Price expectations in IPV sequential second-price auctions

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Abstract

In the symmetric IPV sequential second-price auction with k units, equilibria with decreasing expected prices exist for all numbers $n > k$ of bidders if and only if the value complementary distribution function is not $-(k+1)$-concave everywhere.

Keywords: sequential second-price auction; decreasing price; Milgrom and Weber's model; independent and private values; homogeneous bidders; risk neutrality; inefficient equilibria.

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1. Introduction

The seminal work on sequential auctions of identical items by Milgrom and Weber (2000),[1] henceforth referred to as MW, predicts nondecreasing prices along the symmetric equilibrium of their symmetric model with interdependent values, affiliated signals, risk neutrality, and single-unit demands. With private and independent values, or IPV, the expected price in MW's equilibrium stays constant as the price becomes a martingale[2]. By augment-

[1] While published in 2000, this work has circulated since 1982. See also Weber (1983) and Mezzetti, Pekeč, and Testlin (2008).

[2] The realized price goes up and down. Price decreases may be more probable than price increases, as in the sequential second-price auction with two units and three bidders when the value density decreases. Indeed, in this case, the median of the second-auction price conditional on the first-auction price is smaller than its expectation, hence than the first-auction price.

ing MW's IPV model in a variety of ways, theorists have provided possible explanations for the many documented instances of changing and even declining prices. What explanation is the most plausible depends on the particular auctions under consideration. Decreasing price expectations already appear in MW's discussion of risk aversion. Later contributions have allowed, for example: other utility functions; entry costs; exogenously or endogenously uncertain supply; multi-unit demands with decreasing marginal values, complementarities, options to buy more than one unit at the current price, budget constraints, or complete information; different items being auctioned with various assumptions on when bidders learn their values; costs of waiting; bidding agents, or nonstrategic bidders. Deltas and Kosmopolou (2004) and Trifunović (2014) survey many of these contributions.

More recently, Ghosh and Liu (2021) have found decreasing prices in the sequential first-price auction with ambiguity-averse bidders.[3] Rosato (2023) has shown that the random price process is a supermartingale if bidders' preferences display expectations-based loss aversion of the Koszegi and Rabin (2006, 2007, 2009)'s type.

Surprisingly, the vast theoretical literature that followed MW has yet to answer the fundamental question of whether MW's original IPV model possesses asymmetric equilibria where the expected price decreases. As I show in this paper, the answer for the sequential second-price auction, or SSPA, is a qualified yes. In fact, such equilibria do exist for any number k of units and all larger numbers of bidders, but only when the value complementary distribution function is not $-(k+1)$-concave everywhere. To obtain this result, I first prove through marginal revenue analysis that a declining expected price can only occur for such value distributions and for some inefficient equilibria. For these distributions, I then construct sequentially rational equilibria with the required properties.

[3] Gosh and Liu (2021) deal with the time inconsistency that may arise in the bidders' strategies through an equilibrium requirement similar to Strotz (1955)'s "consistent planning."

Decreasing prices in asymmetric equilibria also occur in Katzman (1999) and Lorenzon (2023). Both consider IPV models of the two-unit SSPA with two bidders, each with two-unit demand. Katzman (1999) finds that the expected price decreases along an asymmetric equilibrium if the bidders are sufficiently heterogeneous ex ante. Although Lorenzon (2023) assumes ex-ante homogeneous bidders, he obtains a price supermartingale at equilibrium when one of the two bidders learns his value only after winning the first auction. While I too construct asymmetric equilibria, I stay within MW's IPV model, where the bidders are ex-ante homogeneous, learn their own values before the SSPA starts, and have single-unit demands.

In Section 2, I explain the reasons behind the results, using the two-unit three-bidder case as an example. In Section 3, I provide preliminaries of the formal analysis, along with various definitions and reminders. I devote Section 4 to marginal revenue analysis and Sections 5 and 6 to the construction of equilibria. I obtain the main result in Section 7, and conclude in Section 8.

2. First observations on price changes and inefficiency

The results will follow from a link between the equilibrium allocation and the changes in the expected price. Here, I illustrate this link in the simple case of two units and three bidders whose values are independently and identically distributed over $[0, 1]$. In any efficient equilibrium of the SSPA, the two bidders with the highest values receive the two units, and the price at the second auction, where bidders bid their values, is the lowest value, as in MW's equilibrium. From the multi-unit extension of the revenue-equivalence theorem, the sum of the expected prices at both auctions is the same as in MW's equilibrium. The expected price at the first auction must then also be the same.

In the equilibrium in Figure 1 below, a bidder waits until the second auction to submit a serious bid. The two other bidders follow a common bidding function at the first auction. Although asymmetric, this equilibrium is efficient; therefore, the expected price does not change from the first auction to the second. I explain in Appendix A why the bidding functions in Figure 1 form an equilibrium.

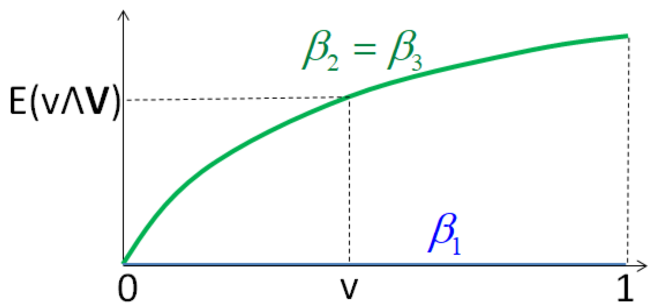

Figure 1: First-auction bidding functions in an efficient asymmetric equilibrium. $E\left(\mathrm{v} \wedge \mathbf{V}\right)$ is the expectation of the minimum of v and a random variable $\mathbf{V}$ with the same probability distribution as the bidders' values.

