Supplementary Material to
"First-Price Auctions with Resale and with Outcomes Robust to Bid Disclosure"

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A. Monopsony Resale

Our analysis translates easily to monopsony resale, where only the auction loser may propose a resale price, not larger than his value. If the bidders' (seller) virtual value functions $v_s + \frac{F_s(v_s)}{f_s(v_s)}$ and $v_s + \frac{F_s(v_s)}{f_s(v_s)}$ are strictly increasing, the formula (2) for the equilibrium bidding functions under ND becomes:

$$\beta_i(v) = \frac{\int_0^{F_i(v)} \rho\left(F_w^{-1}(q), F_s^{-1}(q)\right) dq}{F_i(v)},$$

where, for $v_w \leq v_s$, $\rho(v_w, v_s)$ is the monopsony resale price offer from the strong bidder with value v_s to the weak bidder with value larger than v_w. The strong bidder after losing with his bid $\beta_s(v_s)$ offers to buy at the price $r(\beta_s(v_s)) = \rho\left(F_w^{-1}(F_s(v_s)), v_s\right)$.

Under FD, Step 1 of the randomization procedure produces the strong bidder's revised beliefs after observing a winning bid $b \geq \beta_s(v_s)$ as a conditional distribution $F_w(v_w|b)$ with support $[c, r(b)]$ such that the strong bidder offers the same resale price as under ND[1]. At Step 2, the weak bidder is made to randomize over $[r^{-1}(v_w), \beta_w(d)]$ in a way consistent with the same bid marginal distribution as under ND and the strong bidder's revised beliefs.

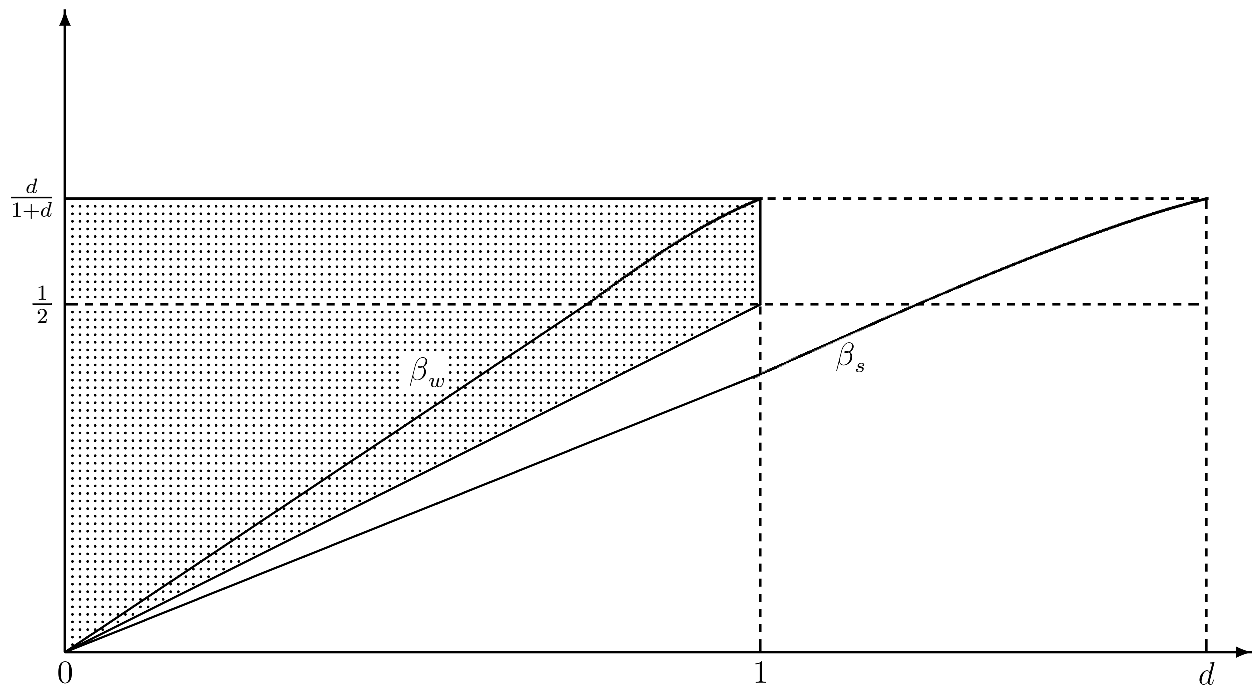
[1] Lemmas similar to Lemmas A1 and A2 hold true for monopsony resale. The statement of Lemma A1 (ii), for example, becomes under monopsony resale: $\frac{\partial_r}{\partial v_s}\rho(v, v) = \frac{1}{2}$, for all v in $[c, d)$.

In our example above with uniform distributions, $\rho\left(F_w^{-1}(q), F_s^{-1}(q)\right)$ is the corner solution 1 when $q > 2/(1+d)$ and the equilibrium bidding functions under ND are:

$$\begin{aligned}
\beta_w(v) &= \beta_s(dv) \\
&= \frac{1+d}{4}v, \text{ if } 0 \leq v \leq \frac{2}{1+d}; \\
&= 1 - \frac{1}{v(1+d)}, \text{ if } \frac{2}{1+d} \leq v \leq 1.
\end{aligned}$$

Under FD, the randomization procedure makes the weak bidder bid over $[v_w/2, d/(1+d)]$ according to the conditional distribution below (see the figure below):

$$\begin{aligned}
&G_w(b|v_w) \\
= \; & 1 - \left(\frac{v_w}{2b}\right)^{2/(d-1)}, \text{ if } \frac{v_w}{2} \leq b \leq \frac{1}{2}; \\
= \; & 1 - \frac{v_w^{2/(d-1)}}{d-1}\left\{1 + d - \frac{1}{1-b}\right\}, \text{ if } \frac{1}{2} \leq b \leq \frac{d}{1+d}.
\end{aligned}$$



B. N Bidders: Conditions for Equilibrium under ND

Assume β_w, β_s are strictly increasing and such that $\beta_w \geq \beta_s$; $\beta_w(c) = \beta_s(c) = c$; $\beta_w(d) = \beta_s(d)$; and their inverses α_w, α_s are differentiable over $(c, d]$.

Denote π_w the optimal resale price function when the potential buyer's value is distributed according to F_w, that is, for all u, v in $[c, d]^2$ such that $u \leq v$, $\pi_w(u, v)$ is the optimal resale price offer in $[u, v]$ from a monopolist with value u to a weak bidder with value not larger than v.

The optimal resale mechanism of a bidder with value v allocates the item to the bidder with the highest conditional virtual value, if larger than v, at the price equal to the smallest value at which he could have been allocated the item.

Derivative a Weak Bidder's Payoff

The derivative of a weak bidder's expected payoff at a bid $b > \beta_w(v)$ is:

$$\int_c^{\alpha_w(b)} \left(\pi\left(\max\left(v, u - \frac{F_w(\alpha_w(b)) - F_w(u)}{f_w(u)}\right), \alpha_s(b)\right) - b\right) dF_w(u)^{n-2} \frac{d}{db} F_s(\alpha_s(b))$$
$$+ \iint\limits_{(u_w,u_s)\in[c,\alpha_w(b)]\times[c,\pi(\alpha_w(b),\alpha_s(b))]}$$
$$\left\{\begin{array}{c} \left(\begin{array}{c}\pi_w\left(\max\left(v, u_w - \frac{F_w(\alpha_w(b))-F_w(u_w)}{f_w(u_w)}, u_s - \frac{F_s(\alpha_s(b))-F_s(u_s)}{f_s(u_s)}\right), \alpha_w(b)\right) \\ -b\end{array}\right) \\ d\left(\frac{F_w(u_w)}{F_w(\alpha_w(b))}\right)^{n-3} dF_s(u_s)\end{array}\right\}$$
$$\frac{d}{db} F_w(\alpha_w(b))^{n-2}$$
$$+ (\pi(\alpha_w(b), \alpha_s(b)) - b)(F_s(\alpha_s(b)) - F_s(\pi(\alpha_w(b), \alpha_s(b)))) \frac{d}{db} F_w(\alpha_w(b))^{n-2}$$
$$-F_w(\alpha_w(b))^{n-2} F_s(\alpha_s(b)).$$

The first term accounts for the event when the strong bidder's value is $\alpha_s(b)$ and hence his virtual value conditional on the interval $[c, \alpha_s(b)]$ is also equal to $\alpha_s(b)$. Since $\alpha_s(b) \geq \alpha_w(b)$, the weak bidder's optimal resale mechanism allocates the item to the strong bidder at the price equal to $\pi\left(\max\left(v, u - \frac{F_w(\alpha_w(b))-F_w(u)}{f_w(u)}\right), \alpha_s(b)\right)$, where u is the maximum of the other weak bidders' values. In fact, at this value, the strong bidder's conditional virtual value is just equal to $\max\left(v, u - \frac{F_w(\alpha_w(b))-F_w(u)}{f_w(u)}\right)$.

