of his own bid, as in Güth and van Damme's auction, or by having bidders receive a uniform proportion of the losing bid, as in Goeree and Offerman's Amsterdam auction, or even by selling uniform toeholds to the bidders prior to the auction. The equilibria of Güth and van Damme's auction with n bidders and of the other auctions with two bidders are identical to equilibria of first-price auctions with transformed value distributions and converge towards truth-bidding if the pay-your-bid component vanishes. With two bidders and under power relation between the bidders' value cumulative or decumulative functions, we obtain explicit expressions for the different first-order effects of this component. **Keywords:** revenues; efficiency; second-price auction; first-price auction; English auction; 2-k-price auction; Amsterdam auction; toeholds; electronic auction; bidder heterogeneity. **J.E.L. Classification code:** D44.

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1. Introduction

In the independent private values model with risk-neutral bidders and when a single indivisible item is for sale, the Vickrey-Clarke-Groves mechanism (see, for example, Milgrom 2004 and Krishna 2009) reduces to a second-price auction, or SPA. Truthful bidding is its only Bayesian-Nash equilibrium in weakly dominant strategies and produces an efficient outcome: the item goes to the highest-value bidder. Truthful bidding above the reserve price is the unique unrestricted Bayesian-Nash equilibrium when the reserve price is binding and there are at least three bidders (see Remark 3.2, page 1488, in Maskin and Riley 1984 and Blume and Heidhues 2004). Furthermore, the rules of the SPA are simple and anonymous.

Increasing auction revenues is obviously a major concern to private sellers (as well as decreasing costs to private buyers, for procurement auctions), but also to governments seeking to rely less on taxes[1]. Efficiency is important to governments selling public assets and, because it may increase participation, even to some private sellers[2].

Although the SPA with an optimally chosen reserve price may maximize the seller's expected revenues when the bidders are ex ante homogeneous[3], this is no longer true if there are "strong" and "weak" bidders, as is the case of many theoretical models and empirical studies[4]. From Myerson

[1]For example, although given a low priority, it was alreday one of the stated goals of the first spectrum auctions. See McMillan (1994) and McAfee and McMillan (1996).

[2]For example, Cramton (1998) writes: "The conflict between revenue maximization and efficiency is further reduced when one considers the desirable effects an efficient auction has on participation. Potential bidders are attracted to the auction based on the expected gains from participation. An efficient auction maximizes the gains from trade, which is the pie to be divided between seller and buyer." As McAfee (2002, p.117) writes about the spectrum auctions, efficiency is important to bidders because it "eliminates unnecessary risk and minimizes the amount of resale that will occur after the auction."

[3]It does maximize expected revenues if the common value distribution is "regular." See Myerson (1981).

[4]For example, the empirical studies of Branmann and Froeb (2000), Krasnokutskaya (2011), and Athey, Levin and Seira (2011) are consistent with such heterogeneous bidders. Among initially homogeneous bidders, efficient collusion or mergers give rise to effective

(1981), the optimal auction allocates the item to the bidder with the highest "virtual value," computed from his actual value and its probability distribution. When a strong bidder's value distribution hazard-rate dominates a weak bidder's, his virtual value is smaller than the weak bidder's with the same actual value and the optimal auction must then be biased towards the weak bidder. Thus, implementing the optimal auction requires complicated and case-specific payment rules[5,6].

With two bidders, one strong and one weak, we show that introducing a small and non-discriminatory pay-your-bid element to the payment rule of the SPA has a strictly positive first-order effect on expected revenues and only a second-order effect on expected total surplus. Thus, even a risk-neutral seller who cares about surplus will want to modify the rules in this way as long as he also cares about revenues[7]. The intuition for our result is simple: in the modified auction, as in the first-price auction or FPA, the weak bidder bids more aggressively than the strong bidder. Because, contrary to

bidders with "power related" value cumulative distributions functions, that is, fixed powers of the same cumulative function. Many important references on the implementation of efficient collusion can be found in Marshall and Marx (2012). See also the paper by Biran and Forges (2011). The effects of efficient mergers have been simulated in Dalkir, Logan, and Masson (2000), Tschantz, Crooke, and Froeb (2000), and Branmann and Froeb (2000). Pesendorfer (2000) and Baldwin, Marshall, and Richard (1997) have detected efficient collusion in actual auctions. Power related cumulative distributions follow from natural axioms in Whaerer and Perry (2003) and Piccione and Tan (1996).

[5]If discrimination among bidders is impossible, for example because illegal, even more complexity may be necessary. For example, an obvious nondiscriminatory optimal mechanism would: 1. ask the bidders to reveal their identities (their value distributions) and their common knowledge (the set of the value distributions of all bidders); 2. cancel the sale if their answers are inconsistent; and 3. otherwise, implement the optimal discriminatory mechanism through a direct mechanism. For the implementation of the optimal mechanism through a particular class of anonymous sealed-bid auctions, see Deb and Pai (2014).

[6]Nevertheless, Lebrun (2012) shows that, in a restricted n-bidder model and in the general 2-bidder model, the standard English auction if followed by resale implements the mechanism that is optimal among all mechanisms selling with certainty. Wilson (1987) advocates the study of the possible interim efficiency, of which optimality is a special case, of simple and familiar trading rules "over a wide class of environments."

[7] I owe this observation to Brent Hickman.

the FPA, the equilibrium allocation changes only when the weak bidder has almost the same value as, hence a larger virtual value than, the strong bidder's, the revenues increase more than the surplus decreases. Obviously, the optimal "dosage" of pay-your-bid depends on the particular value distributions and the seller's utility. For example, from Maskin and Riley (1985), there exist power related distributions for which the seller prefers the SPA to the FPA, as this latter auction brings less revenues. However, the direction for improvement is unambiguous: the addition of any sufficiently small pay-your-bid element will make the seller better off.

Some pay-your-bid feature can be introduced by requiring the winning bidder to pay a proportion of his bid, as in Güth and van Damme (1988)'s auction; or by paying bidders a proportion of the difference between the losing bid and the reserve price, as in Goeree and Offerman (2004)'s second-price Amsterdam auction or AA; or even by first selling at the reserve price "toeholds," that is, shares of ownership in the item for sale, and then running the SPA. Indeed, in all cases, the net payment when winning–the difference between the payments when winning and when losing–includes a proportion of own bid. Furthermore, we establish formal one-to-one relations among the equilibria of these auctions and the FPA. For any two auction procedures, the equilibrium of one is identical to the equilibrium of the other provided the value distributions are changed according to a transformation that we explicitly describe. While general explicit expressions are very rare in the literature on pay-your-bid auctions with heterogeneous bidders, we obtain such expressions for the first-order effects of the pay-your-bid rule on the equilibrium strategies when there are two bidders and their value cumulative or decumulative distribution functions are power related.

Although we assume the bidders' values are distributed over a common interval, our results are robust to small (probability-wise) heterogenous perturbations of the value supports[8].

[8]This is a consequence of the continuity of the equilibrium of the FPA (Lebrun 2002),

Various conditions under which the revenues from the FPA and SPA can be ranked (in either way) are investigated in Maskin and Riley (2000), Cheng (2006), Kirkegaard (2012), and Kaplan and Zamir (2012). Discriminatory ways to increase revenues have been studied empirically and theoretically. Kirkegaard and Overgaard (2008) and Mares and Swinkels (2011) are recent examples of theoretical contributions.

When k is an integer, it is customary (see, for example, Monderer and Tennenholtz 2000 and 2004) to refer to the auction where the price is the kth highest bid as the k-price auction. For k between zero and one, Cramton et al. (1987) call k+1-price auction the auction used to dissolve a partnership among the bidders and where the price is the weighted average of the highest and second highest bids with respective weights $1-k$ and k. Thus, Wasser (2013) calls 2–k-price auction the similar auction where the weights are reversed. As we use the notation k for the weight given to the highest bid, we also call Güth and van Damme (1986)'s auction the 2–k-price auction or 2–k-PA[9]. However, it should be emphasized that, contrary to Cramton et al. (1987) and Wasser (2013), our 2–k-PA is a standard "unilateral" auction, that is, no bidder taking part in it owns any share of the item for sale.

Links similar to our links among the equilibria of FPA's, 2–k-PA's, and AA's were observed by: Bulow, Huang, and Klemperer (1999, page 445) between FPA's and SPA's with common value, identically and independently distributed signals, and possibly asymmetric toeholds; and by de Frutos (2000) and Wasser (2013) among FPA's and "bilateral" auctions.

From Lizzeri and Persico (2000), the equilibrium of the 2–k-PA with heterogenous bidders and interdependent values exists and is unique if the reserve price is binding. In the present paper, we allow a nonbinding reserve price.

Auctions where (typically, relatively small) premiums are handed to some

to which our auctions are related.

[9]Although they do not coin a name for their auction, Güth and van Damme (1986) refer to its payment rule as the "λ-pricing rule" where λ is our $1-k$.

bidders have existed in Europe since the Middle Ages. Bochove, Boerner, and Quint (2013) find that the data from premium auctions of financial securities in eighteenth-century Amsterdam is consistent with a common-value model with independent private signals. Tobay, premium auctions are used to allocate real estate and second-hand machinery, for example. In the words of Goeree and Offerman (2004), the AA is a "stylized version that captures the essential features shared by all premium auctions." With $n > 2$ bidders, it proceeds, similarly to Klemperer (2002)'s Anglo-Dutch auction, as an ascending-price auction up to the last stage, when a sealed-bid premium auction takes place between the two remaining bidders. Through experiments, Goeree and Offerman (2004) study the revenue and efficiency performances of the AA relative to the English and optimal auctions when bidders are asymmetric. For example, in the strongly asymmetric case, with one strong bidder and $n-1$ "very weak" bidders, with no chance of winning the item, average revenues and percentage efficiency are, respectively, 66 and 87.7% for the AA and 44.1 and 96.7% for the English auction[10]. By solving the theoretical model pertaining to this case, the authors show that the competition for the premium among the very weak bidders enhances revenues by creating an endogeneous "reserve price" for the strong bidder. In our present paper, the premiums increase revenues through a different channel, one that is present even with only two bidders, each with some chance of winning. This channel is the bias of the ensuing equilibrium allocation towards the weaker bidder.

In a symmetric model with interdependent values and independent signals, Hu, Offerman, and Zou (2011) extend the rules of the AA to those of the "English premium auction" and prove that it brings more revenues than the standard English auction if the bidders are risk loving. This result is consistent with the experimental study in Brunner, Hu, and Oechssler

[10] The premiums in these experiments are 30%. Interestingly, the average revenues from the "first-price AA," where the last stage is a first-price auction with premium, are 60.1.

(2014).

We observe that the AA with two bidders is equivalent to an auction where bidders receive "price-proportional benefits" because they had previously bought small shares of the item being auctioned. Thus, using premiums may be a convenient way to reach the same outcome as an auction with toeholds when actual toeholds are difficult or impossible to implement (for example, because of legal costs). In a common-value symmetric model, Bulow et al. (1999) find that the seller prefers symmetric bidder toeholds to asymmetric ones. Here, the seller prefers symmetric toeholds to no toeholds.

Because of proxy bidding, the "electronic auctions," on Web-sites such as eBay, Amazon, and Yahoo!, are similar to the SPA. As Hickman (2010) observes, the raising of the price by increment introduces a small pay-your-bid element: the price is the sum of the second-highest bid and the bid increment unless this amount is above the highest bid, in which case the price is the highest bid. When the bidders are homogeneous, Hickman (2010) characterizes the equilibrium and shows that, although significantly different from truth-bidding, it is still increasing and symmetric. Therefore, expected revenues and total surplus are the same as in the SPA. Unfortunately, as the weight (zero or one) given to the own bid in the computation of the price changes with the bids, solving for the equilibrium when bidders are heterogeneous is likely to be even more complicated than in the 2–k-PA, where this weight is fixed. We conjecture that, with strong and weak bidders, the increment rule too increases revenues without significantly affecting efficiency. We leave for future work the formal examination of this conjecture.

In Section 2, we analyze the 2–k-PA with n bidders and, using methods from Lebrun (1999), prove that its equilibrium for given value distributions is identical to the equilibrium of the FPA where the value distributions have been raised to the exponent $1/k$. This link between the two auction procedures allows us to import results from the literature on the FPA with heterogeneous bidders. If there exists a weakest bidder, we prove that as k tends towards zero the bidding functions tend towards the identity function and we obtain bounds on the rates of convergence. Because of the known complexity of the FPA hence the 2–k-PA with heterogeneous bidders, we have to assume two bidders and power related cumulative distributions in order to obtain explicit expressions for the rates of change of the revenues and surplus.

We turn to the AA, with two bidders, in Section 3 and prove that its equilibrium is the equilibrium of the 2–k-PA where the value distributions have been "reflected" through the middle of the value support. Thanks to this new link, our results about the 2–k-PA, such as the convergence of the equilibrium towards truth-bidding when k tends towards zero as well as explicit expressions for the first-order effects, translate to the AA. Thus, the equilibrium converges towards truth-bidding if k tends towards zero and explicit expressions for the first-order effects can be obtained if the value decumulative functions are power related. Section 4 is the conclusion. Proofs similar to proofs already included in the text or to proofs of existing results, as well as proofs of secondary or technical results, have been relegated to the appendices.

2. The 2–k-Price Auction

Bidders $1, ..., n$, with $n \geq 2$, participate in the auction. Their values are randomly and independently drawn from the interval $[c, d]$, with $0 \leq c < d$, according to the probability distributions $F_1, ..., F_n$. We use the same notation for a probability distribution and its (continuous from the right) cumulative function. We assume that, for all $1 \leq i \leq n$, F_i is continuous over $\mathbb{R}$, continuously differentiable over $[c, d]$ with a strictly positive derivative f_i over $(c, d]$, twice-continuously differentiable over $(c, d]$, and such that $\frac{d}{dv} \frac{f_i}{F_i}(v)$ is bounded from above.

As F_i is atomless, the logarithm $\ln F_i(v)$ tends towards $-\infty$ if v approaches c and hence the reverse hazard rate $\frac{d}{dv}\ln F_i(v) = \frac{f_i(v)}{F_i(v)}$ must take on unbounded positive values and its derivative $\frac{d}{dv}\frac{f_i}{F_i}(v)$ unbounded negative values. Our assumption above simply means that $\frac{d}{dv}\frac{f_i}{F_i}(v)$ never takes on unbounded positive values. Obviously, a smooth F_i that is log-concave in some interval $(c, c+\varepsilon)$, with $\varepsilon > 0$, satisfies our assumptions as then $\frac{d}{dv}\frac{f_i}{F_i}(v) \leq 0$ over this interval. From the technical Lemma A1 in Appendix 1, we have $\lim_{v \to > c} \frac{f_i}{F_i}(v) = +\infty$.

The concept of equilibrium we use throughout is Bayesian Nash in weakly undominated strategies.

We will make frequent use of the notations below for the reverse hazard rate, hazard rate, and virtual value of bidder i with actual value v:

$$
\begin{aligned}
\rho_i(v) &= \frac{f_i(v)}{F_i(v)}; \\
\sigma_i(v) &= \frac{f_i(v)}{1 - F_i(v)}; \\
\omega_i(v) &= v - \sigma_i(v)^{-1}.
\end{aligned}
$$

Note that we allow ω_i to be nonmonotonic.

2.1 The Equilibrium with N Bidders and its Properties

Let k be a number between zero and one. In the 2–k-PA with reserve price r in $[c, d)$, the highest bidder wins when his bid is at least equal to r, in which case he pays the weighted average of his own bid and the maximum of the second highest bid and r with respective weights k and $1-k$[11] No payment is made by any other bidder. Thus, if k is small the rules of the

[11] All our results about the 2-k-PA when c is the (nonbinding) reserve price would still hold true for any finite reserve price not larger than c.

2–k-PA are "close" to those of the 2-PA, that is, the SPA. Bidding strictly above one's value is weakly dominated in the 2–k-PA.

We construct in Theorem 1 below a standard FPA that has the same equilibrium as the 2–k-PA. Thanks to this construction, results from the literature on the FPA translate to the 2–k-PA. In particular, the equilibrium, although complex, is unique (see Corollary 1 below).

Theorem 1-Characterization of the equilibrium of the 2–k-PA with n bidders and link with the FPA: *Let k be such that $k \in (0,1)$.*

(i) **Equilibrium of the 2–k-PA:** *An n-tuple of strategies $(\beta_1(.;k), ..., \beta_n(.;k))$ is an equilibrium of the 2–k-PA if and only if:*

1. For all i: $\beta_i(.;k)$ over $[c,r)$ specifies bidding strictly below r and $\beta_i(r;k)$ specifies bidding bids not larger than r.

2. $\beta_1(.;k), ..., \beta_n(.;k)$ are bidding functions over $(r,d]$ such that their inverses $\alpha_1(.;k) = \beta_1^{-1}(.;k), ..., \alpha_n(.;k) = \beta_n^{-1}(.;k)$ exist, are strictly increasing, strictly above the identity function, and form a solution over $(r,\eta(k)]$ of the system of differential equations (1) below

$$\frac{1}{k}\frac{d}{db}\ln\prod_{j\neq i}F_j(\alpha_j(b;k)) = \frac{1}{\alpha_i(b;k)-b}, \; 1 \leq i \leq n, \tag{1}$$

and their continuous extensions satisfy the boundary conditions:

$$\begin{aligned} \alpha_i(r;k) &= r, \text{ for all except possibly one } i; && (2)\\ \alpha_1(\eta(k);k) &= ... = \alpha_n(\eta(k);k) = d, && (3) \end{aligned}$$

for a certain value in (r,d) of the parameter $\eta(k)$.

3. If $\alpha_j(r;k) > r$, then $\beta_j(.;k)$ over $(r,\alpha_j(r;k)]$ takes the constant value r.

Moreover, if $r = c$ or if $n = 2$, (2) can be replaced by (4) below and 3. above never applies:

$$\alpha_1(r;k) = ... = \alpha_n(r;k) = r. \tag{4}$$

(ii) **Relation to the FPA:** *An n-tuple of strategies is an equilibrium of the 2–k-PA if and only if it is an equilibrium of the FPA where the bidders' values are distributed according to $F_1^{1/k}, ..., F_n^{1/k}$.*

Proof: See Appendix 1.

The differential equations (1) in Theorem 1 are the equilibrium FOC's. In fact, the expected payoff of bidder i with value $v > r$ and bid $b > r$ is

$$\int_c^b (v - kb - (1-k)\max(b', r))\, d\prod_{j\neq i} F_j(\alpha_j(b';k)),$$

whose derivative[12] with respect to $b > r$ is

$$(v-b)\frac{d}{db}\prod_{j\neq i} F_j(\alpha_j(b;k)) - k\prod_{j\neq i} F_j(\alpha_j(b;k)).$$

Setting this expression equal to zero at $v = \alpha_i(b;k)$, we indeed find one of the equations in (1)–the same as from the maximization of $(v-b)\prod_{j\neq i} F_j(\alpha_j(b;k))^{1/k}$ in the FPA with the modified value distributions.

As we state in Corollary 1, the equilibrium is (essentially) unique when the reserve price is binding and, under the additional assumption of local strict log-concavity at c, when it is not. This assumption is equivalent to requiring

[12]The differentiability is proved in Appendix 1.

that $\rho_1(v), ..., \rho_n(v)$ tend monotonically towards $+\infty$ as v approaches c on an interval, however small, to the right of c.

Corollary 1–Properties of the equilibrium of the 2–k-PA with n bidders: *Let k be such that $k \in (0,1]$.*

(i) **Existence:** *There exists an equilibrium of the 2–k-PA.*

(ii) **Expression for the bidding functions:** *If $(\beta_1(.;k), ..., \beta_n(.;k))$ is the equilibrium of 2–k-PA, we have:*

$$\beta_i(v;k) = v - \frac{\int_r^v \prod_{j \neq i} F_j\left(\varphi_{ji}(w;k)\right)^{1/k} dw}{\prod_{j \neq i} F_j\left(\varphi_{ji}(v;k)\right)^{1/k}},$$

for all v in $(\alpha_i(r;k), d]$, where $\varphi_{ji}(.;k) = \alpha_j(\beta_i(.;k);k)$ and $\alpha_j(.;k)$ is the inverse of $\beta_j(.;k)$.

(iii) **More aggressive bidding by weaker bidders while preserving reverse-hazard-rate bid dominance:**

(iii.1) If $\rho_i \leq \rho_j$, then *$\beta_i(.;k) \geq \beta_j(.;k)$, over $(r,d]$, and $\frac{d}{db} \ln F_i \alpha_i(.;k) \leq \frac{d}{db} \ln F_j \alpha_j(.;k)$, over $(r, \eta(k)]$, for all $i \leq j$;*

(iii.2) If $\rho_i(v) < \rho_j(v)$, for all v in (r,d), then $\beta_i(v;k) > \beta_j(v;k)$, for all v in (r,d).

(iii.3) If $\rho_i \leq \rho_n$, for all i, *then, in the characterization (i.1, i.2, i.3) in Theorem 1, the initial condition (2) in (i.2) can be replaced by (5) below:*

$$\alpha_1(r;k) = ... = \alpha_{n-1}(r;k) = r; \tag{5}$$

and, in Theorem 1 (i.3), only $j = n$ needs be considered.

(iv) **Uniqueness:** *If $r > c$, the equilibrium is essentially unique*[13]*. It is also essentially unique if $r = c$ and there exists $\varepsilon > 0$ such that $F_1, ..., F_n$*

[13] That is, unique up to the inessential indeterminacy for values not larger than r (see (i.1) in Theorem 1).

are strictly log-concave over $(c, c+\varepsilon)$.

Proof: See Appendix 1.

As we state in Theorem 2 (iii) below, under reverse-hazard-rate dominance of F_1 by the other F_i's, the equilibrium approaches the truth-bidding equilibrium of the SPA if k tends towards zero. The proof proceeds first by showing that the bid shading in Corollary 1 (ii) by bidder 1, who is the weakest by assumption and, from Corollary 1 (iii), the most aggressive bidder, vanishes at the limit. This is a consequence of: 1. the fact that bidder 1 bids higher than in a FPA with $(n-1)/k$ other bidders, with the same value distribution as his; and 2. the obvious convergence towards truth-bidding of the equilibrium of the FPA with ex ante homogeneous bidders when the number of bidders tends towards infinity. Once this is established, the result for the other bidders follows. In fact, if another bidder's bid stayed away from his value, bidding closer to it would be a profitable deviation: it would increase his probability of winning by the probability (which we now know is bounded away from zero) with which bidder 1 bids within the gap and the other bidders below it and it would increase his payment by a negligible amount, as k tends towards zero. Theorem 2 also gives first bounds on rates of convergence.