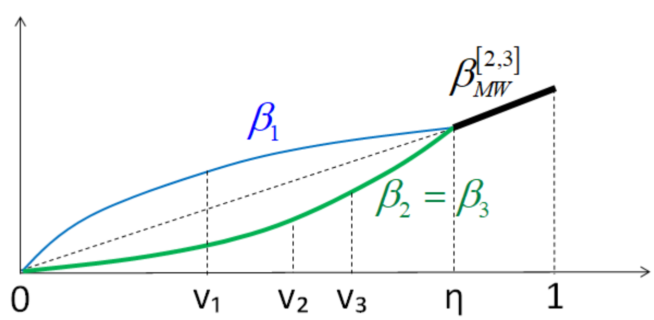

Figure 2: An inefficient equilibrium. $\beta_{MW}^{[2,3]}$ is MW's equilibrium bidding function.

The equilibrium in Figure 2 is inefficient. By bidding more aggressively than bidders 2 and 3 over $[0, \eta]$ at the first auction, bidder 1 can win a unit despite having the lowest value. In such equilibria, whose existence I prove in Appendix A, the expected prices at the two auctions may differ.

Inefficiency increases the second-auction price above the lowest value. If there was an equilibrium with decreasing price expectations, the total revenues would then exceed the revenues from efficient equilibria. Per the revenue-equivalence theorem, a winning bidder's contribution to the expected revenues is the marginal revenue $MR(v)$ of the value complementary distribution function, which is interpreted as a demand function, at this bidder's value v, which is interpreted as a price. Therefore, no decreasing-price equilibrium exists if the marginal revenue increases with the value, as the efficient equilibria maximize revenues in this case.

For some value distributions, the marginal revenue decreases quickly enough that the increase in total revenues due to some inefficient equilibrium can exceed twice that of the second-auction price. When this happens, the expected-price sequence decreases. For example, in Figure 2, bidder 1 wins the first auction although his value v_1 is the lowest, bidder 3 with value v_3 wins the second auction, and the price at the second auction is bidder 2's value v_2. Under efficiency, bidder 2 would have won a unit, as his value is the second highest, and the price at the second auction would have been v_1. Thus, the expected price decreases in this equilibrium if $MR(v_1) - MR(v_2)$ is larger than $2(v_2 - v_1)$ for any such combination of values. That is, if $2v + MR(v)$ decreases over the interval $[0, \eta]$. As I show in the next section, this is the case when the value complementary cumulative distribution function is "convex enough" or, more precisely, strictly -3-convex over this interval.

Later in Sections 5 and 6, I describe another type of inefficient equilibria, where the inefficiency also comes from a bidder following a higher bidding function, but only for values that can be confined to any given subinterval of $[0, 1]$–in particular to an interval where $2v + MR(v)$ would decrease. Consequently, for any value distribution where $2v + MR(v)$ decreases somewhere, an equilibrium of this type will exist where the expected price decreases.

3. Definitions and preliminary results

In this section, I define the model, the equilibrium concept, and some notations. I also obtain preliminary results.

A number $k > 1$ of identical units is put up for sale at an SSPA with $n > k$ potential bidders, whose values are private and independently distributed over $[0,1]$ according to the same continuous cumulative distribution function F that is C^2 over $(0,1]$ and has a strictly positive derivative $F' = f$ over $(0,1)$. The choice of the value interval $[0,1]$ is made only for notational convenience, and all results apply to any bounded interval. By labelling the bidders from 1 to n, their set becomes $\mathcal{N} = \{1, 2, ..., n\}$.

Participation in any auction is voluntary, and who participates is public information before bidding takes place. Only bidders who participated and lost all previous auctions may participate in a given auction. The reserve price is always zero, and fair lotteries break ties. After each auction, the auctioneer announces only the winning bid.

Thus, if bidder i wins the s^{th} auction or does not take part in it, he becomes ineligible to participate in future auctions. Before bidding at the t^{th} auction, a participating bidder i has observed the history $\left(v_i, \left(\mathcal{P}^{(s)}, b_i^{(s)}, b_w^{(s)}\right)_{s<t}, \mathcal{P}^{(t)}\right)$, where v_i is his value; $\mathcal{P}^{(s)}, b_i^{(s)}, b_w^{(s)}$ are the set of participants, his own bid, and the winning bid at the past s^{th} auction; and $\mathcal{P}^{(t)}$ is the set of participants at the current auction.

Only the winner of an auction earns a payoff: the difference between his value and the auction price. Additionally, the bidders are risk neutral. I denote the sequential auction so defined as SSPA(k,n). In some lemmas, I will have to allow the values to be distributed differently. I will clearly indicate when this is the case.

A bidder i's strategy τ_i in SSPA(k,n) recommends permissible participation and bidding decisions, conditional on every possible history he has observed.[4] An undominated strategy prescribes participation when eligible and, with at least one other bidder participating, a bid never exceeding the value at any auction, equal to the value at the last auction, and equal to 0 at any previous auction with no more participating bidders than units.[5]

An equilibrium $\boldsymbol{\sigma}$ of SSPA(k,n) must specify a profile $(\tau_1, ..., \tau_n)$ of undominated strategies that are sequentially rational, that is, such that they recommend optimal responses from any bidder to the other bidders' strategies, given his beliefs about their values and past bids. In addition to consistency between beliefs and strategies, I require that no bidder participating in an auction be believed to have previously deviated, as any undominated strategy defined above is compatible with the bidder having lost all prior auctions. A bidder's beliefs about other participants to an auction are then uniquely determined by the history he has observed and the strategies. Beliefs about nonparticipants are irrelevant, as those are ineligible to take part in any future auction. The profile of strategies will therefore suffice to define any equilibrium. Finally, by dominance of truth-bidding, no attention needs to be paid to the beliefs at the last auction.