The second term accounts for the event when the highest value among the other weak bidders' is $\alpha_w(b)$, also equal to the conditional virtual value, and the strong bidder's conditional virtual value is smaller than $\alpha_w(b)$, that is, his value is smaller than $\pi(\alpha_w(b), \alpha_s(b))$. In this event, the weak bidder with value $\alpha_w(b)$ is allocated the item (since $\alpha_w(b) \geq v$) at the price equal to $\pi_w\left(\max\left(v, u_w - \frac{F_w(\alpha_w(b))-F_w(u_w)}{f_w(u_w)}, u_s - \frac{F_s(\alpha_s(b))-F_s(u_s)}{f_s(u_s)}\right), \alpha_w(b)\right)$, where u_w is the maximum value among the other $n-3$ weak bidders and u_s is the strong bidder's value.

Finally, the event when the highest value among the other weak bidders' is $\alpha_w(b)$ and the strong bidder's conditional virtual value is larger than $\alpha_w(b)$ gives the third term.

With similar interpretations, the derivative at b in $(\beta_s(v), \beta_w(v))$ is:

$$\begin{aligned}
&(\pi(v,\alpha_s(b)) - b) F_w(\alpha_w(b))^{n-2} \frac{d}{db} F_s(\alpha_s(b)) \\
&+ \left\{ \begin{array}{c} (v-b) F_s(\pi(v,\alpha_s(b))) \\ +(\pi(v,\alpha_s(b)) - b)(F_s(\alpha_s(b)) - F_s(v)) \end{array} \right\} \frac{d}{db} F_w(\alpha_w(b))^{n-2} \\
&-F_w(\alpha_w(b))^{n-2} F_s(\alpha_s(b));
\end{aligned}$$

and at $b < \beta_s(v)$:

$$\begin{aligned}
&(v-b) \left\{ \begin{array}{c} F_w(\alpha_w(b))^{n-2} \frac{d}{db} F_s(\alpha_s(b)) \\ +F_s(\alpha_s(b)) \frac{d}{db} F_w(\alpha_w(b))^{n-2} \end{array} \right\} \\
&-F_w(\alpha_w(b))^{n-2} F_s(\alpha_s(b)).
\end{aligned}$$

Derivative of the Strong Bidder's Payoff

The derivative of the strong bidder's expected payoff at $b < p^{-1}(v)$ is:

$$(p(b) - b) \frac{d}{db} F_w(\alpha_w(b))^{n-1} - F_w(\alpha_w(b))^{n-1};$$

at b in $(p^{-1}(v), \beta_w(v))$:

$$(v-b) \frac{d}{db} F_w(\alpha_w(b))^{n-1} - F_w(\alpha_w(b))^{n-1};$$

and at $b > \beta_w(v)$:

$$\int_c^{\alpha_w(b)} \left(\pi_w\left(\max\left(v, u - \frac{F_w(\alpha_w(b)) - F_w(u)}{f_w(u)}\right), \alpha_w(b)\right) - b\right)$$
$$d\left(\frac{F_w(u)}{F_w(\alpha_w(b))}\right)^{n-2} \frac{d}{db} F_w(\alpha_w(b))^{n-1}$$
$$-F_w(\alpha_w(b))^{n-1}.$$

Sufficiency of the Conditions

As in the FD regime, the conditions (8, 9) are sufficient because the derivatives above are nondecreasing in the bidder's value v and reduce to the LHS's of (8, 9) at the bid the strategy specifies, that is, when $v = \alpha_w(b)$ for a weak bidder and when $v = \alpha_s(b)$ for the strong bidder.

C. N Bidders: Existence and Uniqueness of the Equilibrium

The system of differential equations (8, 9) is equivalent to the system (C1, C2) below:

$$\begin{aligned} & \frac{d}{db} \ln F_s(\alpha_s(b)) \\ = \; & \frac{1}{p(b) - b} \\ & \left\{ 1 - \frac{n-2}{n-1} \frac{\left(\begin{array}{c} (\alpha_w(b) - b) F_s(p(b)) \\ + (p(b) - b)(F_s(\alpha_s(b)) - F_s(p(b))) \end{array} \right)}{(p(b) - b) F_s(\alpha_s(b))} \right\}; \end{aligned} \quad \text{(C1)}$$

$$\frac{d}{db}\ln F_w\left(\alpha_w\left(b\right)\right)=\frac{1}{\left(n-1\right)\left(p\left(b\right)-b\right)}. \tag{C2}$$

We have the proposition below:

Proposition: *If*

(a) F_s is twice continuously differentiable over $(c,d]$, and such that $\frac{d}{dv}\left(\frac{f_s(v)(v-c)^2}{F_s(v)}\right)\geq 0$ and $\frac{d}{dv}\left(v-\frac{1-F_s(v)}{f_s(v)}\right)>0$, for all $v>c$;

(b) $\frac{F_s}{F_w}$ is nondecreasing;

(c) $\frac{d}{dv}\frac{F_s}{F_w}\left(d\right)>0$;

then:

(i) (existence and uniqueness) There exists one and only one couple of strictly increasing bidding functions β_w,β_s such that $\beta_w\left(c\right)=\beta_s\left(c\right)=c,\beta_w\left(d\right)=\beta_s\left(d\right)$, $\beta_w\geq\beta_s$, and their inverses α_w,α_s satisfy (C1, C2).

(ii) (stochastically larger bid from the strong bidder) At the unique solution $\left(\beta_w,\beta_s\right)$ in (i), $\frac{F_s(\alpha_s(b))}{F_w(\alpha_w(b))}$ is nondecreasing over $(c,d]$.

Main Ideas of the Proof of the Proposition

The proof of the existence in (i) proceeds by studying the solution to the system (C1, C2) with initial condition $\alpha_w\left(d\right)=\alpha_s\left(d\right)=\eta$, where η is a parameter such that $\eta<d$. Because the system (properly rewritten) is locally Lipschitz at such an initial condition, the standard theory of ordinary differential equations applies and implies that any solution, where defined, is strictly monotonic, and such that $\frac{F_s(\alpha_s(b))}{F(\alpha(b))}$ is nondecreasing and $\alpha_w\leq\alpha_s$. Moreover, the functions α_w,α_s and the lower extremity $\underline{b}\left(\eta\right)$ of their largest definition interval are monotonic with respect to η. We then prove that there exist some values η of the parameter such that $\underline{b}\left(\eta\right)>c$ and others such that $\underline{b}\left(\eta\right)<c$. Finally, we show that there exists an intermediate value of the parameter such that $\underline{b}\left(\eta\right)=c$ and the remaining boundary condition

$\alpha_w(c) = \alpha_s(c) = c$ is satisfied. To this end, we rule out jumps, due to small decreases of η, of the graphs of the functions α_w, α_s from common points on the 45-degree line, where they end up when $\underline{b}(\eta) > c$, to points to the vertical of and away from (c, c).

To prove the uniqueness in (i), we transform the system (C1, C2) into a differential system in $\varphi = \alpha_s \beta_w$ and β_w. We then show that, if there existed two solutions, the function φ that would correspond to the higher value of the parameter η would be smaller. Then, the value $\beta_w(d)$, through its positive relation with φ (obtained by integrating (C2)) would also be smaller, which would contradict the initial condition $\beta_w(d) = \eta$.

Proof of the Proposition

I. Technical Extension of the Function π:

For $u \leq v$, $\pi(u, v)$ is as defined in (3). For technical purposes, extend the function π to $[c, d]^2$, differently than in Subsection 6.2, by setting $\pi(u, v) = \frac{u+v}{2}$, for $u > v$. It is then easy to check that the so defined π is continuously differentiable over $(c, d]^2$. In fact, from assumption (a), (3) satisfies the conditions of the implicit function theorem (the derivative with respect to π of the RHS of (3) is strictly positive). Furthermore, the partial derivatives of the solution π of (3) tend towards 1/2 when (v, w) tend towards towards a couple on the 45-degree line. Since π is continuously differentiable, it is also locally Lipschitz over $(c, d]^2$.