We first recall from Royden (1988) the definition of "derivates," which are generalizations of the concept of one-sided derivative, and introduce notations, one of which for the elasticity of a density with respect to the cumulative probability.

Definition: Let $h(x_1, ..., x_m)$ be a real-valued function defined over a product of intervals with nonempty interiors $P = \prod_{s=1}^{m} [c_s, d_s] \subseteq \mathbb{R}^m$. Then, for all x in P and all s the right-handed partial derivates of h at (x_{-s}, c_s) are as follows:

$$\begin{aligned}
\frac{\partial^{+}}{\partial x_s} h\left(x_{-s}, c_s\right) &= \overline{\lim}_{\Delta x_s \rightarrow > 0} \frac{h\left(x_{-s}, c_s+\Delta x_s\right)-h\left(x_{-s}, c_s\right)}{\Delta x_s} \\
\frac{\partial_{+}}{\partial x_s} h\left(x_{-s}, c_s\right) &= \underline{\lim}_{\Delta x_s \rightarrow > 0} \frac{h\left(x_{-s}, c_s+\Delta x_s\right)-h\left(x_{-s}, c_s\right)}{\Delta x_s}.
\end{aligned}$$

Thus, $\frac{\partial^{+}}{\partial x_s} h\left(x_{-s}, c_s\right)$ and $\frac{\partial_{+}}{\partial x_s} h\left(x_{-s}, c_s\right)$ are the supremum and infimum, respectively, of the rates of increase of h with respect to x_s to the right of c_s. The standard right-handed derivative $\frac{\partial_r}{\partial x_s} h\left(x_{-s}, c_s\right)$ exists if and only if $\frac{\partial^{+}}{\partial x_s} h\left(x_{-s}, c_s\right)$ and $\frac{\partial_{+}}{\partial x_s} h\left(x_{-s}, c_s\right)$ are equal and finite, in which case it is equal to their common value. If there is only one variable, that is, $m=1$, the derivates and the right-handed derivative are denoted $\frac{d^{+}}{dx} h, \frac{d_{+}}{dx} h$, and $\frac{d_r}{dx} h$.

Convention and Notations:

1. We keep denoting $\beta_1(.;k), \ldots, \beta_n(.;k)$ and $\alpha_1(.;k), \ldots, \alpha_n(.;k)$ the unique direct and inverse equilibrium bidding functions of the 2–k-PA, and $\varphi_{ij}(.;k)$ the compound function $\alpha_i\left(\beta_j(.;k);k\right)$ that connects the values at which bidders i and j would tie. For convenience, we extend the inverse bidding function $\alpha_i(.;k)$ as the constant function d above the maximum equilibrium bid $\eta(k)$, that is, we set:

$$\alpha_i(b;k)=d,$$

for all $\eta(k) \leq b \leq d$ and all i.

2. For all i, we denote $\varepsilon_i(p)$ the elasticity of the density $f_i\left(F_i^{-1}(p)\right)$ with respect to the cumulative probability p, that is:

$$\begin{aligned}
\varepsilon_i(p) &= \frac{d \ln f_i\left(F_i^{-1}(p)\right)}{d \ln p} \\
&= \frac{p f_i'\left(F_i^{-1}(p)\right)}{f_i\left(F_i^{-1}(p)\right)^2}.
\end{aligned}$$

Theorem 2–Convergence of the equilibrium of the 2–k-PA with n bidders towards truth-bidding: *Extend the functions* $\beta_i(.;k)$, $\alpha_i(.;k)$, $\varphi_{ij}(.;k)$ *to* $k=0$ *as follows:*

$$\beta_i(v;0)=\alpha_i(v;0)=\varphi_{ij}(v;0)=v,$$

for all v *in* $[r,d]$. *Then:*

(i) **Upper bound on the rate of convergence of the strategy of a bidder who is weaker than some other bidders:** *Let* $i<n$ *be such that* $\rho_i\leq\rho_j$, *for all* $i\leq j$. *Then, we have, for all* v *in* $[r,d]$*:*

$$0\leq-\frac{\partial_+}{\partial k}\beta_i(v;0),\frac{\partial^+}{\partial k}\alpha_i(v;0)\leq((n-i)\rho_i(v))^{-1}.$$

If I *is* $[r,d]$ *when* $r>c$ *and any interval* $[c+\gamma,d]$, *where* $\gamma>0$, *when* $r=c$, *the upper bound above on the derivates is "uniform" in* v *over* I, *that is:*

$$\overline{\lim}_{k\to0}\max_{v\in I}\left(\max\left(\frac{v-\beta_i(v;k)}{k},\frac{\alpha_i(v;k)-v}{k}\right)-((n-i)\rho_i(v))^{-1}\right)\leq0.$$

Moreover, if $r=c$ *and the elasticity* ε_i *is bounded from below, this is also true for* $I=[c,d]$.

(ii) **Lower bound on the rate of convergence of the strategy of a strongest bidder:** *Assume* $\rho_i\leq\rho_n$, *for all* i. *Then, we have*[14]*:*

$$-\frac{\partial_+}{\partial k}\beta_n(v;0),\frac{\partial^+}{\partial k}\alpha_n(v;0)\geq((n-1)\rho_n(v))^{-1}.$$

(iii) **Joint continuity of the bidders' strategies when there exists**

[14]Because we do not need it in our later developments, we have abstained to state the uniformity of the lower bound in (ii) on the derivates $-\frac{\partial_+}{\partial k}\beta_n(v;0)$ and $\frac{\partial^+}{\partial k}\alpha_n(v;0)$, which can be proved similarly to the uniformity of the upper bounds in (i).

a weakest bidder: *Assume* $\rho_1 \le \rho_i$, *for all* i. *For all* $1 \le i, j \le n$, *the functions* $\beta_i(v;k), \alpha_i(v;k), \varphi_{ji}(v;k)$ *are continuous jointly in both variables* v, k *at* $(v;0)$, *for all* v *in* $[r,d]$.

Proof: See Appendix 2.

The assumption in (i) that the elasticity ε_i be bounded from below is equivalent to requiring that there exists a strictly positive number T, however large, such that F_i^T is convex[15].

The analysis of the standard FPA is complicated by the singularity of the system of FOC's at the lower extremity of the winning bid interval. What makes matters even worse in the 2–k-PA with k close to zero is that, as the bidding functions tend towards the identity function, the system (1) is nearly singular everywhere over the bid interval. Partly for this reason, it is difficult to go much further than Theorem 2 while keeping the same level of generality. However, as we show in the next subsection, explicit expressions for the rate of change of expected revenues can be obtained in the two-bidder case.

2.2 The two-bidder case

In addition to assuming two bidders, we also make the additional assumption of strict log-concavity at c from the previous subsection and strengthen

[15]This assumption is also equivalent to the requirement that the "local ρ-concavity" of F_i be bounded from above (for other applications of local ρ-concavity to auction theory, see, for example, Mares and Swinkels 2011).

If F_i is log-concave at c, the elasticity ε_i is bounded from above, as $\varepsilon_i \le 1$ over a small interval $(c, c+\varepsilon)$. Thus, in this case, we could replace the additional assumption that ε_i be bounded from below by the assumption that it be bounded.

Similar assumptions–on the elasticity of the slope of demand functions–have long been standard in the literature on oligopoly (see, for example, Seade 1980 a and b, Suzumura and Kiyono 1987, Besley and Suzumura 1989, Suzumura 1992, Okuno-Fujiwara and Suzumura 1993).

the stochastic dominance relation between the two value distributions to power relation. Thus, $n = 2$ and

$$F_2 = F_1^l, \tag{6}$$

for some constant $l \geq 1$. This assumption of power relation immediately implies $\rho_2(v) = l\rho_1(v)$ and hence reverse-hazard-rate dominance of F_1 by F_2. It also implies[16] hazard-rate dominance, that is, $\sigma_1(v) \geq \sigma_2(v)$ and $\omega_1(v) \geq \omega_2(v)$; with strict inequalities if $l > 1$ and $v \in (c, d)$.

We keep using primes and straight derivative signs for the derivatives with respect to the bid or value. The system (1) of FOC's and the terminal condition (3) in Theorem 1 reduce to the system (7) and condition (8) below:

$$\begin{aligned} \frac{d \ln F_1(\alpha_1(b;k))}{db} &= \frac{1}{(\alpha_2(b;k) - b)/k} \\ \frac{d \ln F_2(\alpha_2(b;k))}{db} &= \frac{1}{(\alpha_1(b;k) - b)/k} \end{aligned} \tag{7}$$

$$\alpha_1(\eta(k);k) = \alpha_2(\eta(k);k) = d, \tag{8}$$

where $\eta(k) \in (r, d)$ is the maximum equilibrium bid. Thanks to the system (7), how the derivatives of the bidding functions evolve with k informs us on the rates of convergence of the bidding functions towards the identity function.

From Corollary 1 (iii), bidder 1 bids more aggressively and, as a consequence, bidder 2 needs a higher value to tie for winner of the auction, that is, $\varphi_{21}(v;k) = \alpha_2(\beta_1(v;k);k) \geq v$. From hereon, we simplify our notations by dropping the subscripts from φ_{21}. The equilibrium allocation of the 2–k-PA is inefficient in the set of value couples (v_1, v_2) bounded from below by the 45-degree line and from above by the graph of $\varphi(.;k)$ (see Figure 1).

[16] Indeed, it implies likelihood-ratio dominance. For these stochastic orders, see, for example, Appendix B in Krishna (2009).

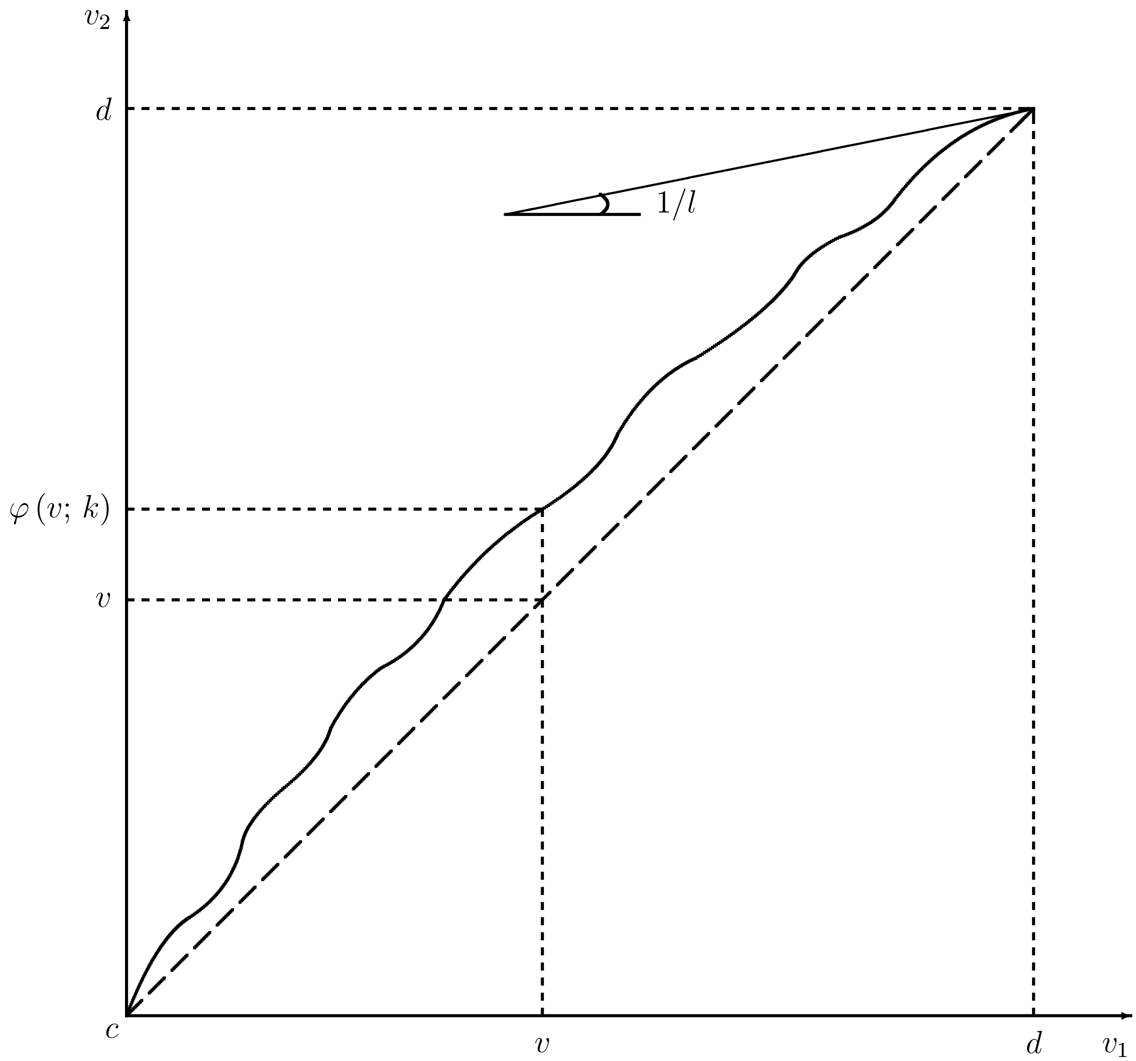

FIGURE 1: Boundary between value couples leading to different allocations in the 2–k-PA and the SPA.

Although, from Theorem 2 (iii), $\varphi\,(.;k)$ approaches the identity function, the value of its derivative does not approach one everywhere. Indeed, dividing the second equation in (7) by the first, we find:

$$\frac{d\ln F_2\left(\alpha_2\left(b;k\right)\right)}{d\ln F_1\left(\alpha_1\left(b;k\right)\right)}=\frac{\alpha_2\left(b;k\right)-b}{\alpha_1\left(b;k\right)-b}; \tag{9}$$

which, together with (6) and the condition (8), implies:

$$\varphi'\left(d;k\right)=\frac{1}{l}.$$

Thus, when the values are differently distributed, $\varphi'\left(d;k\right)$ stays away from one.

To acquire information about the behavior of the derivative of φ, we differentiate (9) and, after rearranging, we find:

$$\begin{aligned}
&\frac{d}{db}\ln\frac{d\ln F_2\left(\alpha_2\left(b\right)\right)}{d\ln F_1\left(\alpha_1\left(b\right)\right)}\\
=\;&\frac{\left(\alpha_2\left(b;k\right)+k\frac{F_2\left(\alpha_2\left(b;k\right)\right)}{f_2\left(\alpha_2\left(b;k\right)\right)}\right)-\left(\alpha_1\left(b;k\right)+k\frac{F_1\left(\alpha_1\left(b;k\right)\right)}{f_1\left(\alpha_1\left(b;k\right)\right)}\right)}{\left(\alpha_1\left(b\right)-b\right)\left(\alpha_2\left(b\right)-b\right)}\\
=\;&\frac{\gamma_1\left(\alpha_2\left(b;k\right);k\right)-\gamma_2\left(\alpha_1\left(b;k\right);k\right)}{\left(\alpha_1\left(b;k\right)-b\right)\left(\alpha_2\left(b;k\right)-b\right)},
\end{aligned} \tag{10}$$

where the functions γ_1,γ_2 are defined as follows:

$$\gamma_i\left(v;k\right)=v+\frac{k}{\rho_j\left(v\right)},$$

for all v in $(c,d]$ and $i,j=1,2,\;\; i\neq j$. Thus, $\gamma_i\left(.;k\right)$ would be bidder i's inverse bidding function if bidder j bid truthfully. The following properties of these functions are simple to prove.

Lemma 1: *There exist $\zeta,\mu>0$ such that $\gamma_1\left(.;.\right)$, $\gamma_2\left(.;.\right)$ can be continuously extended to $[c,d+\mu)\times(-\zeta,\zeta)$ in such a way that*, for $i\neq j$*:*

(i) $\gamma_i\left(c;k\right)=c$*, for all k in $(-\zeta,\zeta)$;*

(ii) Over $(c,d+\mu)\times(-\zeta,\zeta)$: $\gamma_i\left(.;.\right)$ is continuously differentiable;

the derivative $\gamma_i'(.;.)$ with respect to b is strictly positive and bounded away from zero; $\frac{\partial}{\partial k}\gamma_i(.;.)$ is strictly positive and bounded from above; and $\gamma_i(d+\mu;k) > d$, for all k in $(-\zeta,\zeta)$.

(iii) Over $(c,d] \times (-\zeta.\zeta)$: $\frac{\partial}{\partial k}\gamma_i^{-1}(.;.)$ is strictly negative, bounded from below, and equal to $-\left(\gamma_i'\left(\gamma_i^{-1}(v;k);k\right)\rho_j\left(\gamma_i^{-1}(v;k)\right)\right)^{-1}$.

Proof: See Appendix 3.

The sign of the expression above does not depend directly on b: $\ln F_2(\varphi(v;k))$ is concave or convex with respect to $\ln F_1(v)$ depending on whether $\gamma_1(\varphi(v;k);k)$ is smaller or larger than $\gamma_2(v;k)$. We are lead to consider the functions $\gamma_1^{-1}(\gamma_2(.;k);k), \varphi(.;k)$, and the identity function in the space of logarithms of cumulative probabilities[17]. We denote the new functions $\Psi(.;k), \Phi(.;k)$, and Λ and define them formally below. The linearity of Λ follows immediately from the power relation (6).

Definitions: Let k be in $(0,\zeta)$, where $\zeta > 0$ is as in Lemma 1 above. Let $\Psi(.;k), \Phi(.;k)$, and Λ be the following functions:

(i)

$$\Psi(u;k) = \ln F_2\left(\gamma_1^{-1}\left(\gamma_2\left(F_1^{-1}(\exp u);k\right);k\right)\right),$$

for all u in $(-\infty, \ln F_1(x(k)))$, where $x(k) \in (c,d]$ is defined as follows:

$$x(k) = \gamma_2^{-1}(\gamma_1(d;k);k);$$

and

$$\Psi(u;k) = 0,$$

for all u in $[\ln F_1(x(k)), 0]$;

[17] This method of proof appears to be novel in the literature on pay-your-bid auctions.

(ii)

$$\Phi\left(u;k\right)=\ln F_2\left(\varphi\left(F_1^{-1}\left(\exp u\right);k\right)\right),$$

for all u in $\left(\ln F_1\left(r\right),0\right]$ and, if $r>c$, at $u=\ln F_1\left(r\right)$, with $\varphi\left(.;k\right)=\alpha_2\left(\beta_1\left(.;k\right);k\right)$;

(iii)

$$\Lambda\left(u\right)=lu,$$

for all u in $\mathbb{R}_-$.

As $\rho_2=l\rho_1$, we have $\gamma_2\left(.;k\right)\geq\gamma_1\left(.;k\right)$ and $x\left(k\right)$ in (i) above is well defined. All functions above are continuous. The function $\Psi\left(.;k\right)$ is strictly increasing over $\left(-\infty,\ln F_1\left(x\left(k\right)\right)\right)$ and equal to zero over $\left[\ln F_1\left(x\left(k\right)\right),0\right]$. The function $\Phi\left(.;k\right)$ is strictly increasing over its entire definition domain, tends towards $\ln F_2\left(r\right)$ if its argument u tends towards the infimum $\ln F_1\left(r\right)$ of its domain, and vanishes at the supremum (zero).

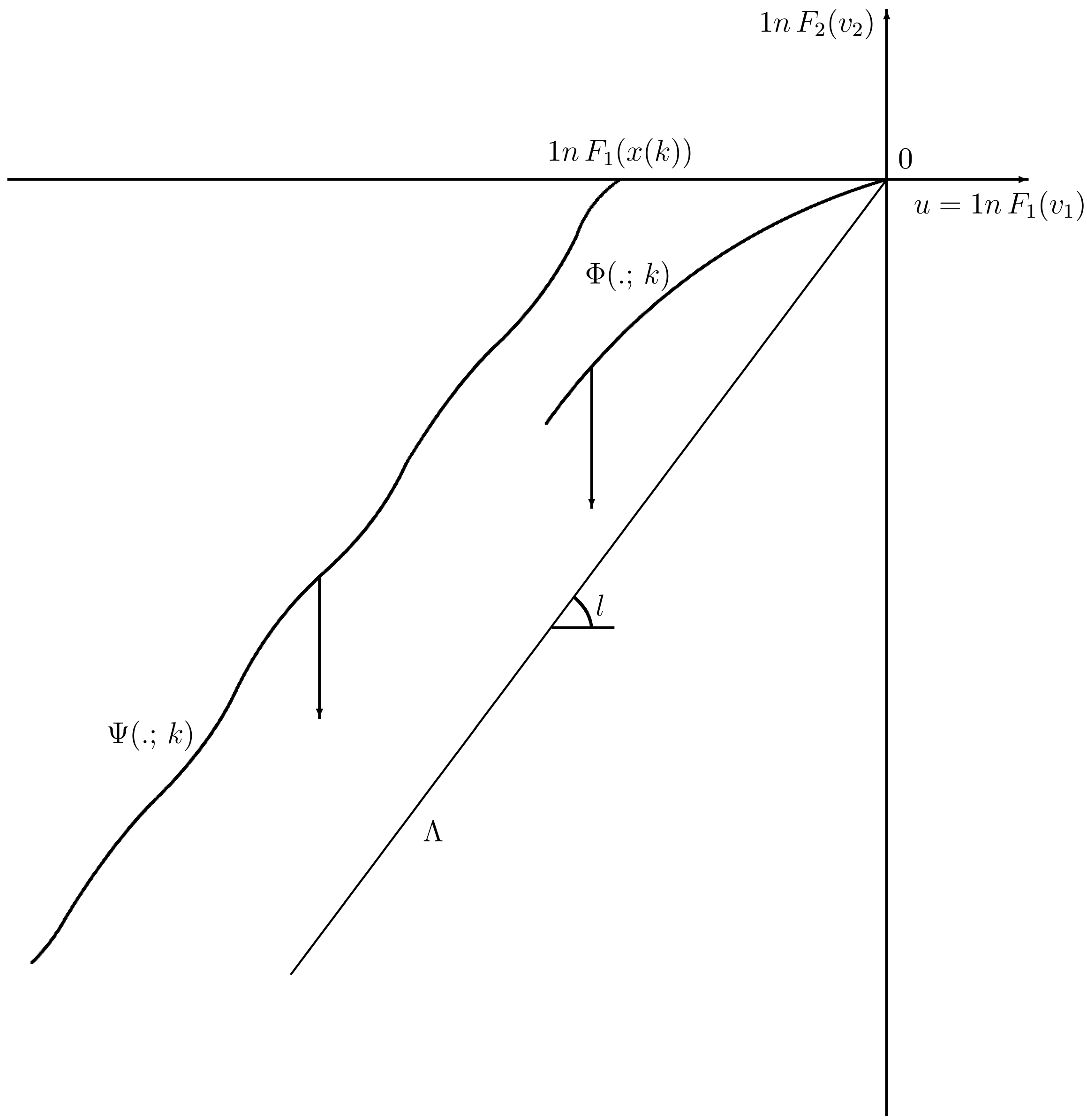

FIGURE 2: Possible configuration of the functions Ψ, Φ, and Λ when $l > 1$ and $r = c$.