An equilibrium is said to have its inefficiency confined to some interval if conditional upon allocating the units inefficiently the values of the inefficient winners and losers almost surely belong to this interval. An inefficient equilibrium allocates units inefficiently with strictly positive probability.

Thanks to the supermodularity I state in Lemma 1 below, verifying first-order optimality conditions or ruling out profitable local deviations suffices to check the global optimality of first-auction nondecreasing bidding functions. From (ii), supermodularity will also occur in the "continuation" sequential auctions, starting after one or more units have already been sold.

[4] The probability distributions over the possible decisions must depend on the observed histories in a Borel measurable way.

[5] This definition encompasses all strategies that are not dominated, with the exception of those that do not recommend participation at value 0.

Lemma 1:[6]

(i) *Given strategies the other bidders are expected to follow, a bidder's total expected payoff in* SSPA(k,n) *is supermodular in his value and first-auction bid if he behaves optimally after losing the first auction.*

(ii) *(i) holds true even if the bidders' values are not distributed identically.*

Proof: See Appendix B.

$\overline{F}$ is the value complementary cumulative function, or ccdf, $1-F$. Throughout the paper, bold Latin letters refer to random variables. Thus, $\mathbf{V}_i$ is bidder i's value viewed as a random variable, and v_i its realization. $\mathbf{V}$ is a generic value, distributed according to F, independently of all other random variables. In addition to the standard order statistics, I also rely on order statistics of values when one of them may be distributed differently. See the notations below.

Order statistics

(i) $\mathbf{V}_{(l,m)}$ is the l^{th} order statistic of m values iid[7] according to F.

(ii) $\widetilde{\mathbf{V}}_{(l,m)}$ is the l^{th} order statistic of m values independently distributed: $m-1$ according to F; and one, I denote $\widetilde{\mathbf{V}}$, according to some, possibly different, probability distribution. I will still use the notation $\mathbf{V}_{(l',m-1)}$ for the order statistics of the $m-1$ iid values, and will indicate the distribution of $\widetilde{\mathbf{V}}$ as a subscript to the expectation sign. For example, $E_{\widetilde{\mathbf{V}}\sim G}\left(\widetilde{\mathbf{V}}_{(5,8)}|\mathbf{V}_{(1,7)}=v\right)$ means the expectation of the fifth order statistic of a complete set of eight values conditional on v's being the maximum of the seven iid values, and G's being the distribution of the value $\widetilde{\mathbf{V}}$.

Given a strategy profile, $\mathbf{P}_t$ is the price at the t^{th} auction when bidders follow their strategies. A decreasing expected-price sequence is a nonincreasing

[6] The assumption below of optimal responses after the first auction implies their existence for all values in the distributions' supports.

[7] Standard notation for "independently and identically distributed."

sequence $E\mathbf{P}_1 \geq ... \geq E\mathbf{P}_k$ such that $E\mathbf{P}_1 > E\mathbf{P}_k$.

I denote τ_{MW} MW's equilibrium strategy. At any auction where $l > 1$ units remain and $m > l$ bidders participate, this undominated strategy recommends the bidding function $\beta_{MW}^{[l,m]}$ such that, for all v in $[0,1]$:

$$\beta_{MW}^{[l,m]}(v) = E\left(\mathbf{V}_{(l,m-1)} | \mathbf{V}_{(1,m-1)} = v\right).$$

For $l \geq 1$, the ccdf of $\mathbf{V}_{(l,l)}$ is $\overline{F}^l$. The marginal-revenue function[8] of this ccdf is $MR_{(l)}$ below:

$$MR_{(l)}(v) = v - \frac{\overline{F}(v)}{lf(v)},$$

with v in $(0,1)$. As in Caplin and Nalebuff (1991), a strictly positive function g is (strictly) ρ-concave if ρg^ρ is (strictly) concave when $\rho \neq 0$ and (strictly) log-concave when $\rho = 0$. The definitions of ρ-convexity and strict ρ-convexity are similar. ρ-concavity implies strict σ-concavity, for all $\sigma < \rho$, and ρ-convexity implies strict ζ-convexity, for all $\zeta > \rho$. A simple calculation and results from McAfee and McMillan (1987) and Ewerhart (2013) imply the technical lemma below, where $\frac{d \ln f}{d \ln \overline{F}}$ measures the convexity of the inverse of $\overline{F}$ as the coefficient of relative risk aversion measures the concavity of a von Neumann-Morgenstern utility function of wealth in standard microeconomic theory.

Lemma 2: *For all $l, l' \geq 1$, v in $(0,1)$, and open subinterval I of $(0,1)$:*

(i) $MR_{(l)}(v) = (1 - l'/l)\, v + (l'/l)\, MR_{(l')}(v)$

(ii) *The following three statements are equivalent: $MR_{(l)}$ is nondecreasing over I; $\overline{F}$ is $-l$-concave over I; $\sup_{w \in I} \frac{d \ln f}{d \ln \overline{F}}(w) \leq l+1$.*

(iii) *If $\frac{d \ln f}{d \ln \overline{F}}(v) > l+1$, there exists an open subinterval I' containing v where $\overline{F}$ is strictly $-l$-convex and $MR_{(l)}$ decreases strictly.*

Throughout the paper, $\vee$ and $\wedge$ are the maximum and minimum opera-

[8] Or "virtual-value" function. See Myerson (1981, 1984), Bulow and Roberts (1989), and Bulow and Klemperer (1996).

tions.