Lemma C.1: *$\frac{\partial}{\partial u}\pi(u, v)$ is bounded away from zero over $[c + \varepsilon, d]^2$, for all $\varepsilon > 0$.*

Proof: For all (u, v) with $u \geq v$, we have, by definition of the extension of π, $\frac{\partial}{\partial u}\pi(u, v) = \frac{1}{2}$. From (3), we have, for $u \leq v$:

$$\left\{2 + \frac{F_s(v) - F_s(\pi(u, v))}{f_s(\pi(u, v))^2} f_s'(\pi(u, v))\right\} \frac{\partial}{\partial u}\pi(u, v) = 1.$$

If, furthermore, $(u,v) \in [c+\varepsilon, d]^2$, then $\pi(u,v) \in [c+\varepsilon, d]^2$. From the equality above, we have:

$$\frac{\partial}{\partial u}\pi(u,v) \geq L(\varepsilon) > 0,$$

for all (u,v) in $[c+\varepsilon, d]^2$such that $u \leq v$, with $L(\varepsilon)$ defined as follows:

$$L(\varepsilon) = \frac{1}{2 + K(\varepsilon)/M(\varepsilon)},$$

with:

$$\begin{aligned} K(\varepsilon) &= \max_{v \in [c+\varepsilon, d]} f_s'(v) \\ M(\varepsilon) &= \min_{v \in [c+\varepsilon, d]} f_s(v) > 0. \end{aligned}$$

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II. Main Lemmas

Through the change of variables $\psi_1 = F_w\alpha_w$, $\psi_n = F_s\alpha_s$, the system (C1, C2) becomes the system below:

$$\begin{aligned}
&\frac{d}{db}\psi_n(b) \\
= \;& \frac{\psi_n(b)}{\pi\left(F_w^{-1}\psi_1(b), F_s^{-1}\psi_1(b)\right) - b} \\
&\left\{1 - \frac{n-2}{n-1}\frac{\begin{pmatrix} \left(F_w^{-1}\psi_1(b) - b\right) F_s\left(\pi\left(F_w^{-1}\psi_1(b), F_s^{-1}\psi_n(b)\right)\right) + \\ \left(\pi\left(F_w^{-1}\psi_1(b), F_s^{-1}\psi_n(b)\right) - b\right) \\ \left(\psi_n(b) - F_s\left(\pi\left(F_w^{-1}\psi_1(b), F_s^{-1}\psi_n(b)\right)\right)\right) \end{pmatrix}}{\left(\pi\left(F_w^{-1}\psi_1(b), F_s^{-1}\psi_n(b)\right) - b\right)\psi_n(b)}\right\}; \\
&\frac{d}{db}\psi_1(b) \\
= \;& \frac{\psi_1(b)}{(n-1)\left(\pi\left(F_w^{-1}\psi_1(b), F_s^{-1}\psi_n(b)\right) - b\right)}.
\end{aligned}$$

By extending the functions F_w^{-1}, F_s^{-1} into locally Lipschitz functions over $(0, 1+\varepsilon)$, where $\varepsilon > 0$, in such a way that $\frac{1-q}{f_s\left(F_s^{-1}(q)\right)}$ is nonincreasing over this interval, the assumptions of the theory of ordinary differential equations are satisfied over the domain[2]. $\mathcal{D} = \left\{(b, \psi_1, \psi_n) \,|\, 0 < \psi_1, \psi_n \leq 1, \rho\left(F^{-1}\psi_1(b), F_s^{-1}\psi_n(b)\right) > b\right\}$. Consequently, for every $\eta < d$, there exists one and one solution in this domain.

Consider the extension (C3, C4) of the original system (C1, C2) over the domain $D = \left\{\begin{array}{c} (b, \alpha_w, \alpha_s) \,| \\ c < \alpha_w, \alpha_s \leq d; \pi(\alpha_w, \alpha_s) > b \end{array}\right\}$ (the image of $\mathcal{D}$ by the change of variables above), with initial condition (C5) below, where η is a

[2]The change of variables allows to apply the theory of ordinary differential equations without making unnecessary Lipschitz assumptions on the density f.

parameter such that $\eta < d$:

$$\begin{aligned}
& \frac{d}{db} \ln F_s\left(\alpha_s(b)\right) \\
= \; & \frac{1}{\pi\left(\alpha_w(b), \alpha_s(b)\right) - b} \\
& \left\{ 1 - \frac{n-2}{n-1} \frac{\left(\begin{array}{c} \left(\alpha_w(b) - b\right) F_s\left(\pi\left(\alpha_w(b), \alpha_s(b)\right)\right) + \\ \left(\pi\left(\alpha_w(b), \alpha_s(b)\right) - b\right) \\ \left(F_s\left(\gamma(b)\right) - F_s\left(\pi\left(\alpha_w(b), \alpha_s(b)\right)\right)\right) \end{array} \right)}{\left(\pi\left(\alpha_w(b), \alpha_s(b)\right) - b\right) F_s\left(\alpha_s(b)\right)} \right\}; \quad \text{(C3)}
\end{aligned}$$

$$\begin{aligned}
& \frac{d}{db} \ln F_w\left(\alpha_w(b)\right) \\
= \; & \frac{1}{(n-1)\left(\pi\left(\alpha_w(b), \alpha_s(b)\right) - b\right)}. \qquad \text{(C4)}
\end{aligned}$$

$$\alpha_w(\eta) = \alpha_s(\eta) = d. \qquad \text{(C5)}$$

It follows immediately from (C4) that $\frac{d}{db}\alpha_w(b) > 0$, for all solution of (C4, C5 in D. Moreover, at the initial condition, the derivative of γ is also strictly positive. In fact, from (C3, C4) and $\pi(d, d) = d$:

$$\frac{d}{db} \ln F_s\left(\alpha_s(b)\right) = \frac{d}{db} \ln F_w\left(\alpha_w(b)\right) = \frac{1}{(n-1)(c-d)}.$$

We have proved Lemma C2 below.

Lemma C2: *Let (α_w, α_s) be a solution of (C3-C5) in the domain D defined over $(b', \eta]$. Then, $\frac{d}{db}\alpha_w(b) > 0$, for all b in $(b', \eta]$, and $\frac{d}{db}\alpha_s(\eta) > 0$.*

From Lemma C2, the solution to (C3-C5) is strictly increasing at η and can be continued within D to the left of this point.

Lemma C3: *Let (α_w, α_s) be a solution of (C3-C5) in the domain D defined over $(b', \eta]$. Since, from Lemma C2, $\frac{d}{db}\alpha_w(b) > 0$, for all b in $(b', \eta]$, the function φ below is well defined and differentiable:*

$$\varphi = \alpha_s \alpha_w^{-1} = \alpha_s \beta_w,$$

where β_w is the inverse of α_w. Then, the inequality below holds true for all v in $(\alpha_w(b'), d]$:

$$\lambda(v) \leq \varphi(v);$$

where λ is defined as follows:

$$\lambda(v) = F_s^{-1}\left(F_w(v) \min_{u \in [v,d]} \frac{F_s(u)}{F_w(u)}\right).$$

Proof: Let v be in $(\alpha_w(b'), d)$, k be such that $0 < k < \min_{u \in [v,d]} \frac{F_s(u)}{F_w(u)}$, and let the function λ_k be defined as follows:

$$\lambda_k(u) = F_s^{-1}(kF_w(u)),$$

for all u in $(v, d]$. From its definition and $k < \frac{F_s(u)}{F_w(u)}$, for all $u \geq v$, we have:

$$\begin{aligned} \frac{d}{dv} \ln F_s(\lambda_k(u)) &= \frac{d}{dv} \ln F_w(u) \\ \lambda_k(u) &< u, \end{aligned}$$

for all $u \geq v$. In particular, $\lambda_k(d) < d$ and, thus,

$$\lambda_k(d) < \varphi(d).$$

The system (C3, C4) can be rewritten under a form similar to (8, 9) with the weak bidder's value v as the variable:

$$
\begin{aligned}
& \left(\pi\left(v, \varphi\left(v\right)\right)-\beta_{w}\left(v\right)\right) \frac{d}{d v} \ln F_{s}\left(\varphi\left(v\right)\right)+ \\
& (n-2)\left\{\begin{array}{c}
\left(v-\beta_{w}\left(v\right)\right) \frac{F_{s}\left(\pi\left(v, \varphi\left(v\right)\right)\right)}{F_{s}\left(\varphi\left(v\right)\right)}+ \\
\left(\pi\left(v, \varphi\left(v\right)\right)-\beta_{w}\left(v\right)\right)\left[1-\frac{F_{s}\left(\pi\left(v, \varphi\left(v\right)\right)\right)}{F_{s}\left(\varphi\left(v\right)\right)}\right]
\end{array}\right\} \frac{d}{d v} \ln F_{w}\left(v\right) \\
= & \frac{d}{d v} \beta_{w}\left(v\right), \qquad \text{(C6)}
\end{aligned}
$$