Figure 2 displays a possible configuration of the graphs of these functions. From Corollary 1 (iii), $\Phi\,(.;k)$ is at least equal to Λ, and strictly above it over the interior of its definition domain if $l > 1$. Because $\Psi\,(.;k)$ reaches zero already at $\ln F_1(x(k))$ and $\Phi(.;k)$ only at zero, $\Psi(.;k)$ is above $\Phi(.;k)$ in the neighborhood of the origin. If $l > 1$, $\gamma_2(.;k) > \gamma_1(.;k)$ and $\Psi(.;k)$ is strictly above Λ. As we already inferred from (10), the position of $\Phi(.;k)$ relative to $\Psi(.;k)$ determines the direction of its concavity. That is, Lemma 2 below holds true.

Lemma 2: *If $r = c$, assume F_1 is strictly log-concave over $(c, c+\varepsilon)$, for some $\varepsilon > 0$. Then, when $l > 1$:*

(i) $\Psi(u;k) > \Lambda(u)$, for all u in $\mathbb{R}_-$;

(ii) $\Psi(u;k) > \Phi(u;k)$, for all u in $(\ln F_1(x(k)), 0)$; and if $r > c$: $\Psi(\ln F_1(r);k) > \Phi(\ln F_1(r);k) = \Lambda(\ln F_1(r))$.

(iii) $\Phi(u;k) > \Lambda(u)$, for all u in $(\ln F_1(r), 0)$;

(iv) $\Phi''(u;k) > (<; =)\ 0$ if and only if $\Phi(u;k) > (<; =)\ \Psi(u;k)$, for all u in $(\ln F_1(r), 0)$.

From the explicit expression in the definition of $\Psi(.;k)$, it is simple to prove that it and its derivative tend towards Λ and its derivative. From its definition and Theorem 2 (iii), $\Phi(.;k)$ tends towards Λ. The downward pointing arrows in Figure 2 above represent these convergences.

From Lemma 2, we can prove that the derivative of $\Phi(.;k)$ converges towards the derivative of Λ compactly over $(\ln F_1(r), 0)$, that is, uniformly over every compact subinterval K of $(\ln F_1(r), 0)$. The main ideas of the proof are as follows. From the convergence of $\Psi(.;k)$ and its derivative, there exists k' such $\Psi'(.;k)$ is close to l over K and smaller or only possibly slightly larger than l over $[\min K, 0]$, for all $0 < k < k'$. If, at some point[18] s in K and for small k, the derivative of $\Phi(.;k)$ was further above l than $\Psi'(.;k)$ is over $[\min K, 0]$, $\Phi(.;k)$ could not at the same time tend towards

[18] The full formal argument rules out the existence of a sequence $(s_t;k_t) \to (u;0)$, where $u \in K$, such that $\lim_{t\to+\infty} \Phi'(s_t;k_t)$ exists in the weak sense (that is, can be infinite) and is different from l. See Appendix 3.

Λ, remain above it, and be strictly increasing. For example, if $\Phi'(.;k)$ at a point s was further above l and if $\Phi(s;k)$ was not smaller than $\Psi(s;k)$, $\Phi(.;k)$ would be convex at s (from Lemma 2 (iv)), its derivative would even be higher and $\Phi(.;k)$ would hence never meet $\Psi(.;k)$ to the right of s. As depicted in Figure 3 (for the case $r = c$), $\Phi(.;k)$ would then be equal to zero to the left of $\ln F_1(x(k))$; something which is impossible as $\Phi(u;k)$ vanishes only at $u = 0$.

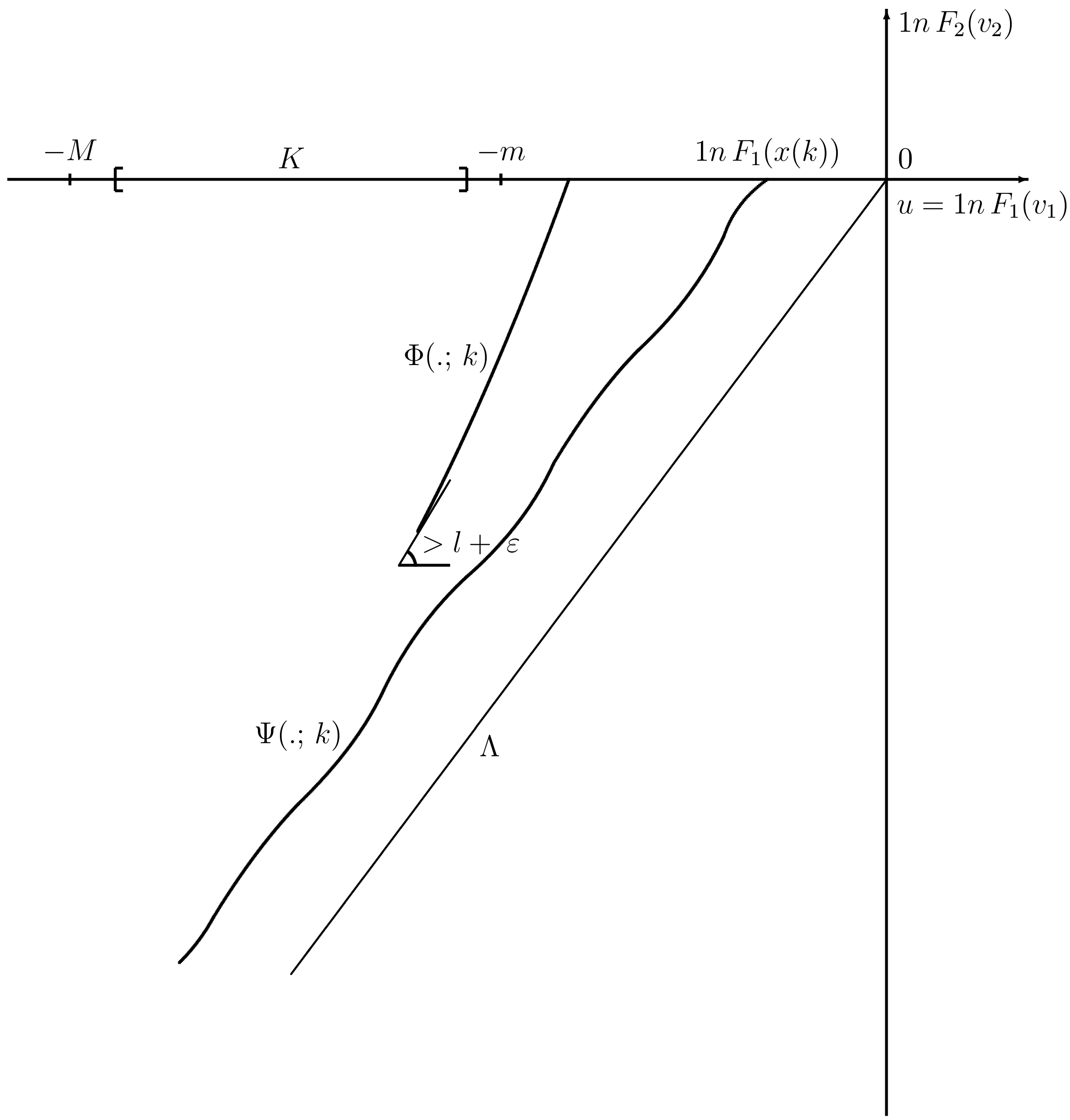

FIGURE 3: Ruling out $\Phi'(.;k)$ further above l than $\Psi'(.;k)$ is while $\Phi(.;k)$ is not smaller than $\Psi(.;k)$.

If $\Phi'(.;k)$ at a point s was again further above l than $\Psi'(.;k)$ is over $[\min K, 0]$, but if $\Phi(s;k)$ was now less than or equal to $\Psi(s;k)$, $\Phi(.;k)$ would be concave at s (from Lemma 2 (iv)), its derivative would be higher and hence

$\Phi(.;k)$ would never meet $\Psi(.;k)$ and would remain concave to the left of s. However, as the derivative of Λ is the constant l, a derivative of $\Phi(.;k)$ uniformly bounded away from l to the left of s is incompatible with $\Phi(.;k)$ tending everywhere towards Λ. For an illustration of this case, see Figure 4 below. Ruling out $\Phi'(.;k)$ further below l than $\Psi'(.;k)$ is over K proceeds along similar lines.

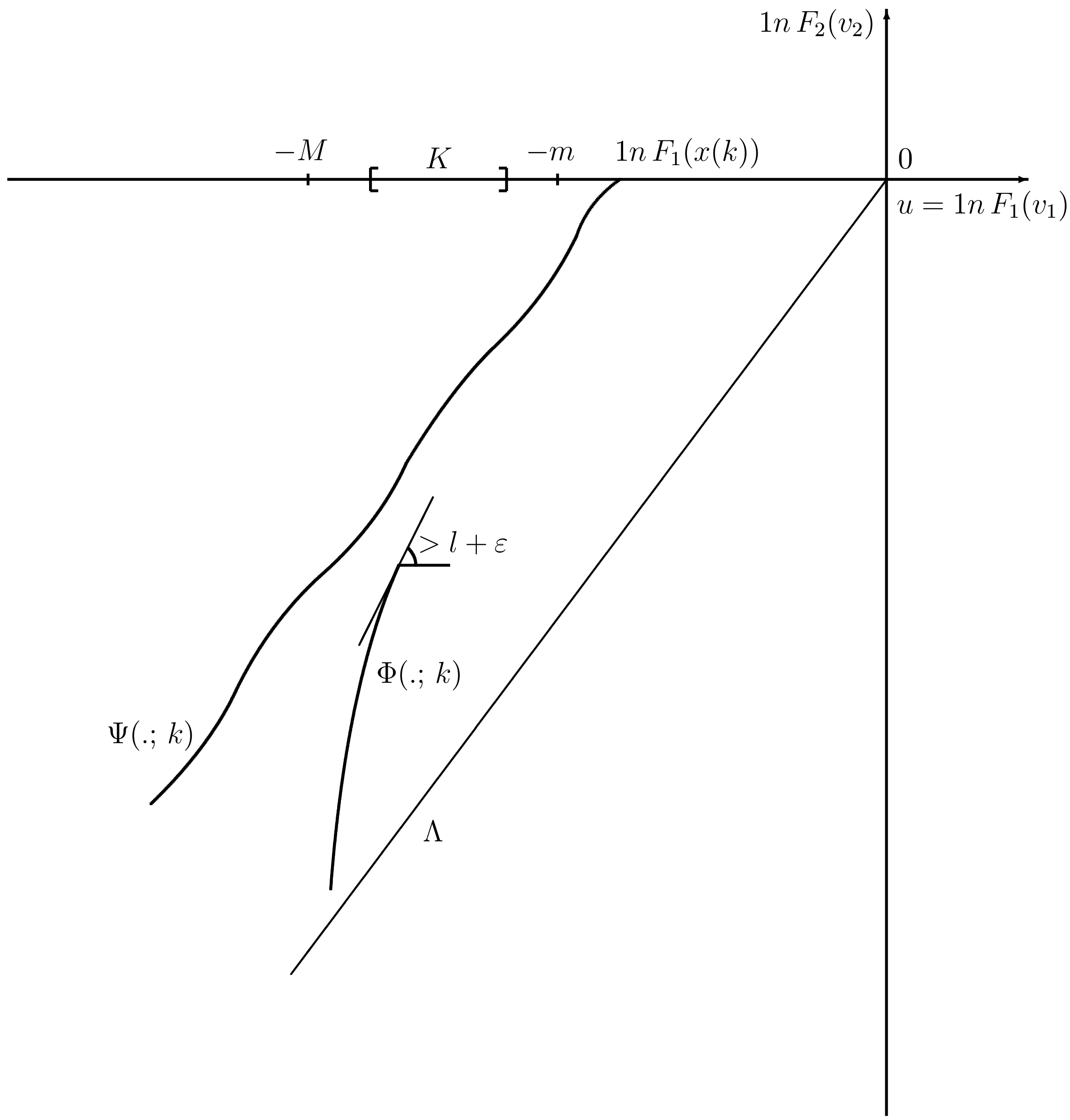

FIGURE 4: Ruling out $\Phi'(.;k)$ further above l than $\Psi'(.;k)$ is while $\Phi(.;k)$ is not larger than $\Psi(.;k)$.

We have Lemma 3 below[19], whose detailed proved is in Appendix 3.

[19]Because of the discontinuity of the derivative of $\Psi(.;k)$ at $\ln F_1(x(k))$, we express statement (ii) below in terms of the inverse function $\Psi^{-1}(.;k)$. As it can be easily

When $r = c$, in order to bound $\Phi'(s;k)$ for s tending towards $-\infty$, that is, for bidder 1's value approaching c, we assume in (v) below that the elasticity ε_1 of the density f_1 with respect to F_1 is bounded from below.

Lemma 3: *If $r = c$, assume F_1 is strictly log-concave over $(c, c+\varepsilon)$, for some $\varepsilon > 0$. Then, when the function spaces below are endowed with the compact-open topology (the topology of uniform convergence over every compact subset):*

(i) $\lim_{k\to 0} x(k) = d$;

(ii) $\Psi^{-1}(.;k)$ *tends towards* Λ^{-1} *in* $C^1(\mathbb{R}_-, \mathbb{R})$;

(iii) $\Phi(.;k)$ *tends towards* Λ *in* $C^1((\ln F_1(r), 0), \mathbb{R})$; *if* $r > c$, $\Phi(.;k)$ *also tends towards* Λ *in* $C^0([\ln F_1(r), 0], \mathbb{R})$;

(iv) $\overline{\lim}_{(s;k)\to(0;0)} \Phi'(s;k) \le l$;

(v) If the elasticity ε_1 *is bounded from below, then:* $\overline{\lim}_{(s;k)\to(-\infty;0)} \Psi'(s;k) \le l$ *and, when* $r = c$, $\overline{\lim}_{(s;k)\to(-\infty;0)} \Phi'(s;k) \le l$.

Proof: See Appendix 3.

Thanks to Lemma 3, we can prove that, away from the extremities of the value interval, the slope of $\varphi(.;k)$ tends towards one. We list this and other useful consequences from Lemma 3 about the limits of derivatives in Theorem 3 below.

Theorem 3–Convergence of the derivatives with respect to value or bid: *If $r = c$, assume F_1 is strictly log-concave over $(c, c+\varepsilon)$, for some $\varepsilon > 0$. Extend the derivatives $\beta_i'(.;k)$, $\alpha_i'(.;k)$, $\varphi'(.;k)$ to $k = 0$ as follows:*

$$\beta_i'(v;0) = \alpha_i'(v;0) = \varphi'(v;0) = 1,$$

verified by appealing to (i), (ii) is equivalent to the uniform convergence over any interval $[\underline{u}, \ln F_1(x(k))]$ of $\Psi(.;k)$ and its derivative towards Λ and its derivative, that is, to the following statement: (ii)' $\max_{u\in[\underline{u},\ln F_1(x(k))]} \max(|\Psi'(u;k) - l|, |\Psi(u;k) - lu|)$ tends towards zero with k, for all finite $\underline{u} < 0$.

for all v in (r,d) and all $i=1,2$. Then:

(i)[20] $\beta_i'(v;k), \alpha_i'(v;k), \varphi'(v;k)$ *are continuous jointly in both variables at $(v;0)$, for all v in (r,d);*

(ii) $\overline{\lim}_{(v;k)\to(d;0)}\varphi'(v;k)\leq 1$*;*

(iii) If $r=c$ and the elasticity ε_1 is bounded from below, then $\overline{\lim}_{(v;k)\to(c;0)}\frac{d\ln F_2(\varphi(v;k))}{d\ln F_1(v)}\leq l$*.*

Proof: See Appendix 4.

From the initial system of differential equations (7) and the convergence of the derivatives, we obtain in Theorem 4 below the rates of convergence of the bidding functions and the function φ, which determines the equilibrium allocation, and hence of the seller's expected revenues. Let $ER(k)$ and $ES(k)$ be the expected revenues and total surplus from the equilibrium of the 2–k-PA. Thus, $ER(0)$ and $ES(0)$ are the expected revenues and total surplus from the SPA.

Theorem 4–Rates of convergence with respect to k: *If $r=c$, assume F_1 is strictly log-concave over $(c,c+\varepsilon)$, for some $\varepsilon>0$. Then:*

(i) For all v in (r,d) and all $i,j=1,2$ with $i\neq j$, we have:

$$\lim_{(b;k)\to(v;0)}\frac{\alpha_i(b;k)-b}{k} = \lim_{(u,k)\to(v,0)}\frac{u-\beta_i(u;k)}{k}=\rho_j(v)^{-1};$$
$$\lim_{(u,k)\to(v,0)}\frac{\varphi(u;k)-u}{k} = \frac{l-1}{l}\rho_1(v)^{-1}.$$

(ii) For all $\gamma>0$, $\overline{\lim}_{k\to 0}\max_{u\in[r+\gamma,d]}\left(\frac{\varphi(u;k)-u}{k}-(l-1)\frac{F_1(u)}{f_1(u)}\right)\leq 0$. *When $r=c$, this inequality also holds true for $\gamma=0$ if the elasticity ε_1 is bounded from below.*

[20] As we already observed, $\varphi'(d;k)=\frac{1}{l}$, for all k. Consequently, $\varphi'(v;k)$ is discontinuous at $(d;0)$ if $l>1$.

(iii) If $r = c$ and the elasticity ε_1 is bounded from below, then $\frac{d_r}{dk}ER(0)$ and $\frac{d_r}{dk}ES(0)$ exist and we have:

$$\frac{d_r}{dk}ER(0) = (l-1)\int_c^d (\omega_1(v) - \omega_2(v)) F_2(v)\, dF_1(v) \geq 0, \quad (11)$$

$$\frac{d_r}{dk}ES(0) = 0; \quad (12)$$

and $\frac{d_r}{dk}ER(0) > 0$ if and only if F_1, F_2 are not identical.

(iv) If $r > c$, we have:

$$\frac{d_+}{dk}ER(0) \geq (l-1)\int_r^d (\omega_1(v) - \omega_2(v)) F_2(v)\, dF_1(v) \geq 0;$$

and $\frac{d_+}{dk}ER(0) > 0$ if and only if F_1, F_2 are not identical.

Proof: See Appendix 4.

When $r = c$, assume the seller maximizes the expectation of a nonnegative linear combination $\mu_R R + \mu_S S$ of revenues and total surplus. If he is not indifferent towards revenues, that is, if $\mu_R > 0$, he will want to increase k from zero, as, from this last theorem, $\mu_R \frac{d_r}{dk}ER(0) + \mu_S \frac{d_r}{dk}ES(0) = \mu_R \frac{d_r}{dk}ER(0) > 0$. Although the optimal size of k depends on the particular coefficients μ_R, μ_S and value distributions, it is always strictly positive. Example 2 below, which we alluded to in the introduction, demonstrates that even if the seller cares only about revenues, that is, $\mu_S = 0$, he may not want to increase k all the way to 1. This was to be expected as the bias of the FPA equilibrium allocation towards the weak bidder is in general not confined to instances where this bidder's virtual value is the larger one.

Because we cannot uniformly bound the rate of increase of φ in a neighborhood of the reserve price if it is binding and because bidder 2's virtual value is bounded above bidder 1's in such a neighborhood, we cannot rule out rates of increase even higher than the expression in (iv) above.

Example 1: Consider the example where the values are distributed over $[0,1]$ according to power distributions $F_i(v) = v^{ei}$ with $e_i \geq 1$, for $i = 1, 2$. From Theorem 4 (iii), we can easily compute the logarithmic derivative of the expected revenues with respect to k. We find[21] $\frac{d_r}{dk} \ln ER(0) = \frac{(e_2 - e_1)^2}{e_1 e_2 (e_1 + e_2 + 2)}$.

Example 2: The FPA brings more revenues than the SPA in Example 1 above[22]. Maskin and Riley (1985) show that the FPA brings strictly less revenues with two ex-ante different bidders and two possible values, one of which strictly above the reserve price. Any two such distributions are power related and can hence be approximated (for the topology of the weak topology, the weak-* topology) by power-related differentiable cumulative distributions with a common full interval support and that satisfy all our assumptions. Lebrun (2002) then guarantees the existence of such distributions for which the unique equilibrium of the FPA is sufficiently close (for the weak-* topology) to the original equilibrium for the SPA to bring strictly more revenues. Nevertheless, for these same approximating distributions, Theorem 4 implies that the revenues from the 2–k-PA exceed those from the SPA (and hence the FPA) for all small $k > 0$.

Remark: Our results depend on full value support. In the two-bidder two-value example of Maskin and Riley (1985), a high-value bidder's equilibrium bid distribution in the 2–k-PA becomes concentrated at the high and low values and the revenues tend towards the, smaller, revenues from the FPA as k approaches zero. As we mentioned in Example 2 above, our results

[21] This formula gives $\frac{d}{dk} \ln ER(0) = 0.3214$ when $e_1 = 1$ and $e_2 = 4$. As the numerical estimations in Marshall, Meurer, Richard and Stromquist (1994) imply $\ln ER(1) - \ln ER(0) = 0.0803$, the relative rate of increase $\frac{d}{dk} \ln ER(k)$ must obviously decrease at some k's between 0 and 1. This is also the case for $e_1 = 2$ and $e_2 = 3$, where $\ln ER(1) - \ln ER(0) = 0.0071$ while $\frac{d}{dk} \ln ER(0) = 0.0238$.