4. Marginal revenue analysis

Lemma 3 below follows from the identity $\sum_{t=1}^{k-1} E(\mathbf{P}_t) - (k-1)\,E\mathbf{P}_k = E\left(\sum_{t=1}^{k} \mathbf{P}_t\right) - kE\mathbf{P}_k$; the revenue-equivalence theorem; the observation that, under efficiency, the price at the last auction is the $k+1^{\text{th}}$ highest value; and Lemma 2. $INEF(\boldsymbol{\sigma})$ in the lemma is the event under which the equilibrium $\boldsymbol{\sigma}$ allocates the units inefficiently.

Lemma 3: *For $2 \leq k < n$, let $\boldsymbol{\sigma}$ be an equilibrium of SSPA(k,n) with inefficiency confined to an interval I. Conditional on $INEF(\boldsymbol{\sigma})$, let $\mathbf{L}_1 \leq k$ be such that $\mathbf{V}_{(\mathbf{L}_1,n)}$ is the highest value among the inefficient losers.' Then:*

$$
\begin{aligned}
&(k+1)\,E\left(MR_{(k+1)}\left(\mathbf{V}_{(k+1,n)}\right) - MR_{(k+1)}\left(\mathbf{V}_{(\mathbf{L}_1,n)}\right)|INEF(\boldsymbol{\sigma})\right)\Pr(INEF(\boldsymbol{\sigma})) \\
= &\sum_{t=1}^{k-1} E(\mathbf{P}_t) - (k-1)\,E\mathbf{P}_k, \textit{ if } k = n-1; \\
\leq &\sum_{t=1}^{k-1} E(\mathbf{P}_t) - (k-1)\,E\mathbf{P}_k, \textit{ if } \overline{F} \textit{ is } -1\textit{-convex over } I.
\end{aligned}
$$

Proof: See Appendix B.

From the equality in the lemma above when $k = n-1$ and from Lemma 2, strict $-(k+1)$-convexity in some interval is necessary for the last expected price to be below average, and, hence, also for the expected price sequence to be decreasing. Lemma 4 follows.

Lemma 4: *If an equilibrium of the* SSPA(k,$k+1$) *exists where $E\mathbf{P}_k < \left(\sum_{t=1}^{k-1} E(\mathbf{P}_t)\right)/(k-1)$, then $\overline{F}$ is not $-(k+1)$-concave everywhere.*

From the inequality in Lemma 3 when $k \leq n-1$ and from Lemma 2, strict $-(k+1)$-convexity of $\overline{F}$ over an interval that contains the inefficiency of an inefficient equilibrium suffices to obtain a last expected price below average.[9]

Example: $k = 2$.

With $k = 2$, the RHS's of the equality and inequality in Lemma 3 reduce to $E\mathbf{P}_1 - E\mathbf{P}_2$. The equality is the link I alluded to in Section 2 between equilibrium allocation and the expected difference between the auction prices when there are $n = 3$ bidders. In this case, $\mathbf{L}_1 = 2$.

For any number $n \geq 3$ of bidders, Lemma 3 implies a decreasing expected-price sequence if the equilibrium allocation is inefficient with strictly positive probability and only for values in an interval where $\overline{F}$ is strictly -3-convex. From Lemma 2 in the previous section and the similarity between $\frac{d \ln f}{d \ln \overline{F}}$ and the coefficient of relative risk aversion, it is simple to find distributions for which $\overline{F}$ is strictly -3-convex throughout an interval of probability close to 1. For example, for any distribution $F(v) = \left(1 - v^{1/(1-c)}\right) / \left(1 - \overline{v}^{1/(1-c)}\right)$, with $c > 4$, $\overline{F}$ is strictly -3-convex over an interval $[1, d(\overline{v})]$ whose probability tends towards 1 as the upper extremity $\overline{v}$ of the support $[1, \overline{v}]$ becomes larger[10]. Thus, the expected price decreases from the first auction to the second along any equilibrium that is inefficient, but only within the interval $[1, d(\overline{v})]$.

5. Construction of equilibria: the strategies at the first auction

Inefficiency in the equilibria I now construct occurs when bidder 1 with a lower value outbids all k highest-value bidders at the first auction. The inefficiency is localized to a value interval where bidder 1 follows a bidding function higher than the other bidders.' The strategies at the subsequent

[9] Under inefficiency, $\mathbf{V}_{(k+1,n)}$ also belongs to this interval.

[10] $d(\overline{v}) = (1 - (4/c))^{c-1} \overline{v}$ and $F(d(\overline{v})) = \frac{1 - \frac{c}{c-4} \overline{v}^{1/(1-c)}}{1 - \overline{v}^{1/(1-c)}}$, for $\overline{v} > (1 - (4/c))^{1-c}$. To obtain an example where $[0, 1]$ is the value support, simply replace v in the definition of $F(v)$ with $1 - v + v\overline{v}$.

auctions then efficiently allocate the remaining units among the remaining bidders. When $k > 2$, the efficient equilibria that the strategies define from the second auction on are asymmetric and similar to the equilibrium in Figure 1 (Section 2).