$$
\begin{aligned}
& \left(\pi\left(v, \varphi\left(v\right)\right)-\beta_{w}\left(v\right)\right) \frac{d}{d v} \ln F_{w}\left(v\right)+ \\
& (n-2)\left\{\begin{array}{c}
\left(\pi\left(v, \varphi\left(v\right)\right)-\beta_{w}\left(v\right)\right) \frac{F_{s}\left(\pi\left(v, \varphi\left(v\right)\right)\right)}{F_{s}\left(\varphi\left(v\right)\right)}+ \\
\left(\pi\left(v, \varphi\left(v\right)\right)-\beta_{w}\left(v\right)\right)\left[1-\frac{F_{s}\left(\rho\left(v, \varphi\left(v\right)\right)\right)}{F_{s}\left(\varphi\left(v\right)\right)}\right]
\end{array}\right\} \frac{d}{d v} \ln F_{w}\left(v\right) \\
= & \frac{d}{d v} \beta_{w}\left(v\right). \qquad \text{(C7)}
\end{aligned}
$$

Suppose there exists u in $(v, d]$ such that $\varphi\left(u\right)=\lambda_{k}\left(u\right)$. Since $\lambda_{k}\left(u\right)<u$, we have $\varphi\left(u\right)<u$ and, consequently, $\pi\left(\varphi\left(u\right), u\right)<u$ (because $\min\left(v, w\right)<\pi\left(v, w\right)<\max\left(v, w\right)$, for all (v, w) such that $v \neq w$). From (C6) and (C7), we then have $\frac{d}{dv} \ln F_{s}\left(\varphi\left(u\right)\right)<\frac{d}{dv} \ln F_{w}\left(u\right)$ and, since $\frac{d}{dv} \ln F_{s}\left(\lambda_{k}\left(u\right)\right)=\frac{d}{dv} \ln F_{w}\left(u\right)$, $\frac{d}{dv} \ln F_{s}\left(\varphi\left(u\right)\right)<\frac{d}{dv} \ln F_{s}\left(\lambda_{k}\left(u\right)\right)$.

From an (elementary) technical lemma, we obtain $\varphi\left(v\right) \geq F_{s}^{-1}\left(k F_{w}\left(v\right)\right)$. The result then follows by taking the limit for k tending towards $\min_{w \in[v, d]} \frac{F_{s}(w)}{F_{w}(w)}$. ||

Lemma C4: *Let (α_{w}, α_{s}) be a solution of (C3-C5) in D defined over $(b', \eta]$. Then, the inequality below holds true for all v in $(\alpha_{w}(b'), d]$:*

$$v \leq \varphi\left(v\right),$$

and

$$\alpha_{w}\left(b\right) \leq \alpha_{s}\left(b\right),$$

for all b in $(b', \eta]$.

Proof: It suffices to apply the previous lemma and to notice that, under our assumption of stochastic dominance (b), $\min_{w \in [v,d]} \frac{F_s(w)}{F_w(w)} = \frac{F_s(v)}{F_w(v)}$. ||

Lemma C5: *Let (α_w, α_s) be a solution of (C3-C5) in D defined over $(b', \eta]$. Then, $\frac{d}{db}\alpha_w(b), \frac{d}{db}\alpha_s(b) > 0$, $\frac{d}{db}\frac{F_s(\alpha_s(b))}{F_w(\alpha_w(b))} \geq 0$ and $F_s(\alpha_s(b)) \leq F_w(\alpha_w(b))$, for all b in $(b', \eta]$.*

Proof: From Lemma C2, $\frac{d}{db}\alpha_w(b) > 0$, for all b in $(b', \eta]$. From Lemma C4, $\alpha_w(b) \leq \alpha_s(b)$. Consequently, the sum of $(\alpha_w(b) - b)\frac{F_s(\pi(\alpha_w(b), \alpha_s(b)))}{F_s(\alpha_s(b))}$ and $(\pi(\alpha_w(b), \alpha_s(b)) - b)\left[1 - \frac{F_s(\pi(\alpha_w(b), \alpha_s(b)))}{F_s(\alpha_s(b))}\right]$ is not smaller than $\pi(\alpha_w(b), \alpha_s(b))$ $-b$, and the factor between braces in the RHS of (C3) is not smaller than $1 - \frac{n-2}{n-1} > \frac{1}{n-1} > 0$. We then also have $\frac{d}{db}\alpha_s(b) > 0$, for all b in $(b', \eta]$. Moreover, from (C3) and (C4), we find $\frac{d}{db}\ln F_s(\alpha_s(b)) \geq \frac{1}{\pi(\alpha_w(b), \alpha_s(b)) - b}\frac{1}{n-1} = \frac{d}{db}\ln F_w(\alpha_w(b))$, for all b in $(b', \eta]$, and consequently $\frac{F_s(\alpha_s(b))}{F_w(\alpha_w(b))}$ is nondecreasing over this interval. Since, from (C5), $\frac{F_s(\alpha_s(\eta))}{F_w(\alpha_w(\eta))} = 1$, we obtain $F_s(\alpha_s(b)) \leq F_w(\alpha_w(b))$, for all b in $(b', \eta]$. ||

Lemma C6 (Monotonicity of the solution of (C3-C5) with respect to η): *Let (α_w, α_s) and $(\widetilde{\alpha}_w, \widetilde{\alpha}_s)$ be the solutions of (C3, C4) in D and the initial condition (C5) for η and $\widetilde{\eta}$, respectively, with $\widetilde{\eta} < \eta$. Assume further that (α_w, α_s) and $(\widetilde{\alpha}_w, \widetilde{\alpha}_s)$ are defined over $(\underline{b}, \widetilde{\eta}]$, where $\underline{b} \geq c$. Then, we have:*

$$\begin{aligned} \widetilde{\alpha}_w(b) &> \alpha_w(b) \\ \widetilde{\alpha}_s(b) &> \alpha_s(b), \end{aligned}$$

for all b in $(\underline{b}, \widetilde{\eta}]$.

Proof: There exists no b in $(\max(c, \underline{b}), \widetilde{\eta}]$ such that $\widetilde{\alpha}_w(b) = \alpha_w(b)$ and $\widetilde{\alpha}_w(b) = \alpha_w(b)$. Otherwise, (α_w, γ) and $(\widetilde{\alpha}_w, \widetilde{\gamma})$ could be extended over the unions of their definition domains and would coincide over this union.

However, this is impossible since, from (C5) and Lemma C2 :

$$\widetilde{\alpha}_w(\widetilde{\eta}) = d > \alpha_w(\widetilde{\eta}).$$

Let b' be defined as follows:

$$b' = \inf\left\{b \in [\underline{b}, \widetilde{\eta}]\, |\widetilde{\alpha}_w(b'') > \alpha_w(b''), \widetilde{\alpha}_s(b'') > \alpha_s(b''), \text{ for all } b'' \text{ in } (b, \widetilde{\eta}]\right\}.$$

From our assumptions and by continuity, there exists $\varepsilon > 0$ such that $[\widetilde{\eta} - \varepsilon, \widetilde{\eta}]$ is included in the set in the definition above of b'. We want to prove that $b' = \underline{b}$. Suppose $b' > \underline{b}$. Then, by continuity and from the observation above only the two cases below are possible:

Case 1: $\widetilde{\alpha}_w(b') = \alpha_w(b')$ and $\widetilde{\alpha}_s(b') > \alpha_s(b')$.
Case 2: $\widetilde{\alpha}_w(b') > \alpha_w(b')$ and $\widetilde{\alpha}_s(b') = \alpha_s(b')$.

We investigate each case in turn.

Case 1. From (C4) and because π is strictly increasing, we have:

$$\begin{aligned} & \frac{d}{db}\ln F_w(\alpha_w(b')) \\ = & \frac{1}{(n-1)(\pi(\alpha_w(b'), \alpha_s(b')) - b')} \\ > & \frac{1}{(n-1)(\pi(\widetilde{\alpha}_w(b'), \widetilde{\alpha}_s(b')) - b')} \\ = & \frac{d}{db}\ln F_w(\widetilde{\alpha}_w(b')). \end{aligned}$$

Then, since $\widetilde{\alpha}_w(b') = \alpha_w(b')$ and $\frac{d}{db}\widetilde{\alpha}_w(b') < \frac{d}{db}\alpha_w(b')$, we would have $\widetilde{\alpha}_w(b) < \alpha_w(b)$, for some b to the right of b', which would contradict the definition of b'.