[22] See, for example, Lebrun (1996).

imply the existence of small revenue-improving k's for any given approximation of this discrete example by distributions that satisfy our assumptions, in particular of full interval support.

3. The Second-Price Auction with Toeholds and the Amsterdam Auction with Two Bidders

The rules of the (second-price) Amsterdam auction or AA with two bidders are identical to those of the SPA except that the seller hands both bidders a premium equal to the share k of the difference between the losing bid and the reserve price c[23]. We assume that bidders cannot submit bids above a very large (larger than d) but finite bound B[24,25]. The AA can be considered a slight variant of the SPA when k is small. As it can be easily verified, bidding strictly below value is weakly dominated in the AA.

This auction is equivalent to a SPA preceded by the sale to every bidder and at the unit price c of a share k of ownership in the item to be auctioned[26]. Indeed, every bidder would still receive as additional benefit the proportion k of the difference between the auction price and c, although now as profits from his ownership share.

As in the previous subsection, F_1 is continuous over $\mathbb{R}$ and continuously differentiable over $[c, d]$. Because we wish to apply the results of Subsection 2.2 to the value distributions "reflected" through the middle $(c + d)/2$ of the value support, we apply this same transformation to our assumptions. Thus,

[23] As there are two bidders, the auction proceeds directly to the "second-stage" where c is the "bottom price," in the terminology of Hu, Offerman, and Zhou (2011).

[24] Or we may assume that they become infinitely risk-averse above a certain high bid threshold.

[25] If unboundedly large bids were allowed, multiple equilibria would exist (as they would exist in the FPA if there was no lower bound on the allowable bids). See also Footnote 28.

[26] A bidder is not allowed to bid if he refuses to buy his share (when only one bidder bids, the auction price is the reserve price c). Not buying one's share is then weakly dominated.

in this section, F_1 is twice-continuously differentiable over $[c, d)$ and has its derivative f_1 strictly positive over $[c, d)$.

Similarly to the 2–k-PA, the equilibrium of the AA is related to the equilibrium of a FPA with modified value distributions[27]. We have Theorem 5 below, where $\theta(k)$ is the bidders' common equilibrium bid at the smallest value c.

Theorem 5: *Let k be in $(0, 1]$.*

(i) **Equilibrium of the AA:** *A couple of strategies $(\beta_1(.;k), \beta_2(.;k))$ is an equilibrium of the AA if and only if:*

1. For all $i = 1, 2$: $\beta_i(d;k)$ specifies bidding bids not smaller than d.

2. $\beta_1(.;k), \beta_2(.;k)$ are bidding functions over $[c, d)$ such that their inverses $\alpha_1(.;k) = \beta_1^{-1}(.;k), \alpha_2(.;k) = \beta_2^{-1}(.;k)$ exist, are strictly increasing, strictly below the identity function, and form a solution over $[\theta(k), d)$ of the system of differential equations (13) below

$$\begin{aligned} \frac{d \ln(1 - F_1(\alpha_1(b;k)))}{db} &= \frac{1}{(\alpha_2(b;k) - b)/k} \\ \frac{d \ln(1 - F_2(\alpha_2(b;k)))}{db} &= \frac{1}{(\alpha_1(b;k) - b)/k} \end{aligned} \tag{13}$$

[27] If the premium is paid only to the loser and not to the winner, submitting a bid strictly smaller than the value divided by $1 + k$ is weakly dominated. The equilibrium $\left(\widehat{\beta}_1(.;k), \widehat{\beta}_2(.;k)\right)$ of this variant of the AA for the value distributions F_1, F_2 is related to the equilibrium $\left(\widetilde{\beta}_1(.;k), \widetilde{\beta}_2(.;k)\right)$ of the AA, the way we have defined it (following Goeree and Offerman, 2004), for the value distributions $\widetilde{F}_1, \widetilde{F}_2$ over $[c/(1+k), d/(1+k)]$ such that $\widetilde{F}_1(x) = 1 - (1 - F_1((1+k)x))^{1/1+k}$, for all $x \in [c/(1+k, d/(1+k))]$, $i = 1, 2$. The relation between the equilibria is as follows: $\widehat{\beta}_i(x;k) = \widetilde{\beta}_i(x/(1+k);k)$, for all x in $[c, d]$ and $i = 1, 2$. In particular, every bidder with value d bids $d/(1+k)$ in this variant of the AA.

and their continuous extensions satisfy the boundary conditions:

$$\begin{aligned} \alpha_1(\theta(k);k) &= \alpha_2(\theta(k);k) = c, && (14) \\ \alpha_1(d;k) &= \alpha_2(d;k) = d, && (15) \end{aligned}$$

for a certain value in (c,d) of the parameter $\theta(k)$.

(ii) **Relations to the 2–k-PA and the FPA:** *Let $(\beta_1(.;k), \beta_2(.;k))$ be a couple of strategies. Then, the three statements below are equivalent:*

1. $(\beta_1(.;k), \beta_2(.;k))$ *is an equilibrium of the AA;*
2. $\left(\widetilde{\beta}_1(.;k), \widetilde{\beta}_2(.;k)\right)$ *is an equilibrium of the FPA where the bidders' values are distributed according to $\widetilde{F}_1^{1/k}, \widetilde{F}_2^{1/k}$;*
3. $\left(\widetilde{\beta}_1(.;k), \widetilde{\beta}_2(.;k)\right)$ *is an equilibrium of the 2–k-PA where the bidders' values are distributed according to $\widetilde{F}_1, \widetilde{F}_2$;*

where the transformed strategies and value distributions are defined as follows:

$$\begin{aligned} \widetilde{\beta}_i(\widetilde{v};k) &= c + d - \beta_i(c + d - \widetilde{v};k) \\ \widetilde{F}_i(\widetilde{v}) &= 1 - F_i(c + d - \widetilde{v}), \end{aligned}$$

for all $\widetilde{v}$ in $[c,d]$ and $i = 1,2$.

Proof: Similar to proof of Theorem 1[28].

If we perform the change of variables $\widetilde{v} + v = b + \widetilde{b} = c + d$ into the

[28] Here, the proof of the "terminal" condition $\beta_i(d;k) = d$, $i = 1,2$, in the AA is similar to the proof of the "initial" condition $\widetilde{\beta}_i(c;k) = c$, $i = 1,2$, in a FPA where the reserve price is not larger and possibly strictly smaller than c. If the condition at d was not satisfied in the AA, a bidder who would win for sure for some value $v < d$ (from our assumption that bids must not exceed B, such a bidder would exist) would increase his payoff by decreasing his bid to a bid $b > v$ that would lose with strictly positive probability.

expression $\int_c^b (v - b' + k(b' - c))\, dF_j(\alpha_j(b')) + \int_b^d k(b-c)\, dF_j(\alpha_j(b'))$ of the expected payoff of bidder i with value v and bid b in the AA, we obtain:

$$\int_c^{\widetilde{b}} \left(\widetilde{v} - (1-k)\widetilde{b}' - k\widetilde{b}\right) d\widetilde{F}_j\left(\widetilde{\alpha}_j\left(\widetilde{b}'; k\right)\right)$$
$$+ \int_c^d (v - b' + k(b' - c))\, dF_j(\alpha_j(b'; k)),$$

where the second term–bidder i's payoff when winning with certainty–does not depend on bidder i's decision variable; and the relation with the 2–k-PA with value distributions $\widetilde{F}_1, \widetilde{F}_2$ becomes apparent.

Theorem 5 allows to translate to the AA our results from the previous section, for example, those on the existence and uniqueness of the equilibrium and those on the convergence towards truth bidding (Corollary 1 and Theorem 2). We focus on the results about the rates of the change of expected revenues ER^A and total surplus ES^A. Thus, we assume the *decumulative* functions are power related, that is:

$$(1 - F_2) = (1 - F_1)^{1/l},$$

for some constant l, from which $\sigma_1 = l\sigma_2$ follows. Again, we assume bidder 2 is strong, that is, $l \geq 1$[29].

Furthermore, the decumulative function $1 - F_1$ is strictly log-concave over $(d - \varepsilon, d)$, for some $\varepsilon > 0$, and such that the elasticity $\widetilde{\varepsilon}_1(q)$ of its density $f_1\left(F_1^{-1}(1-q)\right)$ with respect to the decumulative probability q is bounded from below. This last assumption is equivalent to the existence of $T > 0$ such that $(1 - F_1)^T$ is convex. The local strict log-concavity of $1 - F_1$ at d implies that the hazard rate $\sigma_1(v)$ increases to $+\infty$ as v approaches d.

As $\widetilde{F}_1 = \widetilde{F}_2^l$, bidder 1 is the strong bidder and bids less aggressively than bidder 2 in the 2–k-PA and, as the bidding functions in the two auctions are

[29] Again, F_2 likelihood-ratio dominates F_1 and, in particular, we have $\rho_2 \geq \rho_1$.

inversely related, he bids more aggressively in the AA. See Figure 5.

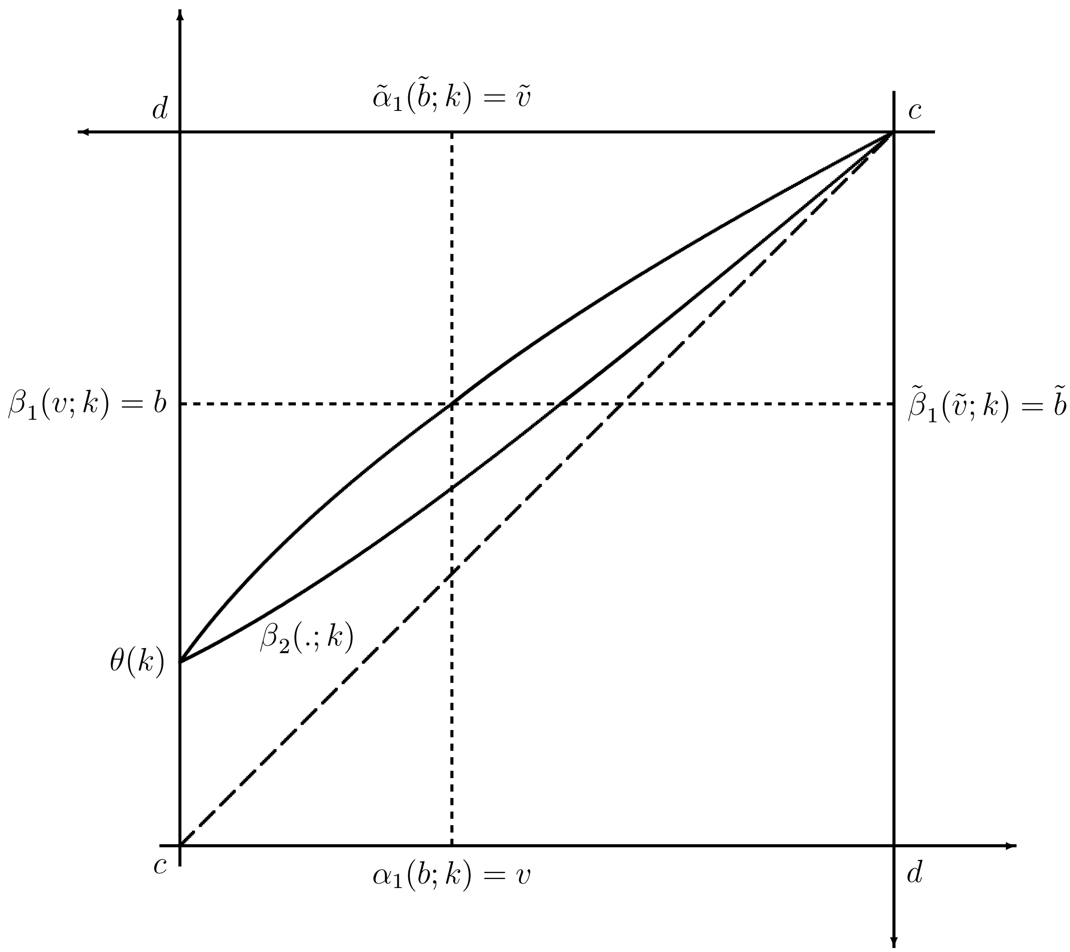

FIGURE 5: Relation between the equilibria of the 2–k-PA and AA. Values and bids are measured along the lower and left-hand axes for the AA and along the upper and right-hand axes for the 2–k-PA.

We have Theorem 6 below.

Theorem 6–Rates of change with respect to k:

(i) For all v in (c,d) and all $i,j=1,2$ with $i\neq j$, we have:

$$\begin{aligned}
\lim_{(b;k)\to(v;0)}\frac{b-\alpha_i(b;k)}{k} &= \lim_{(u,k)\to(v,0)}\frac{\beta_i(u;k)-u}{k}=\sigma_j(v)^{-1};\\
\lim_{(u,k)\to(v,0)}\frac{\varphi(u;k)-u}{k} &= (l-1)\,\sigma_1(v)^{-1}.
\end{aligned}$$

(iii) The right-hand derivatives $\frac{d_r}{dk}ER^A(0)$ and $\frac{d_r}{dk}ES^A(0)$ exist and we have:

$$\frac{d_r}{dk}ER^A(0) = (l-1)^2\int_c^d \sigma_1(v)^{-1}(1-F_1(v))\,dF_2(v)\geq 0, \tag{16}$$

$$\frac{d_r}{dk}ES^A(0) = 0; \tag{17}$$

and $\frac{d_r}{dk}ER^A(0)>0$ if and only if F_1,F_2 are not identical.

Proof: See Appendix 5.

The proof of Theorem 6 is similar to the proof of Theorem 4 with one significant difference: in the AA, contrary to the 2–k-PA, either bidder with the lowest value c makes some expected payoff–the premium he receives. However, the effect on the expected revenues is only of the second-order[30]. In fact, the premium is the product of the share k and the difference $\theta(k)-c$ between the minimum bid and c, which is also the difference between d and the maximum bid $\widetilde{\beta}_i(d;k)$ in the 2–k-PA for the transformed distributions and, consequently, vanishes as k becomes small (from Theorem 2).

It is easy to design examples similar to Examples 1 and 2 in the previous section. Note that, as the decumulative functions too are power related in the discrete 2-value example of Maskin and Riley (1985), it can also be approximated by continuous distributions satisfying all the assumptions of this section.

[30] This is is similar to the observation in Bulow, Huang and Klemperer (1999, page 442).

6.Conclusion

From Myerson (1981), improving on the revenues from the SPA would require "handicapping" the strong bidder. However, the implementation of handicaps may significantly decrease efficiency and may require information the auctioneer does not possess. With two bidders, we showed through explicit expressions for the first-order effects that revenues are higher, without significant change in efficiency, in the non-discriminatory 2–k-PA (under power related value cumulative functions) and the SPA with toeholds and the AA (under power related value decumulative functions) if the link between the winner's net payment and own bid is small and positive. The equilibrium allocation is slightly biased towards the weak bidder as the strong bidder bids less aggressively in reaction to this link. As the bidders' virtual values differ even when their actual values are identical, this slight bias produces an increase in expected revenues of a higher magnitude than the decrease in expected total surplus. We conjecture that the similar link implemented by the price increment rule in the electronic auction has similar effects[31].

We characterized the equilibrium of the 2–k-PA with n bidders and proved that, if there exists a weakest bidder, it tends towards truth-bidding when k vanishes.

Appendix 1

Lemma A1: $\lim_{v\rightarrow>c}\rho_i(v)=+\infty$

[31] This is consistent with the following intuition. If bidder i with value v and bid b faced bidder j who bid "close" to his value, bidder i's expected payoff would be close to $\int_c^{b-\Delta}(v-w-\Delta)\,dF_j(w)+(v-b)(F_j(b)-F_j(b-\Delta))$, where Δ is the price increment. The FOC would then be $\alpha_i(b;\Delta)=b+(F_j(b)-F_j(b-\Delta))/f_j(b)$, a strong bidder would shade his bid more, and higher revenues would ensue for small Δ's. Here, the effect on revenues would actually be of the second-order in Δ and the effect on surplus of a higher order. Whether this heuristic argument can be made rigorous is left for future work.

Proof: As we noticed after stating our assumptions, $\overline{\lim}_{v \to > c} \rho_i (v) = +\infty$. If $\underline{\lim}_{v \to > c} \rho_i (v) < +\infty$, the mean value theorem would imply $\underline{\lim}_{v \to > c} \frac{d}{dv} \rho_i (v) = -\infty$, which would contradict our assumptions. ||

Proof of Theorem 1:

Proof of the characterization in Theorem 1 (i):

We present only the substance of the arguments. They can easily be made more formal along the lines in Lebrun (1997, 1999).

Sketch of the proof of the necessity of the characterization in Theorem 1 (i.1, i.2, i.3):

A. A bidder's weakly undominated strategy must not recommend bidding strictly above his value. (i.1) follows.

B. As in Lemmas A1-1 and A1-2 in Lebrun (1997), r is the minimum of the support of any bidder's bid above r and any bidder with value $v > r$ has a strictly positive expected payoff. As in Lemma A1-4 in Lebrun (1997), Myerson (1981) implies the continuity and monotonicity with respect to the value of any bidder's interim expected payoff.

C. Similarly to the statement of Lemma A1-21 in Lebrun (1997), if $b > r$ is in bidder h's bid support, b must be a point of increase to the left of at least two bidders' bid cumulative functions. That is, there must exist bidders i and j, $j \neq i$, such that $G_i (b - \varepsilon; k) < G_i (b; k)$ and $G_j (b - \varepsilon; k) < G_j (b; k)$, for all $\varepsilon > 0$, where G_i, G_j are bidders i and j's cumulative functions that are continuous from the right. Otherwise, there would exist a gap $(b - \varepsilon, b]$ where no other bidder bids and a bidder would increase his payoff strictly if he bid $b - \varepsilon$ when he is supposed to bid close to b. Notice that bidder h may be one of the two bidders i and j[32].

[32] The proof here is different from the proof of Lemma A1-21 in Lebrun (1997). This latter proof uses previously established properties of the equilibria of the FPA.

D. There does not exist a bid $b > r$ that is a mass point of the bid distributions of two or more bidders. Because the value distributions are atomless, if there existed such a bid, a bidder would submit b with a strictly positive probability for some values strictly smaller than b. This bidder would increase strictly his payoff if he bid slightly above b instead.

E. Similarly to the statement of Lemma A1-7 in Lebrun (1997) (the proof, however, is different), the bid distributions are atomless strictly above r. In fact, from D. above, there could only exist an atom $b > r$ of a single bidder's bid distribution. From C., there exists another bidder, say bidder i, who bids at or below and arbitrarily close to b. For a deviation slightly above b by bidder i not to be strictly profitable, his value must approach b when his bid approaches b. From the continuity in B., his payoff when his value is b must therefore be:

$$(1-k)\int_c^b (b - \max(w, r))\, d\prod_{j \neq i} G_j(w; k);$$

while if he submits bids close to and above bid $b - \varepsilon$, with $\varepsilon > 0$, it would tend towards:

$$
\begin{aligned}
& k\varepsilon \prod_{j\neq i} G_j\left(b-\varepsilon;k\right) + (1-k)\int_c^{b-\varepsilon} \left(b-\max\left(w,r\right)\right) d\prod_{j\neq i} G_j\left(w;k\right) \\
= \;& (1-k)\int_c^{b} \left(b-\max\left(w,r\right)\right) d\prod_{j\neq i} G_j\left(w;k\right) \\
& + \left\{ k\varepsilon \prod_{j\neq i} G_j\left(b-\varepsilon;k\right) - (1-k)\int_{(b-\varepsilon,b)} \left(b-\max\left(w,r\right)\right) d\prod_{j\neq i} G_j\left(w;k\right) \right\} \\
\geq \;& (1-k)\int_c^{b} \left(b-\max\left(w,r\right)\right) d\prod_{j\neq i} G_j\left(w;k\right) \\
& + \varepsilon \left\{ k \prod_{j\neq i} G_j\left(b-\varepsilon;k\right) - (1-k)\left(\prod_{j\neq i} G_j\left(b^{-};k\right) - \prod_{j\neq i} G_j\left(b-\varepsilon;k\right) \right) \right\} \\
> \;& (1-k)\int_c^{b} \left(b-\max\left(w,r\right)\right) d\prod_{j\neq i} G_j\left(w;k\right),
\end{aligned}
$$

where the last inequality holds for all $\varepsilon > 0$ sufficiently small. Bidder i would then have a strictly profitable deviation, which is impossible at an equilibrium.

This also proves that the bid cumulative functions $G_1\left(b;k\right), ..., G_n\left(b;k\right)$ and hence the bidder's expected payoffs are continuous in $b > r$.

F. Because a bidder's expected payoff when he bids strictly above r has strictly increasing differences in his bid and value, equilibrium bidding strategies must be nondecreasing (the proof of the monotonicity in Lemma A1-8 of Lebrun 1997 can also be easily adapted to the 2–k-PA) and consequently, from E., strictly increasing. Theorem 1 (i.3), where $\alpha_i\left(.;k\right)$ is the uniquely determined inverse of $\beta_i\left(.;k\right)$, follows. (2) also follows as any bidder with a value $v > r$ would increase his bid slightly instead of bidding r if a tie could occur there with strictly positive probability.

G. Similarly to Lemma A1-22 in Lebrun (1997) for the FPA (see also Appendix A and Figure 1 in Lebrun, 1999), we can also prove that if bidder

j's strategy $\beta_j(.;k)$ is continuous at v_j and specifies bidding $\beta_j(v_j;k) = b$ strictly inside a discontinuity jump (b', b'') of the strategy of bidder i with value v_i, then bidder j has a strictly higher value, that is, $v_j > v_i$.