The following functions will enter into the construction of the equilibrium bidding functions at auctions with l units and m participants.

Definitions: *Let w be in $(0,1)$, l,m be such that $1 < l < m$, and G be a probability distribution over $[0,1]$.*

(i) $\widetilde{\beta}^{[l,m;G]}$ *is the function whose value at v in $[0,1]$ is:*

$$E_{\widetilde{\mathbf{V}}\sim G}\left(\widetilde{\mathbf{V}}_{(l,m-1)}|\mathbf{V}_{(1,m-2)}=v\right).$$

(ii) $\beta^{[l,m;w]}$ *is the function* $\widetilde{\beta}^{[l,m;G]}$ *where* $G = F|_{[0,w]}$.[11]

Therefore, $\widetilde{\beta}^{[l,m;G]}(v)$ is the expectation of the l^{th} highest among $m-1$ independent values, $m-2$ of which are iid according to F conditional on their maximum's being v, and the remaining one $\widetilde{\mathbf{V}}$ is distributed according to G (see Section 3 for the notations for order statistics).

The first-auction bidding function I now construct from a y in $(0,1)$ agrees with MW's bidding function outside a neighborhood $[x,z]$ of y, whose extremities x, z satisfy the condition (1) below:

$$\int_x^z \left(E\left(y\wedge\mathbf{V}_{(k,n-1)}|\mathbf{V}_{(1,n-1)}=u\right)-\beta^{[k,n;y]}(u)\right)dF(u)^{n-1}=0. \quad (1)$$

Lemma 5 below implies the existence of such neighborhoods $[x,z]$ that can be arbitrarily small, and where the integrand in the LHS above changes its sign only once, from negative to positive.

[11] Alternative expressions for $\beta^{[l,m;w]}$ are: $E\left(\mathbf{V}_{(l-2,m-3)}\wedge\left(\mathbf{V}_{(l-1,m-3)}\vee\mathbf{V}\right)|\mathbf{V}_{(1,m-3)}\leq v;\mathbf{V}\leq w\right)$ if $2 < l < m$; $E\left(v\wedge\left(\mathbf{V}_{(1;m-3)}\vee\mathbf{V}\right)|\mathbf{V}_{(1;m-3)}\leq v;\mathbf{V}\leq w\right)$ if $l=2$ and $m>3$; and $E\left(v\wedge\mathbf{V}|\mathbf{V}\leq w\right)$ if $l=2$ and $m=3$.

Lemma 5: *Let k, n be such that $2 \leq k < n$.*

(i) *For all y in $(0,1)$, the integrand in the LHS of (1) is a strictly negative function of u in $(0,y)$.*

(ii) *For all y in $(0,1)$ and all x in $(0,y)$ sufficiently close to y, there exists a solution z in $(y,1)$ to (1) that tends towards y with x and such that the integrand is strictly positive for all u in (y,z).*

Proof: See Appendix B.

From a triple (x,y,z) as in Lemma 5 (ii) and with x close enough to y that $\beta_{MW}^{[k,n]}(z) \leq y$, I define the bidders' strategies $\tau_1^{[x,y,z]}, ..., \tau_n^{[x,y,z]}$ first by what they recommend at the first auction and then at all subsequent auctions. The strategy $\tau_i^{[x,y,z]}$ of bidder i in $\mathcal{N} = \{1, ..., n\}$ is an undominated strategy that recommends to bid at the first auction according to MW's equilibrium strategy when not all bidders participate and according to the bidding function β_i below when they all participate:

$$\begin{aligned}
\beta_2 &= ... = \beta_n = I_{[x,z]}\beta^{[k,n;y]} + \left(1 - I_{[x,z]}\right)\beta_{MW}^{[k,n]}, \\
\beta_1 &= \beta_{MW}^{[k,n]}(x)\, I_{[x,y)} + \beta_{MW}^{[k,n]}(z)\, I_{[y,z]} + \left(1 - I_{[x,z]}\right)\beta_{MW}^{[k,n]},
\end{aligned}$$

where $I_{[x,y)}$, $I_{[y,z]}$, and $I_{[x,z]}$ are indicator functions[12]. Thus, the bidding functions differ from MW's equilibrium bidding function $\beta_{MW}^{[k,n]}$ only within $[x,z]$. Bidder 1's bidding function jumps up at y from the constant $\beta_{MW}^{[k,n]}(x)$ to the constant $\beta_{MW}^{[k,n]}(z)$. The other bidders' common bidding function jumps up at x and z and is equal to $\beta^{[k,n;y]}$ between these values. See Figure 3 below.

[12] For example, $I_{[x,y)}(v)$ is equal to 1 if v belongs to $[x,y)$, and 0 otherwise.

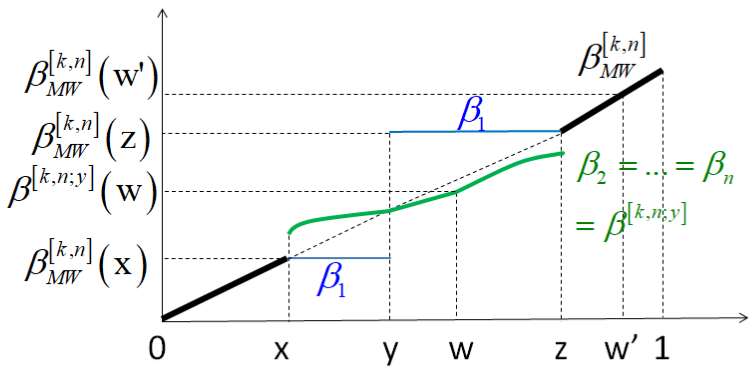

Figure 3: Equilibrium bidding functions at the first auction with $n > 3$.