Case 2. (C3) can be rewritten as follows:

$$\begin{aligned} & \frac{d}{db} \ln F_s \left(\gamma \left(b \right) \right) \\ = \quad & \frac{1}{\left(n-1 \right) \left(\pi \left(\alpha_w \left(b \right), \alpha_s \left(b \right) \right) - b \right)} \\ & + \frac{n-2}{n-1} \frac{\pi \left(\alpha_w \left(b \right), \alpha_s \left(b \right) \right) - \alpha_w \left(b \right)}{\left(\pi \left(\alpha_w \left(b \right), \alpha_s \left(b \right) \right) - b \right)^2} \frac{F_s \left(\pi \left(\alpha_w \left(b \right), \alpha_s \left(b \right) \right) \right)}{F_s \left(\alpha_s \left(b \right) \right)}. \end{aligned} \qquad \text{(C8)}$$

From Lemma C4, $\alpha_w(b) \leq \alpha_s(b)$. From the definition (3) of π, $\pi(\alpha_w(b), \alpha_s(b)) - \alpha_w(b)$ is equal to $\frac{F_s(\alpha_s(b)) - F_s(\pi(\alpha_w(b), \alpha_s(b)))}{f_s(\rho(\alpha_w(b), \alpha_s(b)))}$ and the second term in the RHS of (C8) is equal to:

$$\begin{aligned} & \frac{n-2}{n-1} \frac{F_s \left(\alpha_s \left(b \right) \right) - F_s \left(\pi \left(\alpha_w \left(b \right), \alpha_s \left(b \right) \right) \right)}{F_s \left(\alpha_s \left(b \right) \right)} \\ & \frac{F_s \left(\pi \left(\alpha_w \left(b \right), \alpha_s \left(b \right) \right) \right)}{f_s \left(\pi \left(\alpha_w \left(b \right), \alpha_s \left(b \right) \right) \right) \left(\pi \left(\alpha_w \left(b \right), \alpha_s \left(b \right) \right) - c \right)^2} \left(1 + \frac{b-c}{\pi \left(\alpha_w \left(b \right), \alpha_s \left(b \right) \right) - b} \right)^2, \end{aligned}$$

which, from our assumption (a) (and $b > c$), is nonincreasing with respect $\alpha_w(b)$. Because the first term is strictly decreasing in $\alpha_w(b)$, we find, under the assumptions of Case 2: $\frac{d}{db} \ln F_s(\widetilde{\alpha}_s(b')) < \frac{d}{db} \ln F_s(\alpha_s(b'))$, which, together with $\widetilde{\alpha}_s(b') = \alpha_s(b')$, contradicts the definition of b'. ||

Lemma C7: *Let $\underline{b}(\eta)$ be the lower-extremity of the maximal definition interval of the solution (α_w, γ) in D of the system (C3, C4) with initial condition (C5) for the value η of the parameter. Then, $\underline{b}(\eta)$ is strictly increasing when strictly above c. Furthermore, if $\underline{b}(\eta) > c$, then $\alpha_w(\underline{b}(\eta)), \alpha_s(\underline{b}(\eta)) > c$ and $\pi(\alpha_w(\underline{b}(\eta)), \alpha_s(\underline{b}(\eta))) = \underline{b}(\eta)$.*

Proof: Let η be such that $\eta < d$ and $\underline{b}(\eta) > c$. From the definition of D and Lemma C5, we have $\alpha_w(\underline{b}(\eta)) = c, \alpha_s(\underline{b}(\eta)) = c$, or $\pi(\alpha_w(\underline{b}(\eta)), \alpha_s(\underline{b}(\eta))) = \underline{b}(\eta)$, where $\alpha_w(\underline{b}(\eta)), \alpha_s(\underline{b}(\eta))$ are the values of the continuous extensions of the solution (α_w, α_s) to the system (C3, C4) with initial condition (C5) with the value η of the parameter. From Lemmas C4 and C5, $\alpha_w(\underline{b}(\eta)) = c$ if and only if $\alpha_s(\underline{b}(\eta)) = c$, in which case $\pi(\alpha_w(\underline{b}(\eta)), \alpha_s(\underline{b}(\eta))) = c < \underline{b}(\eta)$, contrary to the definition of D. Consequently, we have

$$\begin{aligned} \alpha_w(\underline{b}(\eta)), \alpha_s(\underline{b}(\eta)) &> c \qquad \text{(C9)} \\ \pi(\alpha_w(\underline{b}(\eta)), \alpha_s(\underline{b}(\eta))) &= \underline{b}(\eta). \end{aligned}$$

Let $\widetilde{\eta}$ be such that $\eta < \widetilde{\eta} < d$. Let $(\widetilde{\alpha}_w, \widetilde{\alpha}_s)$ be the solution of the (C3-C5) for the values $\widetilde{\eta}$ of the parameter in the initial condition (C5). The inequality $\underline{b}(\widetilde{\eta}) < \underline{b}(\eta)$ is impossible. In fact, from Lemma C6, we have:

$$\alpha_s(b) > \widetilde{\alpha}_s(b), \qquad \text{(C10)}$$

$$\alpha_w(b) > \widetilde{\alpha}_w(b), \qquad \text{(C11)}$$

for all b in $(\max(\underline{b}(\widetilde{\eta}), \underline{b}(\eta)), \eta)$. Suppose $\underline{b}(\widetilde{\eta}) < \underline{b}(\eta)$. By making b in these inequalities tend towards $\underline{b}(\eta)$, we would obtain $\widetilde{\gamma}(\underline{b}(\eta)) \leq \alpha_s(\underline{b}(\eta))$ and $\widetilde{\alpha}_w(\underline{b}(\eta)) \leq \alpha_w(\underline{b}(\eta))$ and, consequently, $\underline{b}(\eta) = \pi(\alpha_w(\underline{b}(\eta)), \alpha_s(\underline{b}(\eta))) \geq \pi(\alpha_w(\underline{b}(\widetilde{\eta})), \alpha_s(\underline{b}(\widetilde{\eta}))) = \underline{b}(\widetilde{\eta})$, a contradiction.

We have proved $\underline{b}(\eta) \leq \underline{b}(\widetilde{\eta})$. We now prove $\underline{b}(\eta) < \underline{b}(\widetilde{\eta})$ by showing that the equality $\underline{b}(\eta) = \underline{b}(\widetilde{\eta})$ is impossible. Suppose $\underline{b}(\eta) = \underline{b}(\widetilde{\eta})$. We then have, from the definition of D and from (C10) and (C11):

$$\begin{aligned} \widetilde{\alpha}_s(\underline{b}(\eta)) &\leq \alpha_s(\underline{b}(\eta)) \\ \widetilde{\alpha}_w(\underline{b}(\eta)) &\leq \alpha_w(\underline{b}(\eta)) \\ \pi(\widetilde{\alpha}_w(\underline{b}(\eta)), \widetilde{\alpha}_s(\underline{b}(\eta))) &= \pi(\alpha_w(\underline{b}(\eta)), \alpha_s(\underline{b}(\eta))) = \underline{b}(\eta). \end{aligned}$$

Since π is strictly increasing, we find:

$$\begin{aligned} \widetilde{\alpha}_s(\underline{b}(\eta)) &= \alpha_s(\underline{b}(\eta)) \\ \widetilde{\alpha}_w(\underline{b}(\eta)) &= \alpha_w(\underline{b}(\eta)). \qquad \text{(C12)} \end{aligned}$$

From (C4) and (C10, C11), we have:

$$\begin{aligned}
& \frac{d}{db}\ln F_w\left(\alpha_w\left(b\right)\right) \\
= & \frac{1}{\left(n-1\right)\left(\pi\left(\alpha_w\left(b\right),\alpha_s\left(b\right)\right)-b\right)} \\
< & \frac{1}{\left(n-1\right)\left(\pi\left(\widetilde{\alpha}_w\left(b\right),\widetilde{\alpha}_s\left(b\right)\right)-b\right)} \\
= & \frac{d}{db}\ln F_w\left(\widetilde{\alpha}_w\left(b\right)\right),
\end{aligned}$$

for all b in $\left(\underline{b}\left(\eta\right),\eta\right]$, and, consequently, $\frac{F_w(\alpha_w(b))}{F_w(\widetilde{\alpha}_w(b))}$ is strictly decreasing over this interval.