In fact, bidder j's expected payoff when he submits b must be at least what he obtains with the bid b', that is:

$$\begin{aligned}
& \int_c^b (v_j - kb - (1-k)\max(w,r))\, d\prod_{h\neq j} G_h(w;k) \\
= \;& (v_j - b)\prod_{h\neq j} G_h(b;k) + (1-k)\int_r^b \prod_{h\neq j} G_h(w;k)\, dw \\
= \;& (v_j - v_i)\prod_{h\neq j} G_h(b;k) + (v_i - b)\prod_{h\neq j} G_h(b;k) \\
& + (1-k)\int_r^b \prod_{h\neq j} G_h(w;k)\, dw \\
\geq \;& \int_c^{b'} (v_j - kb' - (1-k)\max(w,r))\, d\prod_{h\neq j} G_h(w;k) \\
= \;& (v_j - v_i)\prod_{h\neq j} G_h(b';k) + (v_i - b')\prod_{h\neq j} G_h(b';k) \\
& + (1-k)\int_r^{b'} \prod_{h\neq j} G_h(w;k)\, dw,
\end{aligned}$$

and therefore:

$$\begin{aligned}
& (v_j - v_i)\prod_{h\neq j} G_h(b) + (v_i - b)\prod_{h\neq j} G_h(b) \\
& + (1-k)\int_{b'}^{b} \prod_{h\neq j} G_h(w)\, dw \\
\geq \;& (v_j - v_i)\prod_{h\neq j} G_h(b') + (v_i - b')\prod_{h\neq j} G_h(b'),
\end{aligned}$$

and, after dividing both sides by $G_i(b;k) = G_i(b';k)$:

$$
\begin{aligned}
& (v_j - v_i) \prod_{h \neq i,j} G_h(b;k) + (v_i - b) \prod_{h \neq i,j} G_h(b;k) \\
& + (1-k) \int_{b'}^{b} \prod_{h \neq i,j} G_h(w;k)\, dw \\
\geq\ & (v_j - v_i) \prod_{h \neq i,j} G_h(b';k) + (v_i - b') \prod_{h \neq i,j} G_h(b';k). \qquad \text{(A1.1)}
\end{aligned}
$$

On the other hand, b' must be at least as good as b for bidder i and we have:

$$
\begin{aligned}
& (v_i - b') \prod_{h \neq i} G_h(b';k) \\
\geq\ & (v_i - b) \prod_{h \neq i} G_h(b;k) + (1-k) \int_{b'}^{b} \prod_{h \neq i} G_h(w;k)\, dw \\
\geq\ & G_j(b';k) \left\{ (v_i - b) \prod_{h \neq i,j} G_h(b;k) + (1-k) \int_{b'}^{b} \prod_{h \neq i,j} G_h(w;k)\, dw \right\},
\end{aligned}
$$

and dividing both sides by $G_j(b';k)$ we find:

$$
\begin{aligned}
& (v_i - b') \prod_{h \neq i,j} G_h(b';k) \\
\geq\ & (v_i - b) \prod_{h \neq i,j} G_h(b;k) + (1-k) \int_{b'}^{b} \prod_{h \neq i,j} G_h(w;k)\, dw. \qquad \text{(A1.2)}
\end{aligned}
$$

(A1.1) and (A1.2) imply $(v_j - v_i) \prod_{h \neq i,j} G_h(b;k) \geq (v_j - v_i) \prod_{h \neq i,j} G_h(b';k)$, that is:

$$
(v_j - v_i) \left(\prod_{h \neq i,j} G_h(b;k) - \prod_{h \neq i,j} G_h(b';k) \right) \geq 0.
$$

From C., $\prod_{h \neq i,j} G_h(b;k) - \prod_{h \neq i,j} G_h(b';k) > 0$ and we must have $v_j \geq v_i$. If v_j

was equal to v_i, there would exist another $v_j' < v_j$ where bidder j's strategy β_j would be continuous and recommend a bid within bidder i's jump, which (as we have just shown) would be impossible. Consequently, $v_j > v_i$.

H. The supports of the bidders' bid distributions have the same maximum. In fact, if this was not the case by monotonicity (F. above) there would exist a bidder i who submits b' when his value is d and a bidder $j \neq i$ whose strategy is continuous at a $v_j < d$ and such that $\beta_j(v_j) = b > b'$. The same proof as in G. above rules out such a possibility. (3) follows.

I. As in the proof of Lemma A1-18 in Lebrun (1997), which is similar to the proof of Lemma 3.6 in Griesmer, Levitan, and Shubik (1967), we can prove the differentiability of a bidder's probability of winning with respect to his own bid where his bidding strategy is locally continuous strictly above r. The main idea of the proof is to express that an optimal b is better for bidder i with value v than a bid b' as the inequality below:

$$\frac{\prod_{j\neq i} G_j(b;k) - \prod_{j\neq i} G_j(b';k)}{b-b'} \geq \frac{\prod_{j\neq i} G_j(b';k)}{v-b} - \frac{1-k}{v-b}\frac{\int_{b'}^{b} \prod_{j\neq i} G_j(w;k)\,dw}{b-b'},$$

if $b' \leq b$ and the reverse inequality if $b' \geq b$. Making $b - b'$ tend towards zero in such inequalities and appealing to the continuity of $G_j(.;k)$ above r (from E. above) gives the result.

J. As in the proof of Lemma A1-19 in Lebrun (1997), from I. above the inverse bidding functions that are strictly increasing over an interval satisfy a system of differential equations similar to (1). In fact, the interval is the range of the corresponding direct continuous bidding functions and is included in a discontinuity jump of the other bidding functions. The probability of outbidding the bidders whose bidding functions jump is constant within the jump and can be factored out of the first-order conditions of the bidders bidding inside the jump.

K. From G., the only remaining possible kind of discontinuity is when some bidders have "nested" discontinuity jumps and the other bidders bid continuously within the smallest jump (b', b'') for strictly higher values. Ruling out this last type of discontinuity proceeds as in the proof of Lemma A1-24 in Lebrun (1997) (see also Appendix A and Figure 2 in Lebrun 1999) by showing that increasing his bid from b' or decreasing it from b'' must be strictly profitable to the bidder with the smallest jump. In turn this follows from expressions, obtained from the system in J. above, for the first-order effects of these two deviations on this bidder's expected payoff.

More precisely, suppose bidder j's bidding strategy β_j jumps from b' to b'' at v_j, the jump (b', b'') is included in discontinuity jumps of bidders $1, ..., j-1$, and bidders $j+1, ..., n$ bid continuously within this jump. The derivative with respect to the bid b within the jump of bidder j's payoff is:

$$\begin{aligned} & K \prod_{h>j} G_h(b;k) \left\{ -k + (v_j - b) \sum_{h>j} \frac{d}{db} \ln G_h(b;k) \right\} \\ = \; & K \prod_{h>j} F_h(\alpha_h(b;k)) \left\{ -k + (v_j - b) \sum_{h>j} \frac{d}{db} \ln F_h(\alpha_h(b;k)) \right\}, \end{aligned}$$

where K is $\prod_{h<j} G_h(b;k) = \prod_{h<j} F_h(\alpha_h(b;k))$, which is constant in (b', b''). From the system of differential equations the functions α_h, $h > j$, satisfy (according to J.), this is also equal to:

$$\begin{aligned} & kK(v_j - b) \prod_{h>j} F_h(\alpha_h(b;k)) \\ & \left\{ -\frac{1}{v_j - b} + \frac{1}{n-j-1} \sum_{h>j} \frac{1}{\alpha_h(b;k) - b} \right\}. \end{aligned}$$

A nonpositive right-hand derivative at the lower extremity b' of the jump and a nonnegative left-hand derivative at the upper extremity b'', as an equilibrium would require, would imply:

$$\left(\sum_{h>j} \frac{v_j - b}{\alpha_h (b; k) - b}\right)_{b=b'} \leq \left(\sum_{h>j} \frac{v_j - b}{\alpha_h (b; k) - b}\right)_{b=b''},$$

which is impossible because, from G. above, $\alpha_h (b; k) > v_j$ and consequently $\sum_{h>j} \frac{v_j - b}{\alpha_h(b;k)-b}$ is strictly decreasing in b.

All bidding functions are therefore continuous and the system (1) is satisfied by all inverse bidding functions. We have proved that any equilibrium in weakly undominated strategies satisfies (i.1, i.2, i.3).

Sketch of the proof of the sufficiency of the characterization (i.1, i.2, i.3)[33]

It is weakly dominated to submit a bid strictly above one's value. It is unprofitable to submit a bid strictly below r when one's value is strictly above r or to submit a bid strictly above the common maximum bid $\eta (k)$. The sufficiency of the system (1) of FOC's then follows from the strict monotonicity of the strategies and the nonnegativity of the cross second-order partial derivative, equal to $\frac{d}{db} \ln \prod_{j \neq i} F_j (\alpha_j (b))$, of any bidder i's payoff with respect to his own bid b in $(r, \eta (k))$ and own value (a direct proof may proceed as the proof in Appendix A in Lebrun 1999 and Section 3 in Lebrun 1997).

Proof of Theorem 1 (ii): From Theorem 1 in Lebrun (1999), the differential system and boundary conditions in (i) are the same as those that characterize the equilibria of the FPA where the value distributions are $F_1^{1/k}, ..., F_n^{1/k}$ and (ii) follows.

[33]The requirement that if $\alpha_i (r) > r$ then $r > c$, as in Theorem 1 (3) in Lebrun (1999), is not necessary. In fact, it is implied by the system of differential equations (as proved in Lemma A2-7 in Lebrun, 1997).

End of the proof of Theorem 1: That all bidding functions must start rising from r if $r = c$ or if $n = 2$ then follows from Theorem 1 (3)[34] and Corollary 6 in Lebrun (1999). (4) therefore holds true. ||

Proof of Corollary 1:

Proof of (i): This follows from Theorem 1 (ii) above and Lebrun (1999) (or other papers on the FPA, such as Lebrun 1997 or Maskin and Riley 2000)[35].

Proof of (ii): The formula is a direct consequence of Theorem (ii) above and the envelope theorem or Myerson (1981).

Proof of (iii): Because $F_j^{1/k}/F_i^{1/k} = (F_j/F_i)^{1/k}$, $F_i \preceq_{rh} F_j$ implies $F_i^{1/k} \preceq_{rh} F_j^{1/k}$. From Corollary 3 (ii) in Lebrun (1999), we then have $\beta_i(.;k) \geq \beta_j(.;k)$ or, equivalently, $\alpha_j(.;k) \geq \alpha_i(.;k)$. From the formula (A2.2) in Lebrun (1999) or from the difference between the equations (1) for i and j, we find:

$$
\begin{aligned}
& \frac{d}{db} \ln \frac{F_j \alpha_j}{F_i \alpha_i}(b;k) \\
= \; & \frac{d}{db} \ln F_j \alpha_j(b;k) - \frac{d}{db} \ln F_i \alpha_i(b;k) \\
= \; & \frac{k}{\alpha_i(b;k) - b} - \frac{k}{\alpha_j(b;k) - b} \\
\geq \; & 0,
\end{aligned}
$$

and (iii.1) is proved.

Assume $\rho_i(v) < \rho_j(v)$, for all v in (r,d). If there existed b in $(r, \eta(k))$ such that $\alpha_i(b;k) = \alpha_j(b;k)$, we would have from (1) (see the proof of (iii.1) above) $\alpha_j'(b;k)\rho_j(\alpha_j(b;k)) = \frac{d}{db}\ln F_j(\alpha_j(b;k)) = \frac{d}{db}\ln F_i(\alpha_i(b;k)) =$

[34] Also from Lemma A2-7 in Lebrun (1997).

[35] Of course, it could also follow from the literature (for example, Reny 1999) on the existence of a Nash equilibrium in discontinuous games.

$\alpha_i'(b;k)\,\rho_i(\alpha_i(b;k))$. From $\rho_i < \rho_j$ over (r, d), we would then have $\alpha_j'(b;k) < \alpha_i'(b;k)$ and $\alpha_j(.;k)$ would be strictly larger than $\alpha_i(.;k)$ to the right of b, which would contradict (iii.1). No such b then exists and we have proved (iii.2).

From (iii.1), all bidders 1 to $n-1$ must bid at least as aggressively as bidder n and hence, from Theorem 1 (i), only bidder n's bidding function may take the constant value r and the modification (iii.3) to the characterization in Theorem 1 follows.

Proof of (iv): Uniqueness when $r > c$ follows from Lebrun (1997, 1999) (specifically, Corollary 2 (i) in Lebrun 1999 or Corollary 2 in Lebrun 1997). Strict log-concavity of F_i and strict log-concavity of $F_i^{1/k}$ are equivalent. Uniqueness when $r = c$ then follows from Lebrun (2006). ||

Appendix 2

Proof of Theorem 2:

Proof of (i): For all $w \leq v$, the assumption of reverse-hazard-rate dominance and Corollary 1 (iii) imply:

$$\frac{\prod\limits_{j>i} F_j\left(\varphi_{ji}(w;k)\right)}{\prod\limits_{j>i} F_j\left(\varphi_{ji}(v;k)\right)} \leq \left(\frac{F_i(w)}{F_i(v)}\right)^{n-i}.$$

Obviously, we also have:

$$\frac{\prod\limits_{j<i} F_j\left(\varphi_{ji}(w;k)\right)}{\prod\limits_{j<i} F_j\left(\varphi_{ji}(v;k)\right)} \leq 1.$$

Consequently, from Corollary 1 (ii) we find:

$$0 \leq \frac{v - \beta_i(v;k)}{k} \leq \frac{1}{k}\int_r^v \left(\frac{F_i(w)}{F_i(v)}\right)^{(n-i)/k} dw, \tag{A2.1}$$

from which it follows:

$$\begin{aligned}
0 &\leq \overline{\lim}_{k\to 0}\frac{v-\beta_i\,(v;k)}{k} \\
&\leq \lim_{k\to 0}\frac{1}{k}\int_r^v\left(\frac{F_i\,(w)}{F_i\,(v)}\right)^{(n-i)/k}dw \\
&= \frac{F_i\,(v)}{(n-i)\,f_i\,(v)},
\end{aligned}$$

where the equality is from Lemma A2 (i) below.

That the upper bound is uniform over I, that is,

$$\overline{\lim}_{k\to 0}\max_{v\in I}\left(\frac{v-\beta_i\,(v;k)}{k}-\frac{F_i\,(v)}{(n-i)\,f_i\,(v)}\right)\leq 0, \tag{A2.2}$$

for $r>c$ and $I=[r,d]$ is a consequence of (A2.1) and Lemma A2 (iv) and for $r=c$ and $I=[c+\gamma,d]$, where $\gamma>0$, of (A2.1) and Lemma A2 (iii). From Lemma A2 (v), it is uniform over $[c,d]$ if $r=c$ and ε_i is bounded from below. We have proved the statements in (i) about $\beta_i\,(v;k)$.

Let ε and γ be arbitrary strictly positive numbers. From (A2.2), for all $I=[r,d]$ if $r>c$ or $I=[c+\gamma,d]$ with $\gamma>0$ if $r=c$, there exists $k'>0$ such that for all $0<k<k'$ we have:

$$\frac{v-\beta_i\,(v;k)}{k}\leq\frac{F_i\,(v)}{(n-i)\,f_i\,(v)}+\varepsilon,$$

for all v in I. Because $b\leq\alpha_i\,(b;k)$, we have $\alpha_i\,(b;k)\in I$ and the inequality above applies to $v = \alpha_i(b;k)$, for all b in I. We find[36]:

$$\begin{aligned} 0 &\leq \frac{\alpha_i(b;k) - b}{k} \\ &\leq \frac{\alpha_i(b;k) - \beta_i(\alpha_i(b;k);k)}{k} \\ &\leq \frac{F_i(\alpha_i(b;k))}{(n-i)\, f_i(\alpha_i(b;k))} + \varepsilon. \qquad \text{(A2.3)} \end{aligned}$$

From the continuity of $\frac{F_i}{(n-i)f_i}$, we then obtain that $\alpha_i(b;k)$ tends towards b uniformly over I when k tends towards zero and the statements in (i) about $\alpha_i(b;k)$ then follow from (A6.3). If ε_i is bounded from below, the same proof (but using the convergence of $F_i(v)/f_i(v)$ towards zero when v tends towards c) goes through with $I = [c,d]$ and establishes the remainder of (i).

Proof of (ii): From the assumption of reverse-hazard-rate dominance and Corollary 1 (iii), we have:

$$\frac{\prod_{j<n} F_j\left(\varphi_{jn}(w;k)\right)}{\prod_{j<n} F_j\left(\varphi_{jn}(v;k)\right)} \geq \left(\frac{F_n(w)}{F_n(v)}\right)^{(n-1)},$$

and consequently:

$$\frac{v - \beta_i(v;k)}{k} \geq \frac{1}{k}\int_r^v \left(\frac{F_n(w)}{F_n(v)}\right)^{(n-1)/k} dw,$$

[36] $\beta_i(\alpha_i(b;k);k) = b$ only when $b \leq \eta(k)$. Otherwise, $\beta_i(\alpha_i(b;k);k) \leq b$.

for all $w \leq v$. From Corollary 1 (ii), we then find:

$$\begin{aligned} & \underline{\lim}_{k\to 0} \frac{v-\beta_n(v;k)}{k} \\ \geq\ & \lim_{k\to 0} \frac{1}{k} \int_r^v \left(\frac{F_n(w)}{F_n(v)} \right)^{(n-1)/k} dw \\ =\ & \frac{F_n(v)}{(n-1)\, f_n(v)}. \end{aligned}$$

(ii) follows.

Proof of (iii): We first prove the statement about β_1 and α_1. If $r > c$, from (i) we have that $\beta_1(v;k)$ and $\alpha_1(v;k)$ are continuous in k uniformly for all v in $[r,d]$ and thereby jointly continuous. We may then assume $r = c$. From (i), $\beta_1(v;k)$ and $\alpha_1(v;k)$ are continuous in k locally uniformly with respect to v at $(v;0)$, for all v in $(c,d]$. The joint continuity at all such points follows. As $c = \beta_1(c;k) \leq \beta_1(v;k) \leq v$, for all v and k, the joint continuity of $\beta_1(v;k)$ at $(c;0)$ is obvious. Take any $w > c$. Then, for all v in $[c, \beta_1(w;k)]$ we have, from the monotonicity of α_1, $c = \alpha_1(c;0) \leq \alpha_1(v;k) \leq \alpha_1(\beta_1(w;k);k) = w$. As $\beta_1(w;k)$ tends towards c as $(w;k)$ tends towards $(c;0)$, $\alpha(v;k)$ is continuous with respect to both variables at $(c;0)$.

Let i be different from 1. For all $v > r$, $\beta_i(v;k)$ must be optimal for bidder i with value v and in particular better than the bid v. After integration by parts, his expected payoff $(1-k)\int_c^v (v - \max(b,r))\, d\prod_{j\neq i} F_j(\alpha_j(b))$ if he submits v is:

$$(1-k)\int_r^v \prod_{j\neq i} F_j(\alpha_j(b;k))\, db$$

and if he submits $\beta_i(v;k)$:

$$\begin{aligned}
& k\left(v-\beta_i(v;k)\right)\prod_{j\neq i}F_j\left(\alpha_j\left(\beta_i(v;k);k\right)\right) \\
& +(1-k)\int_c^{\beta_i(v;k)}\left(v-\max(b,r)\right)d\prod_{j\neq i}F_j\left(\alpha_j(b;k)\right) \\
= \quad & \left(v-\beta_i(v;k)\right)\prod_{j\neq i}F_j\left(\alpha_j\left(\beta_i(v;k);k\right)\right)+(1-k)\int_r^{\beta_i(v;k)}\prod_{j\neq i}F_j\left(\alpha_j(b;k)\right)db.
\end{aligned}$$

Consequently, we must have:

$$\begin{aligned}
& \left(v-\beta_i(v;k)\right)\prod_{j\neq i}F_j\left(\alpha_j\left(\beta_i(v;k);k\right)\right) \\
= \quad & \int_{\beta_i(v;k)}^{v}\prod_{j\neq i}F_j\left(\alpha_j\left(\beta_i(v;k);k\right)\right)db \\
\geq \quad & (1-k)\int_{\beta_i(v;k)}^{v}\prod_{j\neq i}F_j\left(\alpha_j(b;k)\right)db.
\end{aligned}$$

and then:

$$\begin{aligned}
& k(d-c) \\
\geq \quad & k\int_{\beta_i(v;k)}^{v}\prod_{j\neq i}F_j\left(\alpha_j(b;k)\right)db \\
\geq \quad & \int_{\beta_i(v;k)}^{v}\left(\prod_{j\neq i}F_j\left(\alpha_j(b;k)\right)-\prod_{j\neq i}F_j\left(\alpha_j\left(\beta_i(v;k);k\right)\right)\right)db \\
\geq \quad & 0. \qquad \text{(A2.4)}
\end{aligned}$$

Let $(v_l,k_l)_{l\geq 1}$ be a sequence tending towards $(v,0)$ with v in $(r,d]$ and such that $k_l>0$, for all l. Suppose $\beta_i(v_l;k_l)$ does not tend towards v. Extracting a subsequence if necessary, we may assume that it tends towards $b'\neq v$. As $\beta_i(v_l;k_l)\leq v_l$, we must have $b'<v$. From (A2.4) applied to

(v_l, k_l), we have:

$$\begin{aligned}
& k_l\left(d-c\right) \\
\geq\ & \int_{\beta_i(v_l;k_l)}^{v_l}\left(\prod_{j\neq i}F_j\left(\alpha_j\left(b;k_l\right)\right)-\prod_{j\neq i}F_j\left(\alpha_j\left(\beta_i\left(v_l;k_l\right);k_l\right)\right)\right)db \\
\geq\ & \prod_{j\neq 1,i}F_j\left(\alpha_j\left(\beta_i\left(v_l;k_l\right);k_l\right)\right)\int_{\beta_i(v_l;k_l)}^{v_l}\left(F_1\left(\alpha_1\left(b;k_l\right)\right)-F_1\left(\alpha_1\left(\beta_i\left(v_l;k_l\right);k_l\right)\right)\right)db \\
\geq\ & \prod_{j\neq 1,i}F_j\left(\beta_i\left(v_l;k_l\right)\right)\int_{\beta_i(v_l;k_l)}^{v_l}\left(F_1\left(\alpha_1\left(b;k_l\right)\right)-F_1\left(\alpha_1\left(\beta_i\left(v_l;k_l\right);k_l\right)\right)\right)db.
\end{aligned}$$

Making l tends towards $+\infty$ and using the joint continuity of $\alpha_i\left(b;k\right)$, we find:

$$\prod_{j\neq 1,i}F_j\left(b'\right)\int_{b'}^{v}\left(F_1\left(b\right)-F_1\left(b'\right)\right)db=0,$$

which is possible only if $r=c$, $n\geq 3$, and $b'=c$. Actually, this also is impossible as bidder i's payoff would tend towards zero in this case (while if he bid his value, his payoff $\left(1-k_l\right)\int_c^v\prod_{j\neq i}F_j\left(\alpha_j\left(b;k_l\right)\right)db$ would be at least $\left(1-k_l\right)\int_c^v\prod_{j\neq i}F_j\left(b\right)db$, which would approach $\int_c^v\prod_{j\neq i}F_j\left(b\right)db>0$). We have proved that $\beta_i\left(v;k\right)$ is continuous at $\left(v;0\right)$ jointly in $\left(v;k\right)$, for all $v>r$. Continuity at $\left(r;0\right)$ is proved as for β_1, that is: $r\leq\beta_i\left(v;k\right)\leq v$ implies that $\beta_i\left(v;k\right)$ tends towards r if $\left(v;k\right)$ tends towards $\left(r;0\right)$.