Example: $k = 2$

With two units, the equilibrium requirement of sequential rationality reduces to the optimality of the first-auction bidding functions and only under full participation, as I have assumed the strategies coincide with MW's equilibrium strategy under partial participation.

In the particularly simple case of $n = 3$ bidders, the definition of $\beta^{[k,n;y]}(v)$ reduces to:

$$\beta^{[2,3;y]}(v) = E(v \wedge \mathbf{V} | \mathbf{V} \leq y), \quad (2)$$

as the distributions of $\mathbf{V}$ conditioned on $\mathbf{V} \leq y$ and of $\widetilde{\mathbf{V}}$ are identical. By using y as an intermediate bound of integration[13], the condition (1) becomes (3) below, and Lemma 5 is immediate in this case:

$$\int_x^y (E(\mathbf{V}|\mathbf{V} \leq u) - E(u \wedge \mathbf{V}|\mathbf{V} \leq y))\, dF^2(u) \\ + (y - E(\mathbf{V}|\mathbf{V} \leq y))(F(z) - F(y))^2 = 0. \quad (3)$$

The mathematical expression in (2) for $\beta^{[2,3;y]}(v)$, with v in $[x, z]$, satisfies the first-order condition for optimality, according to which the bid from a bidder

[13] And expanding $E(y \wedge \mathbf{V}|\mathbf{V} < u)$ as the sum of $E(\mathbf{V}|\mathbf{V} < y)F(y)/F(u)$ and $y(F(u) - F(y))/F(u)$.

$i \neq 1$ should be the expected "effective" price he would pay at the second auction–the minimum of his value and the value of his remaining competitor, here bidder 1–after losing a tie for highest bidder at the first auction. As Figure 4 illustrates, $\beta^{[2,3;y]}$ takes the constant value $\beta_{MW}^{[2,3]}(y)$ over $[y, z]$, and bidders 2 and 3 with values in this interval are indifferent between losing and winning a tie between them at this bid. Moreover, any bidder $i \neq 1$ with value v is indifferent about the outcome of a tie with bidder 1 at $\beta_{MW}^{[2,3]}(x)$ when $v = x$ and at $\beta_{MW}^{[2,3]}(z)$ when $v = z$. Supermodularity (Lemma 1) then implies the optimality of bidders 2 and 3's bidding function.[14]

From (3), the expected payoff of bidder 1 with value y reaches its maximum over the jump $\left[\beta_{MW}^{[2,3]}(x), \beta_{MW}^{[2,3]}(z)\right]$ at both its extremities. In fact, the expected payoff decreases over $\left[\beta_{MW}^{[2,3]}(x), \beta_{MW}^{[2,3]}(y)\right)$ and the integral in (3) is its change over this semi-open interval. "Jumping over" $\beta_{MW}^{[2,3]}(y)$ does not change the expected payoff conditional on at most one bid from bidders 2 and 3 being equal to $\beta_{MW}^{[2,3]}(y)$. The second term in the LHS of (3) accounts for the–positive–change in the expected payoff conditional on both bids from bidders 2 and 3 being equal to $\beta_{MW}^{[2,3]}(y)$. Optimality of bidder 1's bidding function follows immediately from supermodularity. Therefore, the triple $(\beta_1, \beta_2, \beta_3)$ defines an equilibrium.

As I show in Lemma C1 in Appendix C, the price process in this equilibrium is not a martingale, a supermartingale, or a submartingale. Figure 4 illustrates this fact, with the arrows indicating the directions from current prices towards the expected future prices.

[14] Optimality could also be obtained via Milgrom (2004, pp.103-104)'s sufficiency theorem, with its assumptions slightly weakened to be satisfied here. However, the simplicity of the bidding functions allows for a direct application of supermodularity.

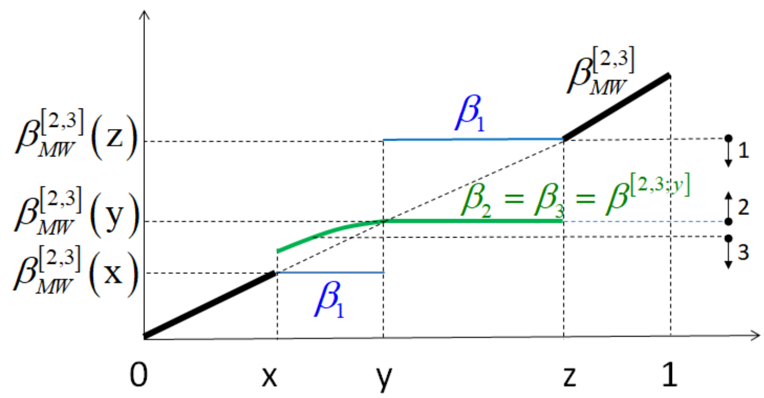

Figure 4: First-auction equilibrium bidding functions under full participation with $k = 2$ and $n = 3$. From current prices, the arrows point towards the expected future prices.

Nevertheless, the equilibrium allocates the units inefficiently if and only if all three bidders' values are in $[y, z]$ and bidder 1 has the lowest value. In this case, bidder 1 is the inefficient winner of the first auction, as he follows a higher bidding function over this interval. When $\overline{F}$ is not -3-concave everywhere, x, y, z exist such that $\overline{F}$ is strictly -3-convex over $[y, z]$. From Lemma 3 in the previous section, the expected price will then decrease strictly along the corresponding equilibrium, and the converse of the implication in Lemma 4 follows for $k = 2$.