From (C9) and (C12), we have $\frac{F_w(\alpha_w(\underline{b}(\eta)))}{F_w(\widetilde{\alpha}_w(\underline{b}(\eta)))}=1$. Thus, $\alpha_w\left(b\right)<\widetilde{\alpha}_w\left(b\right)$, for all b in $\left(\underline{b}\left(\eta\right),\eta\right]$, which contradicts Lemma C6. $||$

In what follows, $\underline{b}\left(\eta\right)$ is as defined in Lemma C7. The sub-lemma below is helpful in the proof of Lemma C8.

Sub-lemma C1: *For all v in $\left(\alpha_w\left(\underline{b}\right),d\right]$, with $\varphi=\alpha_s\beta_w$ and $\underline{b}\geq\underline{b}\left(\eta\right)$:*

$$\begin{aligned}
& \left(\pi\left(v,\varphi\left(v\right)\right)-\beta_w\left(v\right)\right)F_w\left(v\right)^{n-1}-\left(\pi\left(\alpha_w\left(\underline{b}\right),\alpha_s\left(\underline{b}\right)\right)-\underline{b}\right)F_w\left(\alpha_w\left(\underline{b}\right)\right)^{n-1} \\
= & \int_{\alpha_w(\underline{b})}^{v}F_w\left(v\right)^{n-1}\frac{d}{dv}\pi\left(v,\varphi\left(v\right)\right); \qquad \text{(C13)}
\end{aligned}$$

$$\begin{aligned}
& \beta_w\left(v\right)F_w\left(v\right)^n-\underline{b}F_w\left(\alpha_w\left(\underline{b}\right)\right)^n \\
= & \int_{\alpha_w(\underline{b})}^{v}\pi\left(v,\varphi\left(v\right)\right)\frac{d}{dv}F_w\left(v\right)^n. \qquad \text{(C14)}
\end{aligned}$$

Proof: From (C4), we have:

$$\left(\pi\left(v,\varphi\left(v\right)\right)-\beta_w\left(v\right)\right)\frac{d}{dv}F_w\left(v\right)^{n-1}=F_w\left(v\right)^{n-1}\frac{d}{dv}\beta_w\left(v\right),$$

and hence:

$$\frac{d}{dv}\left\{\left(\pi\left(v,\varphi\left(v\right)\right)-\beta_{w}\left(v\right)\right)F_{w}\left(v\right)^{n-1}\right\}=F_{w}\left(v\right)^{n-1}\frac{d}{dv}\pi\left(v,\varphi\left(v\right)\right),$$

for all v in $\left(\alpha_w\left(\underline{b}\right),d\right]$, with $\varphi=\alpha_s\beta_w$ and $\underline{b}\geq\underline{b}\left(\eta\right)$. Integrating this equation from $\alpha_w\left(\underline{b}\right)$ to v in $\left(\alpha_w\left(\underline{b}\right),d\right]$, we find:

$$\begin{aligned}&\left(\pi\left(v,\varphi\left(v\right)\right)-\beta_{w}\left(v\right)\right)F_{w}\left(v\right)^{n-1}-\left(\pi\left(\alpha_{w}\left(\underline{b}\right),\alpha_{s}\left(\underline{b}\right)\right)-\underline{b}\right)F_{w}\left(\alpha_{w}\left(\underline{b}\right)\right)^{n-1}\\=&\int_{\alpha_{w}\left(\underline{b}\right)}^{v}F_{w}\left(v\right)^{n-1}\frac{d}{dv}\pi\left(v,\varphi\left(v\right)\right).\end{aligned}$$

(C13) then follows. Integrating (C13) by parts, we find (C14). ||

Lemma C8:
(i) For all $\eta\leq c$, we have $\underline{b}\left(\eta\right)\leq c$;
(ii) For all η in $\left(d-\int_c^d F_s\left(w\right)^{n-1}dw,d\right)$, we have $\underline{b}\left(\eta\right)>c$.

Proof: **(i)** If $\eta\leq c$, then, since $\left(\eta,d,d\right)$ belongs D, Lemma C5 implies that there exists a strictly increasing solution of (C3-C5) that can be continued strictly to the left of η. Consequently, $\underline{b}\left(\eta\right)<c$ and (i) is proved.

(ii) Let η be in the open interval $\left(d-\int_c^d F_s\left(w\right)^{n-1}dw,d\right)$. We show that $\underline{b}\left(\eta\right)>c$. Suppose that $\underline{b}\left(\eta\right)\leq c$, instead. From (C14) in Sub-lemma C1 with $v=d$ and $\underline{b}=c$ and Lemma C5, which implies $\varphi=\alpha_s\beta_w\leq F_s^{-1}F_w$, we have:

$$
\begin{aligned}
& \eta \\
\leq & \int_{\alpha_w(\underline{b})}^{d} \pi\left(v, \varphi\left(v\right)\right) \frac{d}{dv} F_w\left(v\right)^{n-1} \\
\leq & \int_{\alpha_w(\underline{b})}^{d} F_s^{-1} F_w\left(v\right) \frac{d}{dv} F_w\left(v\right)^{n-1} \\
\leq & \int_{c}^{d} F_s^{-1} F_w\left(v\right) \frac{d}{dv} F_w\left(v\right)^{n-1} \\
= & \; d - \int_{c}^{d} F_s\left(w\right)^{n-1} dw,
\end{aligned}
$$

and $\eta \leq d - \int_c^d F_s\left(w\right)^{n-1} dw$, which contradicts our initial assumption. $||$

Let η^* be defined as follows:

$$\eta^* = \inf\left\{\eta < d | \underline{b}\left(\eta\right) \geq c\right\}.$$

From Lemma C8 (ii), the set in the definition of η^* is not empty and:

$$c \leq \eta^* \leq d - \int_c^d F_s\left(w\right)^{n-1} dw. \tag{C15}$$

Lemma C9: *Let $\left(\alpha_w\left(b;\eta\right), \alpha_s\left(b;\eta\right)\right)$ be the solution of (C3-C5) in the domain D. Suppose $\underline{b}\left(\eta^*\right) > c$. Then, $\underline{b}\left(\eta\right) < c$, for all $\eta < \eta^*$, and :*

$$
\begin{aligned}
& \lim_{\eta \to_< \eta^*} \pi\left(\alpha_w\left(b;\eta\right), \alpha_s\left(b;\eta\right)\right) \\
= & \lim_{\eta \to_< \eta^*} \alpha_w\left(b;\eta\right) \\
= & \lim_{\eta \to_< \eta^*} \alpha_s\left(b;\eta\right) \\
= & \; \underline{b}\left(\eta^*\right),
\end{aligned}
$$

for all $b \in \left(c, \underline{b}\left(\eta^*\right)\right)$.

Proof: For all $\eta \leq \eta^*$, we have $(c, \underline{b}(\eta^*))$ is included the (interior) of the definition domain of the solution (α_w, α_s) of (C3-C5) and

$$\begin{aligned} \alpha_w(b) &\leq \pi(\alpha_w(b), \alpha_s(b)) \\ &\leq \pi(\alpha_w(\underline{b}(\eta^*)), \alpha_s(\underline{b}(\eta^*))) \\ &\leq \pi(\alpha_w^*(\underline{b}(\eta^*)), \alpha_s^*(\underline{b}(\eta^*))) \\ &= \underline{b}(\eta^*), \end{aligned} \tag{C16}$$

for all b in this interval, where the first inequality follows from Lemma C4, the second from Lemma C5, the third from Lemma C6, and the the equality from Lemma C7.

Let b be in $(c, \underline{b}(\eta^*))$. Suppose $\lim_{\eta \to_< \eta^*} \alpha_w(b)$ does not exist or is different from $\underline{b}(\eta^*)$. Then, there exists $v' < \underline{b}(\eta^*)$ and a sequence $(\eta_k)_{k\geq 1}$ such that

$$\begin{aligned} \lim_{k\to+\infty} \eta_k &= \eta^* \\ \alpha_w(b;\eta_k) &< v' \\ \eta_k &< \eta^*, \end{aligned}$$

for all $k \geq 1$.