The continuity of α_i is proved similarly. Assume $\left(b_l,k_l\right)$ tends towards $\left(b,0\right)$ with b in $(r,d]$. Suppose $\alpha_i\left(b_l;k_l\right)$ does not tend towards b. Extracting a subsequence if necessary, we may assume that it tends towards $v'\neq b$. As $\alpha_i\left(b_l;k_l\right)\geq b_l$, we must have $v'>b$. From (A2.4) applied to $\left(\alpha_i\left(b_l;k_l\right),k_l\right)$, we have:

$$
\begin{aligned}
& k_l\left(d-c\right) \\
\geq \ & \int_{\beta_i(\alpha_i(b_l;k_l);k_l)}^{\alpha_i(b_l;k_l)} \left(\prod_{j\neq i} F_j\left(\alpha_j\left(b;k_l\right)\right) - \prod_{j\neq i} F_j\left(\alpha_j\left(\beta_i\left(\alpha_i\left(b_l;k_l\right);k_l\right);k_l\right)\right)\right) db \\
\geq \ & \int_{b_l}^{\alpha_i(b_l;k_l)} \left(\prod_{j\neq i} F_j\left(\alpha_j\left(b;k_l\right)\right) - \prod_{j\neq i} F_j\left(\alpha_j\left(b_l;k_l\right)\right)\right) db \\
\geq \ & \prod_{j\neq 1,i} F_j\left(\alpha_j\left(b_l;k_l\right)\right) \int_{b_l}^{\alpha_i(b_l;k_l)} \left(F_1\left(\alpha_1\left(b;k_l\right)\right) - F_1\left(\alpha_1\left(b_l;k_l\right)\right)\right) db \\
\geq \ & \prod_{j\neq 1,i} F_j\left(b_l\right) \int_{b_l}^{\alpha_i(b_l;k_l)} \left(F_1\left(\alpha_1\left(b;k_l\right)\right) - F_1\left(\alpha_1\left(b_l;k_l\right)\right)\right) db,
\end{aligned}
$$

where we used the inequality $\beta_i\left(\alpha_i\left(b_l;k_l\right);k_l\right) \leq b_l$ and the nonnegativity of the integrand. Making l tends towards $+\infty$ and using the local uniformity of the convergence of $\alpha_1\left(b;k\right)$ towards b, we find:

$$
\prod_{j\neq 1,i} F_j\left(b\right) \int_b^{v'} \left(\prod_{j\neq i} F_j\left(b\right) - \prod_{j\neq i} F_j\left(v'\right)\right) db = 0,
$$

which is impossible as $b > c$.

Let $\left(v_t;k_t\right)_{t\geq 1}$ be a sequence converging towards $(r;0)$. If $\alpha_i\left(v_t;k_t\right)$ did not converge towards towards r, there would exist a subsequence $\alpha_i\left(v_{t_l};k_{t_l}\right)$ converging towards $u > r$. From what we have just proved, $\beta_i\left(\alpha_i\left(v_{t_l};k_{t_l}\right);k_{t_l}\right) = v_{t_l}$ would converge towards u, a contradiction. Consequently, α_i is also continuous at $(r;0)$.

At this point, the joint continuity of φ_{ji} at all $(v;k)$ for all i,j is immediate. $||$

Lemma A2–Technical lemma: *Let G be a continuous function over $[c,d]$ that is continuous differentiable and strictly positive over $(c,d]$ with a* *strictly positive derivative g over this semi-open interval. Then:*

(i) For all v in $(c,d]$, we have:

$$\lim_{l\to+\infty} l\int_c^v \left(\frac{G(w)}{G(v)}\right)^l dw = \frac{G(v)}{g(v)}.$$

(ii) For all v in $(c,d]$, we have:

$$\lim_{l\to+\infty} \int_c^v \left(\frac{G(w)}{G(v)}\right)^l dw = 0,$$

and the convergence is uniform in v over any interval $[c+\gamma,d]$, with $\gamma>0$.

(iii) If G is twice continuously differentiable over $(c,d]$, then the convergence in (i) is uniform in v over any interval $[c+\gamma,d]$, with $\gamma>0$.

(iv) If G is twice continuously differentiable and strictly positive over $[c,d]$ and if $g(c)>0$, then the convergence in (i) is uniform in v over $(c,d]$.

(v) If G is log-concave over an interval $[c,c+\eta]$, with $\eta>0$, $G(c)=0$, and the elasticity ε of g with respect to G is bounded from below, then the convergence in (i) is uniform in v over $(c,d]$.

Proof:

Proof of (i): For all v in $(c,d]$ and all $l>0$, we obviously have:

$$\begin{aligned} & l\int_c^v \left(\frac{G(w)}{G(v)}\right)^l dw \\ = & \int_c^v \frac{G(w)}{g(w)} d\left(\frac{G(w)}{G(v)}\right)^l, \end{aligned}$$

and consequently $\frac{G(w)}{g(w)}$ is integrable for $\left(\frac{G(w)}{G(v)}\right)^l$ over $[c,v]$. The equality in (i) then follows from the weak convergence of $\left(\frac{G(w)}{G(v)}\right)^l$ towards the degenerate distribution δ_v concentrated at v when l tends towards $+\infty$.

Proof of (ii): The first statement in (ii) is an immediate consequence of (i). Consider then an interval $[c+\gamma, d]$ with $\gamma > 0$. Let ε be an arbitrary strictly positive number. From the convergence at the extremities $c+\gamma$ and d, there exists l' such that $\max\left(\int_c^{c+\gamma}\left(\frac{G(w)}{G(c+\gamma)}\right)^l dw, \int_c^d \left(\frac{G(w)}{G(d)}\right)^l dw\right) < \varepsilon$, for all $l > l'$. Let m be the maximum over the interval $[c+\gamma, d]$ of the strictly positive and continuous function $G(v)/g(v)$. Consider any l such that $l > \max(l', m/\varepsilon)$. Then,

$$\max_{v \in [c+\gamma, d]} \int_c^v \left(\frac{G(w)}{G(v)}\right)^l dw < \varepsilon.$$

This is obvious if the maximum is reached at one of the extremities of the interval. Assume then that the maximum is reached at v^* in the interior of the interval. In this case, the FOC is:

$$1 - l\frac{\int_c^{v^*} G(w)^l\, dw}{G(v^*)^l}\frac{g(v^*)}{G(v^*)} = 0,$$

and consequently:

$$\begin{aligned}
&\max_{v \in [c+\gamma, d]} \int_c^v \left(\frac{G(w)}{G(v)}\right)^l dw \\
=\ & \frac{\int_c^{v^*} G(w)^l\, dw}{G(v^*)^l} \\
=\ & \frac{G(v^*)}{l g(v^*)} \\
\leq\ & \frac{m}{l} \\
<\ & \varepsilon.
\end{aligned}$$

Proof of (iii): The proof is similar to the proof of (ii). Consider an arbitrary interval $[c+\gamma, d]$, with $\gamma > 0$, and strictly positive number ε. From the convergence at the extremities $c+\gamma$ and d, there exists l'' such that

$\max\left(\left|l\int_c^{c+\gamma}\left(\frac{G(w)}{G(c+\gamma)}\right)^l dw-\frac{G(c+\gamma)}{g(c+\gamma)}\right|,\left|l\int_c^d G(w)^l dw-\frac{G(d)}{g(d)}\right|\right)<\varepsilon$, for all $l>l''$. Let M be the maximum over the interval $[c+\gamma,d]$ of $\left|\frac{d}{dv}\left(\frac{G(v)}{g(v)}\right)^2\right|/2$. Consider any l such that $l>\max(l'',M/\varepsilon)$. Then,

$$\max_{v\in[c+\gamma,d]}\left|l\int_c^v\left(\frac{G(w)}{G(v)}\right)^l dw-\frac{G(v)}{g(v)}\right|<\varepsilon.$$

This is obvious if the maximum is reached at one of the extremities of the interval. Assume then that the maximum is reached at v^* in the interior of the interval. In this case, the FOC is:

$$l-l\frac{\int_c^{v^*}G(w)^l dw}{G(v^*)^l}\frac{g(v^*)\,l}{G(v^*)}-\frac{d}{dv}\frac{G(v^*)}{g(v^*)}=0,$$

or, equivalently:

$$l\frac{\int_c^{v^*}G(w)^l dw}{G(v^*)^l}=\frac{G(v^*)}{g(v^*)}\left(1-\frac{1}{l}\frac{d}{dv}\frac{G(v^*)}{g(v^*)}\right).$$

Consequently:

$$\begin{aligned}
&\max_{v\in[c+\gamma,d]}\left|l\int_c^v\left(\frac{G(w)}{G(v)}\right)^l dw-\frac{G(v)}{g(v)}\right|\\
=&\ \left|l\int_c^{v^*}\left(\frac{G(w)}{G(v)}\right)^l dw-\frac{G(v^*)}{g(v^*)}\right|\\
=&\ \left|\frac{1}{l}\frac{G(v^*)}{g(v^*)}\frac{d}{dv}\frac{G(v^*)}{g(v^*)}\right|\\
\leq&\ \frac{M}{l}\\
<&\ \varepsilon.
\end{aligned}$$

Proof of (iv): Extend G as a strictly positive and twice continuously differentiable function with strictly positive derivative over an interval $[c-\delta, d]$ where $\delta > 0$. Then, for $v \geq c$, we have:

$$\begin{aligned} & \left| l \int_c^v \left(\frac{G(w)}{G(v)} \right)^l dw - \frac{G(v)}{g(v)} \right| \\ \leq \; & \left| l \int_{c-\delta}^{c-\delta/2} \left(\frac{G(w)}{G(v)} \right)^l dw \right| + \left| l \int_{c-\delta/2}^{v} \left(\frac{G(w)}{G(v)} \right)^l dw - \frac{G(v)}{g(v)} \right|, \end{aligned}$$

and consequently:

$$\begin{aligned} & \max_{v \in [c,d]} \left| l \int_c^v \left(\frac{G(w)}{G(v)} \right)^l dw - \frac{G(v)}{g(v)} \right| \\ \leq \; & l \left(\frac{G(c-\delta/2)}{G(c)} \right)^l + \max_{v \in [c,d]} \left| l \int_{c-\delta/2}^{v} \left(\frac{G(w)}{G(v)} \right)^l dw - \frac{G(v)}{g(v)} \right|. \end{aligned}$$

From (iii), the second term in the RHS above tends towards zero. As the first term obviously also tends towards zero, (iv) follows.

Proof of (v): Let ε be an arbitrary strictly positive number. From the convergence at d, there exists l''' such that $\left| l \int_c^d G(w)^l dw - \frac{G(d)}{g(d)} \right| < \varepsilon$, for all $l > l'''$. Let M be the maximum over the interval $[c, d]$ of $\left| \frac{d}{dv} \left(\frac{G(v)}{g(v)} \right)^2 \right| /2$. We have $M < +\infty$. Indeed, as $G(v)/g(v) = 1/\frac{d}{dv} \ln G(v)$ is nondecreasing in an interval $[c, c+\eta]$ and tends towards 0 when v tends towards c, we have $\frac{d}{dv} \left(\frac{G(v)}{g(v)} \right)^2 \geq 0$ over this interval. Furthermore, we have:

$$\begin{aligned} & \frac{d}{dv} \left(\frac{G(v)}{g(v)} \right)^2 \\ = \; & 2 \left(\frac{G(v)}{g(v)} \right) \left(1 - \frac{d \ln g(v)}{d \ln G(v)} \right) \\ \leq \; & 2(1+B) \left(\frac{G(v)}{g(v)} \right), \end{aligned}$$

with $-B \leq 0$ a lower bound of $\varepsilon(v) = \frac{d \ln g(v)}{d \ln G(v)}$.

Consider then any l such that $l > \max(l''', M/\varepsilon)$. Then,

$$\max_{v \in [c,d]} \left| l \int_c^v \left(\frac{G(w)}{G(v)} \right)^l dw - \frac{G(v)}{g(v)} \right| < \varepsilon.$$

From the definition of l''', the inequality holds true if the maximum is reached at d. The inequality is obviously satisfied if the maximum on the LHS is zero. We may thus that it is strictly positive and, hence, that it is not reached at $v = c$. The rest of the proof then proceeds as in the proof of (iii) above. ||

Appendix 3

Proof of Lemma 1:

(i) follows Lemma A1.

Extend ρ_1, ρ_2 (for example, linearly) as continuously differentiable and strictly positive functions over $(c, d + 2\mu)$, with $\mu > 0$. Finally, use the same formula in the definition to extend $\gamma_i(.;.)$ to $(c, d + 2\mu) \times (-2\zeta', 2\zeta')$, where $\zeta' > 0$. As the partial derivatives will be continuous over $(c, d + 2\mu) \times (-2\zeta', 2\zeta')$, the extension will be continuously differentiable over this product.

From the definition, $\gamma_i'(b; k)$ is equal to $1 - k\rho_j(b)^{-2} \frac{d}{db}\rho_j(b)$. From our assumptions (in particular of local log-concavity at c), $\frac{d}{db}\rho_j(b)$ is bounded from above and consequently there exists $0 < \zeta < \zeta'$ such that $\gamma_i'(b; k)$ is strictly above $\frac{d-c}{d+\mu-c}$, which belongs to $(0, 1)$, over $(c, d + \mu) \times (-\zeta, \zeta)$.

From the definition also, $\frac{\partial}{\partial k}\gamma_i(b; k)$ is equal to $\rho_j(b)^{-1}$, which is bounded from above.

As $\gamma_i(c; k) = c$ and $\gamma_i'(b; k)$ is strictly above $\frac{d-c}{d+\mu-c}$ over $(c, d + \mu) \times (-\zeta, \zeta)$, $\gamma_i(.; k)$ is a strictly increasing function over $(c, d + \mu)$ such that $\gamma_i(d + \mu; k) > d$, for all k in $(-\zeta, \zeta)$. (ii) is proved.

We then have $\gamma_i\left(\gamma_i^{-1}(v;k);k\right) = v$, for all (v,k) in $(c,d] \times (-\zeta,\zeta)$. For all such (v,k), the implicit function theorem implies that $\frac{\partial}{\partial k}\gamma_i^{-1}(v;k)$ exists and is equal to $-\frac{\partial}{\partial k}\gamma_i\left(\gamma_i^{-1}(v;k);k\right)/\gamma_i'\left(\gamma_i^{-1}(v;k);k\right)$. (iii) follows. ||

Proof of Lemma 3:

Proof of (i): Follows directly from the definition of $x(k)$.

Proof of (ii): From Lemma 1 (ii), for all $i = 1, 2$, $\gamma_i'(b;k)$ tends towards one uniformly for b in any compact subinterval of $(c,d]$. As, from Lemma 1 (ii), $\gamma_i(b;k)$ tends towards b for all b, $\left(\gamma_i^{-1}\right)'(b;k) = 1/\gamma_i'\left(\gamma_i^{-1}(b;k);k\right)$ tends towards one and $\gamma_i^{-1}(b;k)$ tends towards b uniformly in $b \in [\underline{b}, \gamma_i(d;k)]$, for all $\underline{b} > c$[37]. As $\gamma_2([\underline{b}, x(k)];k) \subseteq [\underline{b}, \gamma_1(d;k)]$, $\gamma_1^{-1}(\gamma_2(b;k);k)$ tends towards b and its derivative tends towards one uniformly over any interval $[\underline{b}, x(k)]$. (ii) then follows from the definition of $\Psi(.;k)$.

Proof of (iii): That $\Phi(.;k)$ tends to Λ uniformly over any compact subinterval of $(\ln F_1(r), 0)$ follows directly from Theorem 2 (iii) and the definitions of $\Phi(.;k)$ and Λ. With a reserve price $r > c$, Theorem 2 (iii) and the compactness of $[r,d]$ implies that $\Phi(.;k)$ tends to Λ uniformly over $[r,d]$.

Let K be an arbitrary compact subinterval of $(\ln F_1(r), 0)$. We prove first the uniform convergence over K of the derivative $\Phi'(.;k)$ towards the derivative l of Λ. From the compactness of K, it suffices to prove:

$$\lim_{(s,k)\to(u,0)} \Phi'(s;k) = l,$$

for all u in K. Let $-M$ be such that $\ln F_1(r) < -M < \min K$.

Suppose there exists u in K such that $\lim_{(s,k)\to(u,0)} \Phi'(s;k) \neq l$. Then, there exists $\varepsilon > 0$ and a sequence $(s_t;k_t)_{t\geq 1}$ converging towards $(u;0)$ such that:

$$|\Phi'(s_t;k_t) - l| > \varepsilon. \tag{A3.1}$$

[37] That is, for all $\underline{b}$ and for all $\varepsilon > 0$, there exists $k' > 0$ such that $\left|\left(\gamma_i^{-1}\right)'(b;k) - 1\right|, \left|\gamma_i^{-1}(b;k) - b\right| \leq \varepsilon$, for all $0 < k < k'$ and b in $[\underline{b}, \gamma_i(d;k)]$.

From (i), (ii) above and because the left-hand derivative $\Psi'_{lh}(u;k)$ is zero to the right of $\ln F_1(x(k))$, there exists $t' > 0$ and m such that $\max K < -m < 0$ and for all $t > t'$:

$$|\Psi'(s;k_t) - l| < \varepsilon/2, \tag{A3.2}$$

for all s in $[-M,-m]$, and

$$\Psi'_{lh}(u;k_t) < l + \varepsilon/2,$$

for all u in $[-m,0]$. As the limit u of $(s_t)_{t\geq 1}$ belongs to K and hence to the interior of $[-M,-m]$, we may assume that $(s_t)_{t\geq 1}$ is included in $[-M,-m]$. We divide the rest of the proof in four parts.

(a) In (a) and in (b) below, we suppose that $\underline{\lim}_{t\to+\infty}\Phi'(s_t;k_t) < l - \varepsilon$. Extracting a subsequence if necessary, we may assume $\lim_{t\to+\infty}\Phi'(s_t;k_t) < l - \varepsilon$. There then exists $t^* > 0$, which we may assume larger than t', such that:

$$\Phi'(s_t;k_t) < l - \varepsilon,$$

for all $t > t^*$.

Here in (a), we consider the case where there exists a subsequence $(k_{t_r})_{r\geq 1}$ such that $t_r \geq t^*$ and $\Phi(s_{t_r};k_{t_r}) \leq \Psi(s_{t_r};k_{t_r})$, for all $r \geq 1$. Extracting a subsequence again if necessary, we may assume that this the case of the original sequence. For all $t \geq t^*$, as, from (A3.1) and (A3.2), $\Phi'(s_t;k_t) < \Psi'(s_t;k_t)$, there exists $\theta_t > 0$, such that $\Phi(s;k_t) < \Psi(s;k_t)$, for all s in $(s_t, s_t + \theta_t)$. From Lemma 2 (iv), as $\Phi(s;k_t)$, being below $\Psi(s;k_t)$, is concave over $(s_t, s_t+\theta_t)$, we have $\Phi'(s;k_t) < l - \varepsilon$, for all s in $(s_t, s_t+\theta_t)$. As long as it remains strictly below $\Psi(s;k_t)$, $\Phi(s_t;k_t)$ will remain strictly concave and its derivative will decrease and therefore will remain smaller than $l - \varepsilon$ if s increases. As, from (A3.2), $\Psi'(s;k_t) > l - \varepsilon/2$, $\Phi(s;k_t)$ will never meet again $\Psi(s;k_t)$ to the right of s_t. Consequently, for all $t \geq t^*$, $\Phi(s;k_t)$ is strictly concave over $(s_t,-m)$ and $\Phi'(s;k_t) < l - \varepsilon$ over this interval. As

s_t tends towards $u < -m$, for all t large enough $s_t < (u - m)/2$ and hence $\Phi'(s; k_t) < l - \varepsilon$ over $((u - m)/2, -m)$. The different functions are then as in Figure 6 below.