The arguments extend straightforwardly[15] to more than three bidders, and the bidding-function profile $(\beta_1, ..., \beta_n)$ above defines an equilibrium. Contrary to the three-bidder case, no tie occurs with strictly positive probability, as the bidding function of all bidders $i \neq 1$ is strictly increasing (see Figure 3 above). With strictly positive probability, each such equilibrium will result in an inefficient allocation, wherein the two highest-value bidders are outbid by bidder 1 with a lower value, thanks to bidder 1's higher bidding function over the interval $[y, z]$. Lemma 6 below follows.

[15]Except the proof of Lemma 5, which is immediate only when $n = 3$.

Lemma 6: *There exist equilibria of SSPA(2,3) where the expected price sequence is decreasing if and only if* $\overline{F}$ *is not* -3*-concave everywhere, in which case there also exist such equilibria of SSPA(2,n) for all* $n \geq 3$.

6. Construction of equilibria: the strategies after the first auction

The example $k = 2$ in the previous section ended the construction in the two-unit case. Therefore, I may now assume more than two units, that is, $2 < k < n$. By having the strategies recommend MW's equilibrium strategy τ_{MW} at all auctions if not all bidders participated in the first, I may focus on the case of full participation in the first auction. However, even in this case, I need to specify the strategies at all later auctions, whether participation is maximal or not, as beliefs may be asymmetric. From the second auction on, that is, for $s \geq 2$ below, the strategies $\tau_1^{[x,y,z]}, ..., \tau_n^{[x,y,z]}$, with x, y, z as in the previous section, are as follows:

- At the s^{th} auction where $1 \notin \mathcal{P}^{(s)}$, $\tau_1^{[x,y,z]}, ..., \tau_n^{[x,y,z]}$ agree with τ_{MW}.
- At the s^{th} auction where $1 \in \mathcal{P}^{(s)}$, with $k - s + 1 < \left|\mathcal{P}^{(s)}\right|$ and $s < k$:
  - **(a)** $\tau_1^{[x,y,z]}$ recommends to bid 0;
  - **(b)** for $i \in \mathcal{P}^{(s)} \setminus \{1\}$:

    (b.1): if $b_i^{(1)} \notin \left\{\beta_{MW}^{[k,n]}(x), \beta_{MW}^{[k,n]}(z)\right\}$ or if $b_i^{(1)} \neq b_w^{(1)}$, then $\tau_i^{[x,y,z]}$ recommends to follow the bidding function $\widetilde{\beta}^{[l,m;G_i]}$, where $l = k - s + 1$, $m = \left|\mathcal{P}^{(s)}\right|$, and G_i represents the beliefs about bidder 1's value that bidder i revised after having observed the history $\left(v_i, \mathcal{P}^{(1)}, b_i^{(1)}, b_w^{(1)}, \mathcal{P}^{(2)}\right)$.

    (b.2): if $b_i^{(1)} = b_w^{(1)} = \beta_{MW}^{[k,n]}(x)$ or $b_i^{(1)} = b_w^{(1)} = \beta_{MW}^{[k,n]}(z)$, $\tau_i^{[x,y,z]}$ maximizes bidder i's expected payoff, conditionally on his observed history, bidder 1's following $\tau_1^{[x,y,z]}$ in **(a)**, and all other bidders j's following their strategies $\tau_j^{[x,y,z]}$ under the assumption in **(b.1)**.

In agreement with the definition of equilibrium in Section 3, a loser to an auction always revises his beliefs while assuming that all other losers never previously deviated. Also, $\mathcal{P}^{(s)}$ above is the set of participants in the s^{th} auction, $b_i^{(1)}$ and $b_w^{(1)}$ are bidder i's bid and the winning bid at the first auction, and v_i is bidder i's value. Finally, $k - s + 1$ and $\left|\mathcal{P}^{(s)}\right|$ are the numbers of units and participants at the s^{th} auction.

According to the strategies $\tau_1^{[x,y,z]}, ..., \tau_n^{[x,y,z]}$, bidders $i \neq 1$ should follow MW's equilibrium strategy at any auction bidder 1 does not participate in.

From (a), if bidder 1 lost the first auction, he should bid 0 at all other auctions before the last. Consequently, once a bidder $i \neq 1$ has revised his beliefs about bidder 1's value to G_i at the start of the second auction, he will never revise them again. If the assumption in (b.1) holds, bidder $i \neq 1$'s bidding functions $\widetilde{\beta}^{[l,m;G_i]}$ at all remaining auctions will only change with the number l of remaining units and m of participants. Therefore, bidder $i \neq 1$'s behaviour after the first auction should not depend on whether he previously deviated and should no longer be affected by winning bids. If the assumption holds for all bidders $i \neq 1$ active after the first auction, they will all have the same beliefs G about bidder 1's value, and all should use the same bidding functions.

If the first-auction winning bid $b_w^{(1)}$ was $\beta^{[k,n;y]}(w)$, strictly inside β_1's discontinuity jump (see Figure 3 in the previous section), any bidder $i \neq 1$ participating in a subsequent auction along with bidder 1 should then follow $\beta^{[l,m;y]}$, as his revised beliefs G_i would be $F|_{[0,y]}$. If the first-auction winning bid was instead $\beta_{MW}^{[k,n]}(w')$, outside the discontinuity jump, $F|_{[0,w']}$ would be bidder i's beliefs and his bidding function would be $\beta^{[l,m;w']}$. See Figure 5 below, where I denote $\beta_j^{(2)}$ bidder j's bidding function at the second auction, for j in $\mathcal{P}^{(2)}$.[16]

[16] I assume $n > 3$ in Figure 5. If $n = 3$ and hence $k = 2$, $\beta^{[l,m;y]}$ would be constant past y in the left-hand diagram and $\beta^{[l,m;w']}$ would be constant past w' in the right-hand diagram.