Let ε be a strictly positive number. Let (α_w, α_s) be a solution defined over $(\underline{b}, \eta]$ of (C3-C5). Then, (β_w, φ) is the solution, defined over $(\alpha_w(\underline{b}), d]$, of the system below:

$$\begin{aligned} \frac{d}{dv}\varphi(v) &= \frac{f_w(v)}{F_w(v)} \frac{F_s(\varphi(v))}{f_s(\varphi(v))} \\ & \left\{ \begin{array}{c} n-1- \\ (n-2)\dfrac{\left(\begin{array}{c} (v-\beta_w(v)) F_s(\pi(v,\varphi(v))) + \\ (\pi(v,\varphi(v)) - \beta_w(v)) \\ (F_s(\varphi(v)) - F_s(\rho(v,\varphi(v)))) \end{array} \right)}{(\pi(v,\varphi(v)) - \beta_w(v)) F_s(\varphi(v))} \end{array} \right\} \end{aligned} \tag{C17}$$

$$\frac{d}{dv}\beta_w(v) = (n-1)\frac{f_w(v)}{F_w(v)}\left(\pi(v,\varphi(v)) - \beta_w(v)\right). \tag{C18}$$

Through the change of variables $(p,\chi,\zeta) = \left(F_w(v), F_s\varphi F_w^{-1}, \beta_w F_w^{-1}\right)$, the system above is equivalent to the system below:

$$\begin{aligned} \frac{d}{dp}\chi(p) &= \frac{\chi(p)}{p} \\ & \left\{ \frac{n-1-(n-2)\left[\begin{array}{c} \left(F_w^{-1}(p) - \zeta(p)\right) F_s\left(\pi\left(F_w^{-1}(p), \chi(p)\right)\right) + \\ \left(\pi\left(F_w^{-1}(p), \chi(p)\right) - \zeta(p)\right) \\ \left(\chi(p) - F_s\left(\pi\left(F_w^{-1}(p), \chi(p)\right)\right)\right) \end{array}\right]}{\left(\pi\left(F_w^{-1}(p), \chi(p)\right) - \zeta(p)\right)\chi(p)} \right\} \end{aligned} \tag{C19}$$

$$\frac{d}{dp}\zeta(p) = \frac{n-1}{p}\left(\pi\left(F_w^{-1}(p), \chi(p)\right) - \zeta(p)\right), \tag{C20}$$

and $\left(F_s\varphi F_w^{-1}, \beta_w F_w^{-1}\right)$ is a solution over $\left(F_w\left(\alpha_w\left(\underline{b}\right)\right), 1\right]$ to this system and the initial condition below:

$$\chi(1) = 1, \zeta(1) = \eta.$$

For the sake of convenience, denote (α_w^*, α_s^*) the solution to (C3-C5) for η^*. Let w be strictly larger than $\alpha_w^*\left(\underline{b}(\eta^*)\right)$ such that:

$$\left|\pi\left(w, \varphi^*(w)\right) - \beta_w^*(w)\right| < \varepsilon$$

(such a w exists since, from Lemma C7, $\pi\left(\alpha_w^*\left(\underline{b}(\eta^*)\right), \varphi^*\left(\alpha_w^*\left(\underline{b}(\eta^*)\right)\right)\right) = \beta_w^*\left(\alpha_w^*\left(\underline{b}(\eta^*)\right)\right)$). From the continuity of the solution to the system with initial condition above and the continuity of F_s^{-1} and π, for all $\varepsilon' > 0$, there exists $\delta > 0$, such that ζ and χ is defined at $F_w(w)$, and thus β_w and

$\varphi = F_s^{-1}\chi F_w$ are defined at w, and such that:

$$\begin{aligned} & \left|\zeta\left(F_w\left(w\right)\right) - \zeta^*\left(F_w\left(w\right)\right)\right| \\ = & \left|\beta_w\left(w\right) - \beta_w^*\left(w\right)\right| \\ < & \varepsilon, \end{aligned}$$

$$\begin{aligned} & \left|\pi\left(w, \varphi\left(w\right)\right) - \pi\left(w, \varphi^*\left(w\right)\right)\right| \\ = & \left|\pi\left(w, F_s^{-1}\chi F_w\left(w\right)\right) - \pi\left(w, F_s^{-1}\chi^* F_w\left(w\right)\right)\right| \\ < & \varepsilon, \end{aligned}$$

for all η such that $\eta^* - \delta < \eta < \eta^*$ (proceeding in this way, through the system (C19, C20) avoids making Lipschitz conditions on f).

From (C13) in Sub-lemma C1 with $\underline{b} = b$, we find:

$$\begin{aligned} & \int_{\alpha_w(b;\eta_k)}^{w} F_w\left(v\right)^n \frac{d}{dv}\pi\left(v, \varphi\left(v\right)\right) \\ \leq & \ \pi\left(w, \varphi\left(w\right)\right) - \beta_w\left(w\right) \\ \leq & \ 3\varepsilon, \end{aligned}$$

for all $k \geq 1$ such that $\eta^* - \delta < \eta_k$.

From $\alpha_w\left(b; \eta_k\right) < v' < \alpha_w^*\left(\underline{b}\left(\eta^*\right)\right) < w$, we then find:

$$\int_{v'}^{\alpha_w^*(\underline{b}(\eta^*))} F_w\left(v\right)^n \frac{d}{dv}\pi\left(v, \varphi\left(v\right)\right) \leq 3\varepsilon,$$

and, from Lemma C1 and the inequality $\frac{d}{dv}\pi\left(v, \varphi\left(v\right)\right) \geq \frac{\partial}{\partial v}\pi\left(v, \varphi\left(v\right)\right)$, we then obtain:

$$\left(\alpha_w^*\left(\underline{b}\left(\eta^*\right)\right) - v'\right) L \leq 3\varepsilon,$$

where L is a strictly positive lower bound of $\frac{\partial}{\partial v}\pi\left(v, w\right)$ over $\left[v', d\right]^2$. This inequality must hold for all $\varepsilon > 0$, which is clearly impossible since the LHS is a strictly positive constant. We have proved

$$\lim_{\eta \to_< \eta^*} \alpha_w(b;\eta) = \underline{b}(\eta^*),$$

for all b in $(c, \underline{b}(\eta^*))$. From the inequalities (C16), we then find $\lim_{\eta \to_< \eta^*} \pi(\alpha_w(b;\eta), \alpha_s(b;\eta)) = \underline{b}(\eta^*)$ and, consequently (because π is continuous and strictly increasing), $\lim_{\eta \to_< \eta^*} \alpha_s(b;\eta) = \underline{b}(\eta^*)$. $||$

Lemma C10:

$$\underline{b}(\eta^*) = c$$

and the solution of (C3-C5) for the value η^* of the parameter satisfies the boundary conditions $\beta_w(c) = \beta_s(c) = c, \beta_w(d) = \beta_s(d)$.

Proof: Suppose that, as in Lemma C9, $\underline{b}(\eta^*) > c$. For $\eta \leq \eta^*$, consider the function below:

$$\ln(\underline{b}(\eta^*) - b) + n \ln F_w(\alpha_w(b;\eta)), \tag{C21}$$

which is defined for b in $(c, \underline{b}(\eta^*)) \subseteq (\underline{b}(\eta), \underline{b}(\eta^*))$. Let b' be in this interval $(c, \underline{b}(\eta^*))$. Let ε be a (small) strictly positive number. Since, from Lemma C9, we have $\lim_{\eta \to_< \eta^*} \pi(\alpha_w(b;\eta), \alpha_s(b;\eta)) = \lim_{\eta \to_< \eta^*} \alpha_w(b;\eta) = \underline{b}(\eta^*)$, for all b in $(c, \underline{b}(\eta^*))$, there exists $\delta > 0$ such that

$$\begin{aligned} |F_w(\alpha_w(\underline{b}(\eta^*) - \varepsilon;\eta)) - F_w(\underline{b}(\eta^*))| &< \varepsilon, \\ \pi(\alpha_w(b';\eta), \alpha_s(b';\eta)) &> \underline{b}(\eta^*) - \varepsilon/2, \end{aligned} \tag{C22}$$

for all η such that $\eta^* - \delta < \eta < \eta^*$.

From (C4), the derivative, with respect to b, $\frac{-1}{\underline{b}(\eta^*) - b} + n \frac{d}{db} \ln F_w(\alpha_w(b;\eta))$ of the function (C21) is equal to:

$$\frac{-1}{\underline{b}(\eta^*) - b} + \frac{1}{\pi(\alpha_w(b;\eta), \alpha_s(b;\eta)) - b},$$

and consequently, we have:

$$\begin{aligned} & \ln\left(\underline{b}\left(\eta^*\right)-b'\right)+n\ln F_w\left(\alpha\left(b';\eta\right)\right) \\ \leq\ & \ln\varepsilon+n\ln\left\{F_w\left(\underline{b}\left(\eta^*\right)\right)+\varepsilon\right\} \\ & -\int_{b'}^{\underline{b}(\eta^*)-\varepsilon}\left(\frac{1}{\pi\left(\alpha_w\left(b;\eta\right),\alpha_s\left(b;\eta\right)\right)-b}-\frac{1}{\underline{b}\left(\eta^*\right)-b}\right)db, \quad \text{(C23)} \end{aligned}$$

for all η such that $\eta^*-\delta<\eta<\eta^*$.