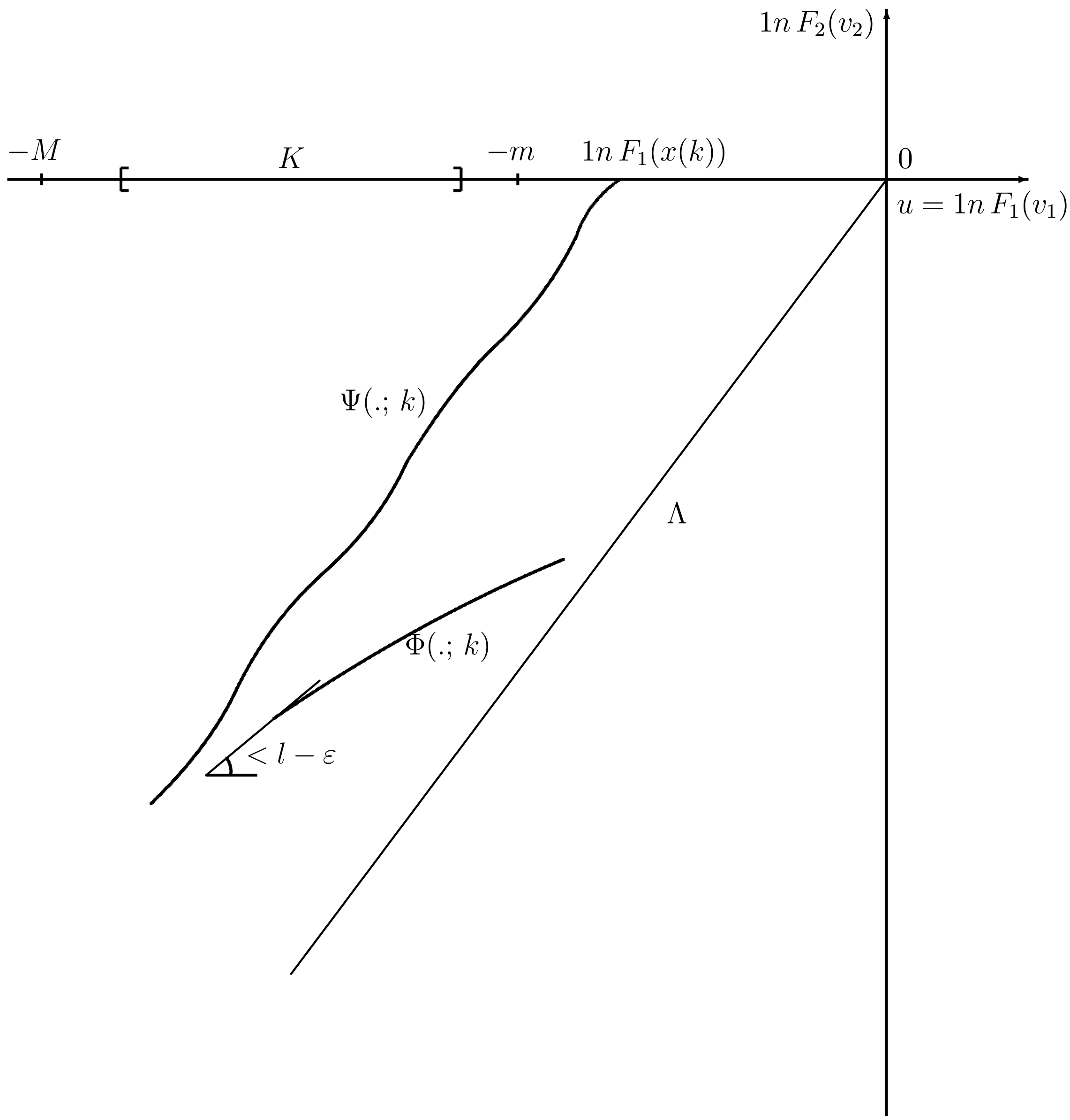

FIGURE 6: Ruling out $\Phi'(.;k)$ further below l than $\Psi'(.;k)$ is while $\Phi(.;k)$ is not larger than $\Psi(.;k)$.

For all t large enough, we then have:

$$\int_{(u-m)/2}^{-m} \Phi'(s;k_t)\,ds < (-u-m)(l-\varepsilon)/2.$$

However, $\Phi(.;k)$ tends towards Λ and consequently we also have:

$$\begin{aligned}
& \lim_{t\to+\infty} \int_{(u-m)/2}^{-m} \Phi'(s;k_t)\,ds \\
= & \lim_{t\to+\infty} \left(\Phi(-m;k_t) - \Phi((u-m)/2;k_t)\right) \\
= & \Lambda(-m) - \Lambda((u-m)/2) \\
= & l(-u-m)/2,
\end{aligned}$$

and we obtain a contradiction.

(b) We consider next the case where there exists a subsequence $(k_{t_r})_{r\geq 1}$ such that $t_r \geq t^*$ and $\Phi(s_{t_r};k_{t_r}) > \Psi(s_{r_r};k_{r_r})$, for all $r \geq 1$. Extracting a subsequence if necessary, we may again assume that this holds true for the original sequence. From Lemma 2 (iv), as long as it remains strictly above $\Psi(s;k_t)$, $\Phi(s;k_t)$ will remain strictly convex and its derivative will decrease and therefore will remain smaller than $l-\varepsilon$ if s decreases. As, from (A3.2), $\Psi'(s;k_t) > l-\varepsilon/2$, $\Phi(s;k_t)$ will never meet $\Psi(s;k_t)$ to the left of s_t. Consequently, for all $t \geq t^*$, $\Phi(s;k_t)$ is strictly convex over $(-M,s_t)$ and $\Phi'(s;k_t) < l-\varepsilon$ over this interval. As s_t converges towards u, for all t large enough $\Phi(s;k_t)$ is strictly convex over $\left(-M,\frac{u-M}{2}\right)$ and $\Phi'(s;k_t) < l-\varepsilon$ over this interval. The configuration of the graphs are as in Figure 7 below.

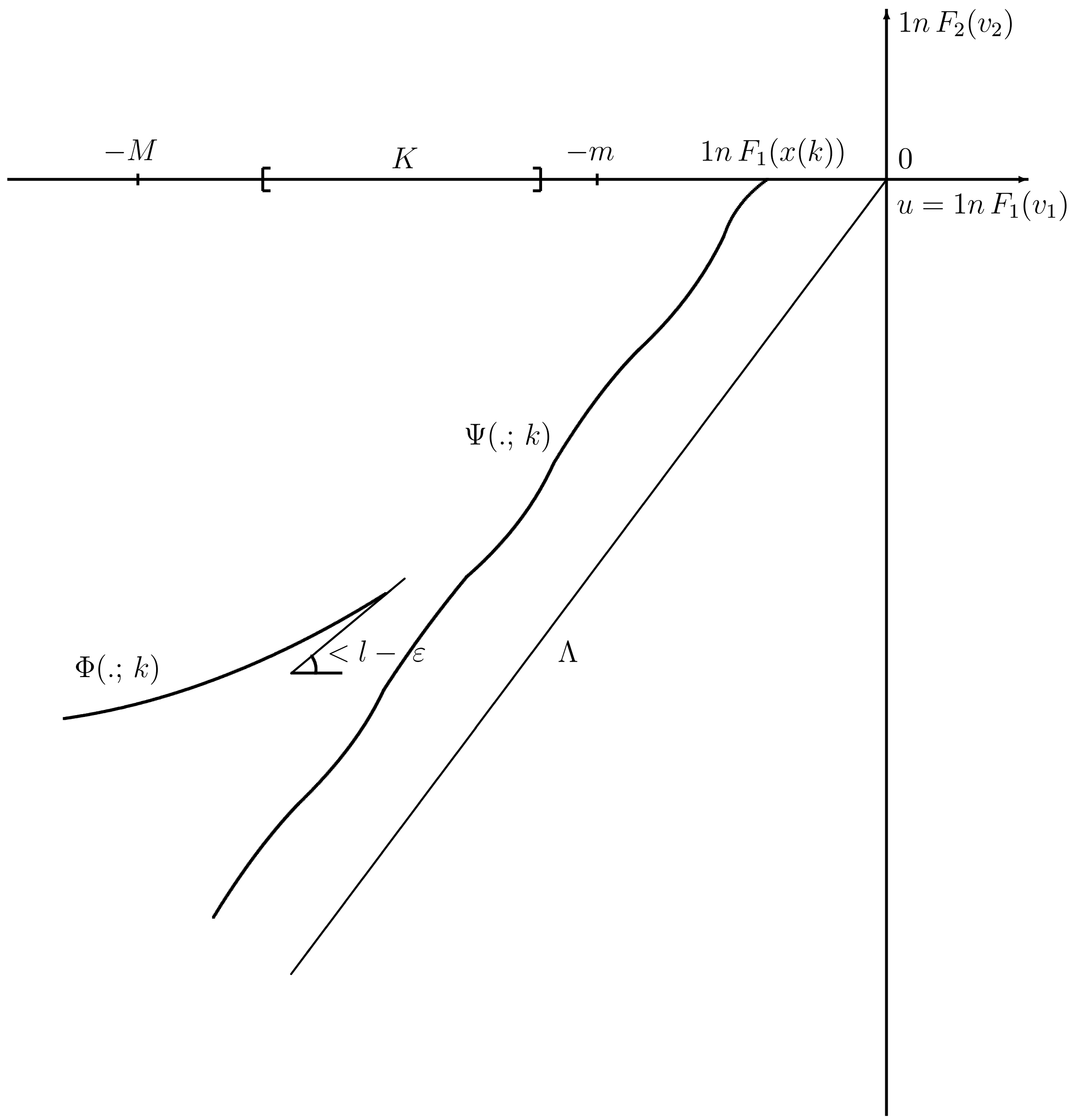

FIGURE 7: Ruling out $\Phi'(.;k)$ further below l than $\Psi'(.;k)$ is while $\Phi(.;k)$ is not smaller than $\Psi(.;k)$.

For all such t, we then have:

$$\int_{-M}^{(u-M)/2} \Phi'(s;k_t)\,ds < (u+M)\,(l-\varepsilon)\,/2.$$

However, we also have:

$$\begin{aligned}
& \lim_{t\to+\infty}\int_{-M}^{(u-M)/2}\Phi'(s;k_t)\,ds \\
= & \lim_{t\to+\infty}\left(\Phi((u-M)/2;k_t)-\Phi(-M;k_t)\right) \\
= & \Lambda((u-M)/2)-\Lambda(-M) \\
= & l(u+M)/2,
\end{aligned}$$

and we obtain a contradiction. As (a) and (b) exhaust all possibilities, we have ruled out $\underline{\lim}_{t\to+\infty}\Phi'(s_t;k_t) < l-\varepsilon$.

(c) Here and in (d) below, we suppose that $\overline{\lim}_{t\to+\infty}\Phi'(s_t;k_t) > l+\varepsilon$. As above, we may assume that[38] $\lim_{t\to+\infty}\Phi'(s_t;k_t) > l+\varepsilon$. Consequently, there exists $t^* > 0$, which we may assume larger than t' such that:

$$\Phi'(s_t;k_t) > l+\varepsilon,$$

for all $t > t^*$.

We show that, for all $t > t^*$, $\Phi(s_t;k_t) < \Psi(s_t;k_t)$. Suppose there exists $t > t^*$ such that $\Phi(s_t;k_t) \geq \Psi(s_t;k_t)$. As $\Phi'(s_t;k_t) > \Psi'(s_t;k_t)$, there exists $\theta_t > 0$ such that $\Phi(s;k_t) > \Psi(s;k_t)$, for all s in $(s_t, s_t+\theta_t)$. From Lemma 2 (iv), $\Phi(s;k_t)$ is convex over $(s_t, s_t+\theta_t)$, and we have $\Phi'(s;k_t) > l+\varepsilon$, for all s in $(s_t, s_t+\theta_t)$. As long as it remains strictly above $\Psi(s;k_t)$, $\Phi(s;k_t)$ will remain strictly convex and its derivative will increase and therefore will remain larger than $l+\varepsilon$ if s increases. As $\Psi'_{lh}(s;k_t) < l+\varepsilon/2$, $\Phi(s;k_t)$ will never meet again $\Psi(s;k_t)$ and therefore will stay strictly above it to the right of s_t. See Figure 4 in the main text. However, this contradicts $\Phi(\ln F_1(x(k_t));k_t) < 0 = \Psi(\ln F_1(x(k_t));k_t)$.

(d) From (c), $\Phi(s_t;k_t) < \Psi(s_t;k_t)$, for all $t > t^*$. As long as it remains strictly below $\Psi(s;k_t)$, $\Phi(s;k_t)$ will remain strictly concave and its derivative will increase and therefore will remain larger than $l+\varepsilon$ if s de-

[38] This limit may be infinite.

creases. As, from (A3.2), $\Psi'(s;k_t) < l+\varepsilon/2$, $\Phi(s;k_t)$ will never meet $\Psi(s;k_t)$ to the left of s_t. Consequently, for all $t \geq t^*$, $\Phi(s;k_t)$ is strictly concave over $(-M, s_t)$ and $\Phi'(s;k_t) > l+\varepsilon$ over this interval. As s_t converges towards u, for all t large enough $\Phi(s;k_t)$ is strictly concave over $(-M,(u-M)/2)$ and $\Phi'(s;k_t) > l+\varepsilon$ over this interval. See Figure 5 in the main text for an illustration of this case.

For all t large enough, we then have:

$$\int_{-M}^{(u-M)/2} \Phi'(s;k_t)\, ds > (u+M)(l+\varepsilon)/2.$$

However, we also have:

$$\begin{aligned} & \lim_{t\to+\infty} \int_{-M}^{(u-M)/2} \Phi'(s;k_t)\, ds \\ = & \lim_{t\to+\infty} \left(\Phi((u-M)/2;k_t) - \Phi(-M;k_t)\right) \\ = & \Lambda((u-M)/2) - \Lambda(-M) \\ = & l(u+M)/2, \end{aligned}$$

and we obtain a contradiction. We have ruled out $\overline{\lim}_{t\to+\infty}\Phi'(s_t;k_t) > l+\varepsilon$ and completed the proof of (iii).

Proof of (iv): Suppose $\overline{\lim}_{(s;k)\to(0;0)}\Phi'(s;k) > l$. There then exist $\varepsilon > 0$ and a sequence $(s_t;k_t)_{t\geq 1}$ tending towards $(0;0)$ and such that $\Phi'(s_t;k_t) > l+\varepsilon$, for all t. We may assume that $(s_t)_{t\geq 1}$ is included in a compact interval $[-M,0]$. As in the proof of (iii) above, from (i) and (ii) there exists t^*, such that $\Psi'_{lh}(s;k_t) < l+\varepsilon/2$,for all $t > t^*$ and s in $[-M,0]$.

As in part (c) of the proof of (iii) above, we can rule out $\Phi(s_t;k_t) \geq \Psi(s_t;k_t)$ for some $t > t^*$. We may then assume $\Phi(s_t;k_t) < \Psi(s_t;k_t)$, for all $t > t^*$. For any $t > t^*$, as longs as $\Phi(s;k_t)$ does not meet $\Psi(s;k_t)$ it will remain strictly concave and therefore its derivative will increase and hence will be larger than $l+\varepsilon$ if s decreases within $[-M,0]$. However, from

$\Psi'_{lh}(s;k_t) < l + \varepsilon/2$, $\Phi(s;k_t)$ will never meet $\Psi(s;k_t)$ to the left of s_t within $[-M,0]$. Consequently, $\Phi'(-M;k_t) > l+\varepsilon$, for all $t > t^*$. This contradicts (iii) and we have proved (iv).

Proof of (v):

(a) Assume the elasticity ε_1 is bounded from below. Differentiating the definition of Ψ, it is straightforward to find the following expression:

$$\begin{aligned}
& \Psi'(s;k) \\
= \; & \frac{f_2\left(F_2^{-1}(\exp \Psi(s;k))\right)\exp s}{f_1\left(F_1^{-1}(\exp s)\right)\exp \Psi(s;k)} \\
& \frac{1+k-k\varepsilon_1(\exp s)}{1+k-k\varepsilon_2(\exp \Psi(s;k))},
\end{aligned} \tag{A3.4}$$

where $\varepsilon_i(p)$ is the elasticity of the density f_i with respect to the cumulative probability p. From our assumption of local log-concavity and the convergence, from (ii), of $\Psi(s;k)$ towards $\Lambda(s)$, there exists $\overline{s} < 0$ such that F_2 is log-concave and hence $\varepsilon_2 \leq 1$ over $\left[c, F_2^{-1}(\exp \Psi(\overline{s};k))\right]$. From Lemma 2 (i), $F_2^{-1}(\exp \Psi(s;k)) \geq F_1^{-1}(\exp s)$. Consequently, for all $s < \overline{s}$, we have:

$$\begin{aligned}
& \frac{f_2\left(F_2^{-1}(\exp \Psi(s;k))\right)\exp s}{f_1\left(F_1^{-1}(\exp s)\right)\exp \Psi(s;k)} \\
\leq \; & \frac{f_2\left(F_1^{-1}(\exp s)\right)\exp s}{f_1\left(F_1^{-1}(\exp s)\right) F_2 F_1^{-1}(\exp s)} \\
= \; & l,
\end{aligned}$$

where the equality follows from $\rho_2 = l\rho_1$. From (A3.4) and $\varepsilon_2 \leq 1$, we then find, for all $s < \overline{s}$:

$$\begin{aligned}
& \Psi'(s;k) \\
\leq \; & l\left(1+k-k\varepsilon_1(\exp s)\right) \\
\leq \; & l\left(1+k+kB\right),
\end{aligned}$$

with $-B$ the lower bound of ε_1. The inequality $\overline{\lim}_{(s;k)\to(-\infty;0)}\Psi'(s;k) \leq l$ follows.

(b) Assume $r > c$ and ε_1 is bounded from below. We will prove $\lim_{k\to 0}\sup_{s\in\mathbb{R}_-}\Phi'(s;k) \leq l$. This will obviously imply (v). First, note $\Phi'(0;k) = 1$, for all k. Let ε be an arbitrary strictly positive number. From (ii) and part (a) above of the current proof, there exists k' such that $\Psi_l'(s;k) < l + \varepsilon/2$, for all $0 < k < k'$ and all s in $(-\infty, 0)$.

Suppose there exists u in $(-\infty, 0)$ and $k < k'$ such that $\Phi'(u;k) > l + \varepsilon$. Assume first $\Phi(u;k) \leq \Psi(u;k)$. Then, proceeding as in the proofs above, $\Phi(.;k)$ remains concave and below $\Psi(.;k)$ and $\Phi'(.;k)$ remains above $l+\varepsilon$ everywhere to the left of u. Consequently, there exists w in $(u - (\Phi(u;k) - \Lambda(u))/\varepsilon, u)$ such that $\Phi(w;k) = \Lambda(w)$. This contradicts Lemma 2 (iii).

Suppose next $\Phi(u;k) > \Psi(u;k)$. Then, $\Phi(s;k)$ remains strictly convex and strictly above $\Psi(s;k)$ everywhere to the right of u. However, this contradicts $\Phi(\ln F_1(x(k));k) < 0 = \Psi(\ln F_1(x(k));k)$.

We have proved $\sup_{s\in\mathbb{R}_-}\Phi'(s;k) \leq l+\varepsilon$, for all $k > k'$, and consequently $\lim_{k\to 0}\sup_{s\in\mathbb{R}_-}\Phi'(s;k) \leq l+\varepsilon$. As ε was arbitrary, the result follows. ||

Appendix 4

Proof of Theorem 3:

Proof of (i):

(a) For all u in $(\ln F_1(r), 0)$, the first equality below follows from Lemma 3 (iii):

$$
\begin{aligned}
& l \\
= & \lim_{(s,k)\to(u,0)} \Phi'(s;k) \\
= & \lim_{(s,k)\to(u,0)} \frac{lf\left(\varphi\left(F^{-1}(\exp s);k\right)\right)}{F\left(\varphi\left(F^{-1}(\exp s);k\right)\right)} \frac{\exp s}{f\left(F^{-1}(\exp s)\right)} \varphi'\left(F^{-1}(\exp s);k\right) \\
= & \ l \lim_{(s,k)\to(u,0)} \varphi'\left(F^{-1}(\exp s);k\right).
\end{aligned}
$$

The second equality follows from the definition of Φ and the third from the joint continuity, according to Theorem 2 (iii), of $\varphi(v;k)$. Consequently, $\lim_{(s,k)\to(u,0)} \varphi'\left(F^{-1}(\exp s);k\right)$, or equivalently $\lim_{(v,k)\to(w,0)} \varphi'(v;k)$, where $w = F^{-1}(\exp u) \in (r,d)$, exists and is equal to 1.

(b) Let v be in (r,d). From Corollary 1 (ii), we have $\beta_1(u;k) = u - \int_r^u \left(\frac{F_2(\varphi(w;k))}{F_2(\varphi(u;k))}\right)^{1/k} dw$, for all u in (r,d). Differentiating this equation, we find:

$$\begin{aligned}
\beta_1'(u;k) &= \frac{1}{k}\int_r^u \left(\frac{F_2(\varphi(w;k))}{F_2(\varphi(u;k))}\right)^{1/k} dw \frac{f_2(\varphi(u;k))}{F_2(\varphi(u;k))}\varphi'(u;k) \\
&= \frac{1}{k}\int_r^e \left(\frac{F_2(\varphi(w;k))}{F_2(\varphi(u;k))}\right)^{1/k} dw \frac{f_2(\varphi(u;k))}{F_2(\varphi(u;k))}\varphi'(u;k) \\
&\quad + \int_e^u \frac{F_2(\varphi(w;k))}{\varphi'(w;k)\, f_2(\varphi(w;k))} d\left(\frac{F_2(\varphi(w;k))}{F_2(\varphi(u;k))}\right)^{1/k} \\
&\quad \frac{f_2(\varphi(u;k))}{F_2(\varphi(u;k))}\varphi'(u;k), \qquad \text{(A4.1)}
\end{aligned}$$

where e is a fixed number strictly between r and v. The first term in the RHS of the last equality above tends towards zero if $(u;k)$ tends towards $(v;0)$. In fact, we have:

$$0 \le \frac{1}{k}\int_r^e \left(\frac{F_2(\varphi(w;k))}{F_2(\varphi(u;k))}\right)^{1/k} dw \frac{f_2(\varphi(u;k))}{F_2(\varphi(u;k))}\varphi'(u;k) \le \frac{1}{k}\left(\frac{F_2(\varphi(e;k))}{F_2(\varphi(u;k))}\right)^{1/k} \frac{f_2(\varphi(u;k))}{F_2(\varphi(u;k))}\varphi'(u;k),$$

and, from Theorem 2 (iii) and what we have just proved in (a) above, $\varphi(u;k)$ and $\varphi'(u;k)$ are jointly continuous in $(u;k)$ at $(v;0)$.