Since $\underline{b}\left(\eta^*\right)-b\geq\varepsilon$ and, from (C22), $\pi\left(\alpha_w\left(b;\eta\right),\alpha_s\left(b;\eta\right)\right)-b\geq\pi\left(\alpha_w\left(b';\eta\right),\alpha_s\left(b';\eta\right)\right)-\left(\underline{b}\left(\eta^*\right)-\varepsilon\right)\geq\varepsilon/2$ over the integration interval in (C23), we have the following bound over this interval:

$$\left|\frac{1}{\pi\left(\alpha_w\left(b;\eta\right),\alpha_s\left(b;\eta\right)\right)-b}-\frac{1}{\underline{b}\left(\eta^*\right)-b}\right|\leq\frac{2}{\varepsilon}+\frac{1}{\varepsilon}.$$

We may thus apply Lebesgue convergence theorem, for example, and we find $\lim_{\eta\rightarrow<\eta^*}\int_{b'}^{\underline{b}(\eta^*)-\varepsilon}\left(\frac{1}{\pi(\alpha_w(b;\eta),\alpha_s(b;\eta))-b}-\frac{1}{\underline{b}(\eta^*)-b}\right)db=0$ and, consequently,:

$$\begin{aligned} & \ln\left(\underline{b}\left(\eta^*\right)-b'\right)+n\ln F_w\left(\underline{b}\left(\eta^*\right)\right) \\ =\ & \lim_{\eta\rightarrow<\eta^*}\left\{\ln\left(\underline{b}\left(\eta^*\right)-b'\right)+n\ln F_w\left(\alpha_w\left(b';\eta\right)\right)\right\} \\ \leq\ & \ln\varepsilon+n\ln\left\{F_w\left(\underline{b}\left(\eta^*\right)\right)+\varepsilon\right\}. \end{aligned}$$

Since this inequality holds for all $\varepsilon>0$, we obtain $\ln\left(\underline{b}\left(\eta^*\right)-b'\right)+n\ln F_w\left(\underline{b}\left(\eta^*\right)\right)=-\infty$ or, equivalently, $\left(\underline{b}\left(\eta^*\right)-b'\right)F_w\left(\underline{b}\left(\eta^*\right)\right)=0$, which is impossible, since $\underline{b}\left(\eta^*\right)>0$ and $b'<\underline{b}\left(\eta^*\right)$. We have proved that $\underline{b}\left(\eta^*\right)>c$ is impossible, that is, we have proved the equality $\underline{b}\left(\eta^*\right)=c$. ||

Lemma C11: *There cannot exist two different values of the parameter η such that the corresponding solutions to (C3-C5) are defined over $(c,\eta]$ and such that $\alpha_w\left(c\right)=\alpha_s\left(c\right)=c$.*

Proof: Suppose there exists two such values η' and $\widetilde{\eta}$, with $\eta'<\widetilde{\eta}<d$. Let α_w',α_s' and $\widetilde{\alpha}_w,\widetilde{\alpha}_s$ the corresponding solutions to (C3-C5). Then, $\beta_w'=$

$\alpha_w'^{-1}, \varphi' = \beta_s' \alpha_w'$, and $\widetilde{\beta}_w = \widetilde{\alpha}_w^{-1}, \widetilde{\varphi} = \widetilde{\beta}_s \widetilde{\alpha}_w$ are solutions to (C17, C18) with initial condition $\varphi(d) = d, \beta_w(d) = \eta$.

From (C17), we have $\frac{d}{d \ln F_w} \varphi'(d) = \frac{d}{d \ln F_w} \widetilde{\varphi}(d) = \frac{1}{f_s(d)}$. Moreover, from (C17, C18) and the differentiability of $\pi(v, \varphi(v))$ and $\pi(v, \widetilde{\varphi}(v))$; $\frac{d}{d \ln F_w} \varphi'$, $\frac{d}{d \ln F_w} \widetilde{\varphi}$, $\frac{d}{d \ln F_w} \beta_w$, $\frac{d}{d \ln F_w} \widetilde{\beta}_w$ are differentiable and, by differentiating (C17), we find:

$$\begin{aligned}
& \frac{d}{dv}\left(\frac{d}{dnF_w}\varphi'(v)\right)_{v=d} - \frac{d}{dv}\left(\frac{d}{d\ln F_w}\widetilde{\varphi}(v)\right)_{v=d} \\
= \; & \frac{1 - \frac{d}{dv}\pi(v, \widetilde{\varphi}(v))_{v=d}}{d - \widetilde{\eta}} - \frac{1 - \frac{d}{dv}\pi(v, \varphi'(v))_{v=d}}{d - \eta'} \\
= \; & \left(1 - \frac{1 + (f_w(d) / f_s(d))}{2}\right)\left(\frac{1}{d - \widetilde{\eta}} - \frac{1}{d - \eta'}\right) \\
> \; & 0,
\end{aligned}$$

since, from assumption (c), $\frac{f_w(d)}{f_s(d)} < 1$. Consequently, there exists $\varepsilon > 0$, such that $\varphi'(v) > \widetilde{\varphi}(v)$ and, from the initial condition $\beta_w(d) = \eta$, $\beta_w'(v) < \widetilde{\beta}_w(v)$, for all v in $(d - \varepsilon, d)$.

Let $\underline{v}$ be defined as follows:

$$\underline{v} = \inf\left\{v \in [c, d] \,|\varphi'(w) > \widetilde{\varphi}(w) \text{ and } \beta_w'(w) < \widetilde{\beta}_w(w), \text{ for all } w \text{ in } (v, d)\right\}.$$

From the previous paragraph, $\underline{v} \leq d - \varepsilon$. Suppose $\underline{v} > c$. Since φ',β_w' and $\widetilde{\varphi}, \widetilde{\beta}_w$ are distinct solutions of the same differential system, the equalities $\varphi'(\underline{v}) = \widetilde{\varphi}(\underline{v})$ and $\beta_w'(\underline{v}) = \widetilde{\beta}_w(\underline{v})$ cannot both hold. Assume first $\varphi'(\underline{v}) > \widetilde{\varphi}(\underline{v})$ and $\beta_w'(\underline{v}) = \widetilde{\beta}_w(\underline{v})$. From (C18), $\frac{d}{dv}\beta_w(\underline{v}) > \frac{d}{dv}\widetilde{\beta}_w(\underline{v})$, which is impossible since $\beta_w'(w) < \widetilde{\beta}_w(w)$ holds true over $(\underline{v}, d)$. Assume next $\varphi'(\underline{v}) = \widetilde{\varphi}(\underline{v})$ and $\beta_w'(\underline{v}) < \widetilde{\beta}_w(\underline{v})$. The factor between braces in (C17) can be rewritten as:

$$n - 1 - (n-2)\left[1 - \frac{\pi(v, \varphi(v)) - v}{\pi(v, \varphi(v)) - \beta_w(v)} \frac{F_s(\pi(v, \varphi(v)))}{F_s(\varphi(v))}\right],$$

and hence is increasing in $\beta_w(v)$. Consequently, $\frac{d}{dv}\varphi'(\underline{v}) < \frac{d}{dv}\widetilde{\varphi}(\underline{v})$, which is impossible since $\varphi'(w) > \widetilde{\varphi}(w)$ holds true over $(\underline{v}, d)$.

We have proved $\underline{v} = c$, which implies $\varphi'(w) > \widetilde{\varphi}(w)$, for all w in (c, d). From (C14) in Sublemma C1, we then have:

$$\begin{aligned} & \beta'_w(d) \\ = & \int_c^d \pi(v, \varphi'(v)) \frac{d}{dv} F_w(v)^n \\ > & \int_c^d \pi(v, \widetilde{\varphi}(v)) \frac{d}{dv} F_w(v)^n \\ = & \widetilde{\beta}_w(d) \end{aligned}$$

and $\beta'_w(d) > \widetilde{\beta}_w(d)$. However, this is impossible since, from the initial condition at d, $\beta'_w(d) = \eta' < \widetilde{\eta} = \widetilde{\beta}_w(d)$. ||

III. Proof of the Proposition

(i) follows from Lemmas C10 and C11 and (ii) from Lemma C5.