We may consider $\left(\frac{F_2(\varphi(w;k))}{F_2(\varphi(u;k))}\right)^{1/k}$ in the second term as a probability distribution over the couples $(w;t)$ that is the product of two distributions: the distribution over $[e,u]$ whose cumulative function is $\left(\frac{F_2(\varphi(w;k))}{F_2(\varphi(u;k))}\right)^{1/k}$ and hence that has a mass point at e; and the degenerate distribution at k. That is, the second term is $\int\int_{(e,u]\times[0,1]} \frac{F_2(\varphi(w;t))}{\varphi'(w;t)f_2(\varphi(w;t))} d\left(\left(\frac{F_2(\varphi(w;k))}{F_2(\varphi(u;k))}\right)^{1/k} \times \delta_{\{k\}}(t)\right)$. As $(u;k)$ tends towards $(v;0)$, this product distribution tends weakly towards the distribution concentrated at $(v;0)$. From Theorem 2 (iii) and (a) above, the integrand is a continuous function of $(u;t)$. Consequently, we have:

$$\begin{aligned}&\lim_{(u;k)\to(v;0)} \int_e^u \frac{F_2(\varphi(w;k))}{\varphi'(w;k)\, f_2(\varphi(w;k))} d\left(\frac{F_2(\varphi(w;k))}{F_2(\varphi(u;k))}\right)^{1/k}\\ = \;& \frac{F_2(v)}{f_2(v)}.\end{aligned}$$

(A4.1) then implies $\lim_{(u;k)\to(v;0)} \beta_1'(u;k) = 1$, for all v in (r,d).

(c) Let b be in (c,d). For all k and all $\widetilde{b}$ such that $c < \widetilde{b} < \beta(d;k)$, we obviously have $\alpha_1'\left(\widetilde{b};k\right) = 1/\beta_1'\left(\alpha_1\left(\widetilde{b};k\right);k\right)$. (b) above then implies $\lim_{\left(\widetilde{b};k\right)\to(b;0)} \alpha_1'\left(\widetilde{b};k\right) = 1$. From the identity $\alpha_2\left(\widetilde{b};k\right) = \varphi\left(\alpha_1\left(\widetilde{b};k\right);k\right)$, we have $\alpha_2'\left(\widetilde{b};k\right) = \varphi'\left(\alpha_1\left(\widetilde{b};k\right);k\right)\alpha_1'\left(\widetilde{b};k\right)$ and consequently $\lim_{\left(\widetilde{b};k\right)\to(b;0)} \alpha_2'\left(\widetilde{b};k\right) = 1$. Finally, from $\beta_2'(u;k) = 1/\alpha_2'(\beta_2(u;k);k)$, for all u in (c,d) and k, we have $\lim_{(u;k)\to(v;0)} \beta_2'(u;k) = 1$, for all v in (c,d).

Proof of (ii): The proof of (ii) from Lemma 3 (iv) is similar to (a) in the proof of (i) above.

Proof of (iii): Proceeding again as in (a) in the proof of (i) above, (iii) is an immediate consequence of Lemma 3 (v). ||

Proof of Theorem 4:

Proof of (i): Let v be in (r,d). From (7), we have $(\alpha_i(b;k) - b)/k = F_j(\alpha_j(b;k)) / \left(f_j(\alpha_j(b;k))\,\alpha_j'(b;k)\right)$, for all b close enough to v and all k small enough. Letting $(b;k)$ tend towards $(v;0)$ and using Theorem 3 (i) and Theorem 2 (iii), we find the statement in (i) about α_i.

We have $(u - \beta_i(u;k))/k = (\alpha_i(\beta_i(u;k);k) - \beta_i(u;k))/k$, for all u close enough to v and k small enough. The statement in (i) about β_i then follows from the statement about α_i and from Theorem 2 (iii).

From (9), we have:

$$\begin{aligned}
& \frac{\varphi(u;k)-u}{k} \\
= \;& \frac{u-\beta_1(u;k)}{k}\left\{\frac{d\ln F_2(\varphi(u;k))}{d\ln F_1(u)}-1\right\} && \text{(A4.2)} \\
= \;& \frac{u-\beta_1(u;k)}{k}\left\{l\varphi'(u;k)\frac{f_1(\varphi(u;k))\,F_1(u)}{F_1(\varphi(u;k))\,f_1(u)}-1\right\}. && \text{(A4.3)}
\end{aligned}$$

Letting $(u;k)$ tend towards $(v;0)$ and using the statement about β_1 and Theorem 3 (i) and Theorem 2 (iii), we obtain the statement in (i) about φ.

Proof of (ii): Let γ and ε be arbitrary strictly positive numbers. From (A4.3), (i) above, Theorem 2 (i), Theorem 3 (i, ii), and the compactness of $[r+\gamma,d]$, there exists $k'>0$ such that for all $0<k<k'$ and u in $[r+\gamma,d]$, we have:

$$\begin{aligned}
& \frac{\varphi(u;k)-u}{k} \\
\leq \;& \left(\frac{F_1(u)}{f_1(u)}+\varepsilon\right)(l-1+\varepsilon) \\
\leq \;& \frac{F_1(u)}{f_1(u)}(l-1)+\varepsilon\max(M,l-1+\varepsilon),
\end{aligned}$$

where M is the maximum of F_1/f_1. Consequently, $\overline{\lim}_{k\to 0}\sup_{u\in[r+\gamma,d]}\left(\frac{\varphi(u;k)-u}{k}-(l-1)\frac{F_1(u)}{f_1(u)}\right)$ $\leq \varepsilon\max(M,l-1+\varepsilon)$ and the first statement in (ii) follows by making ε tend towards zero.

The proof of the second statement proceeds from (A4.2) along similar lines and makes use of Theorem 2 (i) and Theorem 3 (iii).

Proof of (iii): From Corollary 1 (iii), bidder 1 bids more aggressively and the differences $\Delta ER(k)$ and $\Delta ES(k)$ between the expected revenues and total surpluses from the 2–k-PA and the SPA are as follows:

$$\begin{aligned}\Delta ER(k) &= \int_c^d \int_{v_1}^{\varphi(v_1;k)} (\omega_1(v_1) - \omega_2(v_2))\, dF_2(v_2)\, dF_1(v_1) \\ \Delta ES(k) &= \int_c^d \int_{v_1}^{\varphi(v_1;k)} (v_1 - v_2)\, dF_2(v_2)\, dF_1(v_1)\,.\end{aligned}$$

The rate of change of the expected revenues can then be written as:

$$\frac{\Delta ER(k)}{k} = \int_c^d \frac{1}{k} \int_{v_1}^{\varphi(v_1;k)} \Delta(v_1, v_2)\, dv_2 dv_1, \tag{A4.4}$$

where:

$$\begin{aligned}&\Delta(v_1, v_2) \\ =\ & (\omega_1(v_1) - \omega_2(v_2))\, f_2(v_2)\, f_1(v_1) \\ =\ & (v_1 f_1(v_1) - (1 - F_1(v_1)))\, f_2(v_2) \\ & - (v_2 f_2(v_2) - (1 - F_2(v_2)))\, f_1(v_1)\,.\end{aligned}$$

Because the densities f_1, f_2 are continuous over $[c, d]$, so is the difference $\Delta(v_1, v_2)$ over $[c, d]^2$. Let K be the maximum of $|\Delta(v_1, v_2)|$ over this square. From (ii) above, there exists $k' > 0$ such that, for all $0 < k < k'$ and all u in (c, d), we have: $0 \leq (\varphi(u; k) - u)/k \leq 1 + ((l - 1) F_1(u))/f_1(u)$. As $F_1(u)/f_1(u)$ is bounded, there exists L such that $0 \leq (\varphi(u; k) - u)/k \leq L$, for all u in $[c, d]$ and $0 < k < k'$. For all $0 < k < k'$, we then have for all v_1 in $[c, d]$:

$$\begin{aligned}\left| \frac{1}{k} \int_{v_1}^{\varphi(v_1;k)} \Delta(v_1, v_2)\, dv_2 \right| &\leq K \left(\frac{\varphi(v_1; k) - v_1}{k} \right) \\ &\leq KL,\end{aligned}$$

and the integrand in the RHS of (A4.4) is bounded.

From the previous paragraph, if we can find the almost everywhere pointwise limit of the integrand of (A4.4) we will be entitled to apply the Lebesgue dominated convergence theorem. Let v_1 be in (c, d). The integrand of (A4.4) at v_1 can be rewritten as:

$$\begin{aligned} & \frac{1}{k} \int_{v_1}^{\varphi(v_1;k)} \Delta(v_1, v_2)\, dv_2 \\ = & \frac{\varphi(v_1;k) - v_1}{k} \Delta(v_1, v_1) \\ & + \frac{1}{k} \int_{v_1}^{\varphi(v_1;k)} \left(\Delta(v_1, v_2) - \Delta(v_1, v_1)\right) dv_2. \end{aligned}$$

Let ε be an arbitrary strictly positive number. By continuity of $\Delta(v_1, v_2)$ over $[c, d]^2$, there exists $\xi > 0$ such that $|\Delta(v_1, v_2) - \Delta(v_1, v_1)| < \varepsilon$ for all v_2 such that $|v_1 - v_2| < \xi$. From Theorem 2 (iii), there exists k'', which we may assume smaller than k', such that, for all $0 < k < k''$, we have $\varphi(v_1;k) - v_1 < \xi$. We then obtain:

$$\begin{aligned} & \left| \frac{1}{k} \int_{v_1}^{\varphi(v_1;k)} \left(\Delta(v_1, v_2) - \Delta(v_1, v_1)\right) dv_2 \right| \\ \leq & \ \varepsilon \left(\frac{\varphi(v_1;k) - v_1}{k} \right) \\ \leq & \ \varepsilon L, \end{aligned}$$

for all $0 < k < k''$. As ε was arbitrary, we have proved

$$\lim_{k \to +\infty} \frac{1}{k} \int_{v_1}^{\varphi(v_1;k)} \left(\Delta(v_1, v_2) - \Delta(v_1, v_1)\right) dv_2 = 0.$$

From (ii) above, we also have $\lim_{k \to +\infty} \frac{\varphi(v_1;k) - v_1}{k} = \frac{l-1}{l} \frac{F_1(v_1)}{f_1(v_1)}$ and we have proved:

$$
\begin{aligned}
& \lim_{k\to+\infty} \frac{1}{k} \int_{v_1}^{\varphi(v_1;k)} \Delta\left(v_1, v_2\right) dv_2 \\
= & \frac{l-1}{l} \frac{F_1\left(v_1\right)}{f_1\left(v_1\right)} \Delta\left(v_1, v_1\right) \\
= & \frac{l-1}{l} \left(\omega_1\left(v_1\right) - \omega_2\left(v_1\right)\right) F_1\left(v_1\right) f_2\left(v_1\right) \\
= & (l-1)\left(\sigma_2^{-1}\left(v_1\right) - \sigma_1^{-1}\left(v_1\right)\right) F_1\left(v_1\right)^l f_1\left(v_1\right)
\end{aligned}
$$

From the Lebesgue theorem of dominated convergence, $\frac{d_r}{dk} ER(0) = \lim_{k\to>0} \frac{\Delta ER(k)}{k}$ exists and (11) holds true. That $\frac{d_r}{dk} ES(0)$ exists and satisfies (12) can be proved similarly. If F_1, F_2 are different, we have $l > 1$ and $\sigma_2^{-1} > \sigma_1^{-1}$ over (c,d). The strict positivity of $\frac{d_r}{dk} ER(0)$ follows from (11).

Proof of (iv): Here, we have:

$$
\frac{\Delta ER(k)}{k} = \int_r^d \frac{1}{k} \int_{v_1}^{\varphi(v_1;k)} \Delta\left(v_1, v_2\right) dv_2 dv_1. \tag{A4.5}
$$

If $l = 1$, that is, if the distributions F_1, F_2 are identical, all 2–k-PA's and the SPA give the same expected revenues, $\frac{d_r}{dk} ER(0)$ exists and is equal to zero, and the inequality in (iv) is immediate.

We may thus assume $l > 1$. In this case, $\omega_2(r) < \omega_1(r)$. Consequently, there exists $\mu > 0$ such that $\omega_2(v_2) < \omega_1(v_1)$, for all $(v_1, v_2) \in (r, r+\mu)^2$. From the uniform convergence over $[r,d]$ of $\varphi(.;k)$ towards the identity function (see Theorem 2 (iii)), there exists $k' > 0$ such that, for all $0 < k < k'$ and v in $[r, r+\mu/2]$: $0 \leq \varphi(v;k) - v < \mu/2$ and hence $0 \leq w - r \leq \varphi(v;k) - r < \mu$, for all w in $[v, \varphi(v;k)]$. From (A4.5), we then have, for all $0 < k < k'$ and

$0 < \nu < \mu/2$:

$$\begin{aligned}
&\frac{\Delta ER(k)}{k} \\
= \;& \int_r^{r+\nu} \frac{1}{k} \int_{v_1}^{\varphi(v_1;k)} \Delta(v_1, v_2)\, dv_2 dv_1 \\
&+ \int_{r+\nu}^{d} \frac{1}{k} \int_{v_1}^{\varphi(v_1;k)} \Delta(v_1, v_2)\, dv_2 dv_1 \\
\geq \;& \int_{r+\nu}^{d} \frac{1}{k} \int_{v_1}^{\varphi(v_1;k)} \Delta(v_1, v_2)\, dv_2 dv_1.
\end{aligned} \tag{A4.6}$$

By continuity of f_1 and f_2, the maximum of $|\Delta(v_1, v_2)|$ over $[c, d]^2$ exists and is finite. Let K be this maximum. From the bound (ii) above on the rate of convergence of $\varphi(.;k)$, there exists $k'' > 0$, which we may assume smaller than k', such that, for all $0 < k < k''$, we have $0 \leq \frac{\varphi(v_1;k)-v_1}{k} \leq (l-1)\max_{w\in[r,d]} \frac{F(w)}{f(w)} + 1$, for all v_1 in $[r+\nu, d]$. We then have, for all $0 < k < k''$:

$$\begin{aligned}
\left| \frac{1}{k} \int_{v_1}^{\varphi(v_1;k)} \Delta(v_1, v_2)\, dv_2 \right| &\leq K \left(\frac{\varphi(v_1;k) - v_1}{k} \right) \\
&\leq K \left((l-1) \max_{w\in[r,d]} \frac{F(w)}{f(w)} + 1 \right)
\end{aligned}$$

Assume that $0 < k < k''$. The integral in the RHS of (A4.6) therefore satisfies the assumptions of the Lebesgue dominated convergence theorem.

Furthermore, for all v in (c, d), we have:

$$\begin{aligned}
&\frac{1}{k} \int_{v_1}^{\varphi(v_1;k)} \Delta(v_1, v_2)\, dv_2 \\
= \;& \frac{\varphi(v_1;k) - v_1}{k} \Delta(v_1, v_1) \\
&+ \frac{1}{k} \int_{v_1}^{\varphi(v_1;k)} \left(\Delta(v_1, v_2) - \Delta(v_1, v_1)\right) dv_2.
\end{aligned}$$

Let ε be an arbitrary strictly positive number. By uniform continuity of $\Delta(v_1, v_2)$ over $[c,d]^2$, there exists $\rho > 0$ such that $|\Delta(v_1, v_2) - \Delta(v_1, v_1)| < \varepsilon$ for all (v_1, v_2) in that are such that $|v_1 - v_2| < \rho$. From the continuity with respect to $(u;k)$ of $\varphi(u;k)$ at $(v_1;0)$ and the compactness of $[r+\nu, d]$, there exists k''' such that for all $0 < k < k''' < k''$ and all v_1 in $[r+\nu, d]$, we have $\varphi(v_1;k) - v_1 < \rho$. From now on, we assume $k < k'''$. We then have:

$$\begin{aligned} & \left| \frac{1}{k} \int_{v_1}^{\varphi(v_1;k)} \left(\Delta(v_1, v_2) - \Delta(v_1, v_1)\right) dv_2 \right| \\ \leq \;& \varepsilon \left(\frac{\varphi(v_1;k) - v_1}{k} \right) \\ \leq \;& \varepsilon \left((l-1) \max_{w \in [r,d]} \frac{F(w)}{f(w)} + 1 \right). \end{aligned}$$

As ε was arbitrary, we have:

$$\lim_{k \to +\infty} \frac{1}{k} \int_{v_1}^{\varphi(v_1;k)} \left(\Delta(v_1, v_2) - \Delta(v_1, v_1)\right) dv_2 = 0.$$

From the previous result (i) above on the rate of convergence of $\varphi(.;k)$, we have $\lim_{k \to +\infty} \frac{\varphi(v_1;k) - v_1}{k} = \frac{l-1}{l} \frac{F_1(v_1)}{f_1(v_1)}$, for all v_1 in (r,d). From the Lebesgue Theorem of dominated convergence, we then find:

$$\begin{aligned} & \lim_{k \to +\infty} \int_{r+\nu}^{d} \frac{1}{k} \int_{v_1}^{\varphi(v_1;k)} \Delta(v_1, v_2)\, dv_2 dv_1 \\ = \;& \int_{r+\nu}^{d} \frac{l-1}{l} \frac{F_1(v_1)}{f_1(v_1)} \Delta(v_1, v_1)\, dv_1. \end{aligned}$$

and, from (A4.6):

$$\begin{aligned}
&\frac{d_+}{dk}ER(0) \\
=\ &\underline{\lim}_{k\to 0}\frac{\Delta ER(k)}{k} \\
\geq\ &\int_{r+\nu}^{d}\frac{l-1}{l}\frac{F_1(v_1)}{f_1(v_1)}\Delta(v_1,v_1)\,dv_1.
\end{aligned}$$

As ν was an arbitrary strictly positive number smaller than $\mu/2$, this implies:

$$\frac{d_+}{dk}ER(0) \geq \int_{r}^{d}\frac{l-1}{l}\frac{F_1(v_1)}{f_1(v_1)}\Delta(v_1,v_1)\,dv_1. \tag{A4.7}$$

Notice that since $\omega_2(v) < \omega_1(v)$, for all v in (c,d), the integral in (A4.7) is strictly positive. ||

Appendix 5

Proof of Theorem 6:

Proof of (i): It follows directly from Theorem 4 (i) and the previous theorem. For example, from Theorem 4 (i), we have $\lim_{(\widetilde{b};k)\to(\widetilde{v};0)}\frac{\widetilde{\alpha}_i(\widetilde{b};k)-\widetilde{b}}{k} = \widetilde{\rho}_j(\widetilde{v})^{-1}$. However, through the change of variables $\widetilde{b}=c+d-b$, $\widetilde{v}=c+d-v$, we have $\widetilde{\alpha}_i\left(\widetilde{b};k\right)-\widetilde{b}=b-\alpha_i(b)$, $\widetilde{\rho}_j(\widetilde{v})=\frac{\widetilde{f}_j(\widetilde{v})}{\widetilde{F}_j(\widetilde{v})}=\frac{f_j(v)}{1-F_j(v)}=\sigma_j(v)$ and the result follows. The second inequality is proved similarly.

From Theorem 4 (i), we have $\lim_{(\widetilde{u},k)\to(\widetilde{v},0)}\frac{\widetilde{\varphi}(\widetilde{u};k)-\widetilde{u}}{k}=\frac{l-1}{l}\widetilde{\rho}_2(\widetilde{v})^{-1}$, where $\widetilde{\varphi}=\widetilde{\alpha}_1\widetilde{\beta}_2$ (as bidder 2 is the weak bidder for the transformed distributions). Compounding this equality with the joint continuity of $\widetilde{\varphi}^{-1}(\widetilde{u};k)$, from Theorem 2, we find $\lim_{(\widetilde{u},k)\to(\widetilde{v},0)}\frac{\widetilde{u}-\widetilde{\varphi}^{-1}(\widetilde{u};k)}{k}=\frac{l-1}{l}\widetilde{\rho}_2(\widetilde{v})^{-1}$. The last statement of (i) then follows by changing the variables as above.

Proof of (ii):

(a) From Theorem 4 (ii), we have

$\overline{\lim}_{k\to 0}\max_{\widetilde{u}\in[c,d]}\left(\frac{\widetilde{u}-\widetilde{\varphi}^{-1}(\widetilde{u};k)}{k}-(l-1)\,\widetilde{\rho}_2\left(\widetilde{\varphi}^{-1}\left(\widetilde{u};k\right)\right)^{-1}\right)\le 0$, with $\widetilde{\varphi}=\widetilde{\alpha}_1\widetilde{\beta}_2$, and, from the joint continuity of $\widetilde{\varphi}^{-1}\left(\widetilde{u};k\right)$, we then have $\overline{\lim}_{k\to 0}\max_{\widetilde{u}\in[c,d]}\left(\frac{\widetilde{u}-\widetilde{\varphi}^{-1}(\widetilde{u};k)}{k}-(l-1)\,\widetilde{\rho}_2\left(\widetilde{u}\right)^{-1}\right)\le 0$. Changing the variables, we obtain $\overline{\lim}_{k\to 0}\max_{u\in[c,d]}\left(\frac{\varphi(u;k)-u}{k}-l\,(l-1)\,\sigma_1\left(u\right)^{-1}\right)\le 0$.

(b) As the expected payoff of either bidder with value c is the premium $k\left(\theta\left(k\right)-c\right)$ he receives, the differences $\Delta ER^A\left(k\right)$ and $\Delta ES^A\left(k\right)$ between the expected revenues and total surpluses from the AA and the SPA are as follows:

$$\begin{aligned}\Delta ER^A\left(k\right) &= \int_c^d\int_{v_1}^{\varphi(v_1;k)}\left(\omega_1\left(v_1\right)-\omega_2\left(v_2\right)\right)dF_2\left(v_2\right)dF_1\left(v_1\right)-2k\left(\theta\left(k\right)-c\right),\\ \Delta ES^A\left(k\right) &= \int_c^d\int_{v_1}^{\varphi(v_1;k)}\left(v_1-v_2\right)dF_2\left(v_2\right)dF_1\left(v_1\right).\end{aligned}$$

The rate of change of the expected revenues can then be written as:

$$\frac{\Delta ER^A\left(k\right)}{k}=\int_c^d\frac{1}{k}\int_{v_1}^{\varphi(v_1;k)}\Delta\left(v_1,v_2\right)dv_2dv_1-2\left(\theta\left(k\right)-c\right),$$

where $\Delta\left(v_1,v_2\right)$ is defined as in the proof of Theorem 4. From Theorem 2, $\theta\left(k\right)=c+d-\widetilde{\beta}_i\left(d;k\right)$ tends towards c as k tends towards zero. The rest of the proof proceeds as the proof of Theorem 4 and appeals to (i) and (a) above. ||

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