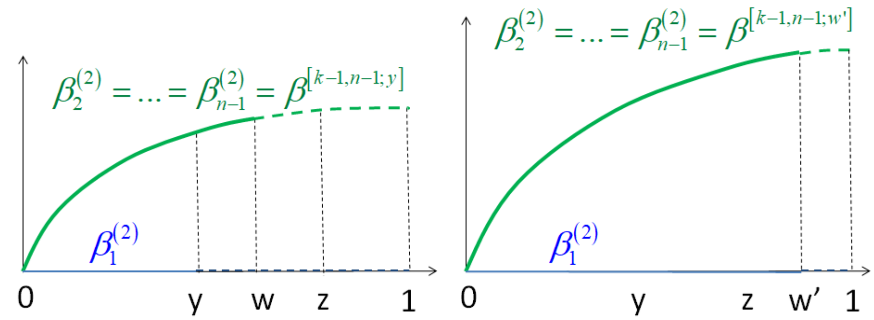

Figure 5: Equilibrium bidding functions at the second auction in which bidders $1, ..., n-1$ participate, when the first-auction winning bid was $\beta^{[k,n;y]}(w)$ (left-hand diagram) and $\beta_{MW}^{[k,n]}(w')$ (right-hand diagram) in Figure 3. Only bidders who deviated at the first auction use the dotted parts of the functions' graphs.

Finally, if a bidder different from bidder 1 won the first auction by having deviated to one of the two mass points of bidder 1's bidding strategy, the revised beliefs about bidder 1's value of any other bidder $i \neq 1$ would no longer be the restriction of F to some interval. For example, if $\beta_{MW}^{[k,n]}(z)$ was the winning bid, they would be a combination $\gamma_i F|_{[0,y]} + (1-\gamma_i) F|_{[y,z]}$ for some $0 < \gamma_i < 1$, as bidder 1's losing the first auction reduces the likelihood of his tying for winner, which would happen if his value belonged to $[y, z]$. If bidder i did not tie with the winner, hence under the assumption in (b.1), $\gamma_i = 2F(y) / (F(z) + F(y))$. If bidder i also deviated and tied with the winner, hence under the assumption in (b.2), $\gamma_i = 3F(y) / (2F(z) + F(y))$. According to (b.2), bidder i's strategy would then solve a dynamic programming problem and could be found by recursion. However, as these last cases occur only if the winner of the first auction deviated, they play no role in the equilibrium condition of no profitable unilateral deviation by bidder i at the first auction.

From Lemma 7 below, $\tau_1^{[x,y,z]}, ..., \tau_n^{[x,y,z]}$ form an equilibrium with inefficiency confined to an arbitrarily small neighborhood of y. Furthermore, the expected price they generate after the first auction stays constant, as the price sequence becomes a martingale.

Lemma 7: *Let y be in $(0,1)$, k, n be such that $2 < k < n$, and ε be strictly positive. Then, there exist x, z as in Lemma 5 (ii) with $x < y < z < y + \varepsilon$ such that $\beta_{MW}^{[k,n]}(z) \leq y$ and:*

(i) $\left(\tau_1^{[x,y,z]}, ..., \tau_n^{[x,y,z]}\right)$ *is an inefficient equilibrium of SSPA(k,n), with inefficiency confined to $[y, z]$;*

(ii) *along its path, the price process starting at the second auction is a martingale.*

Proof: See Appendix D.

To prove Lemma 7, I first show by induction on the number of remaining units that the strategies are sequentially rational after the first auction when (b.1) in the definition of $\tau_i^{[x,y,z]}$ above applies to all bidders $i \neq 1$. I take the distance $z - y$ small enough for the participating bidder 1 with the highest possible value 1 to be willing to bid 0. The first-order optimality condition for the bidding functions of the bidders $i \neq 1$ follows from the property below of the functions $\widetilde{\beta}^{[l,m;G]}$ (Lemma D1 in Appendix D):

$$\widetilde{\beta}^{[l,m;G]}(v) = E\left(\widetilde{\beta}^{[l-1,m-1;G]}\left(\mathbf{V}_{(2,m-2)}\right) | \mathbf{V}_{(1,m-2)} = v\right), \quad (4)$$

for $2 < l < m$. Once I have proved the sequential rationality of the strategies after the first auction, I can use similar arguments to those in the two-unit case to prove their optimality at the first auction.

Along the equilibrium path, the $k-1$ units that remain after the first auction are efficiently allocated to the remaining bidders. This is immediate if bidder 1 wins the first auction, as the other bidders will follow MW's strategy. If bidder 1 loses the first auction, all remaining units except one will be allocated before the last auction to the bidders $i \neq 1$ with highest values. The one remaining unit will go to the bidder with the highest value among all remaining bidders at the last auction. Therefore, inefficiency occurs only at the first auction, when all k highest values and the strictly lower value of bidder 1 belong to $[y,z]$. The martingale property after the first auction follows from (4) with $l>2$ and the definition of $\widetilde{\beta}^{[l,m;G]}$ with $l=2$.

From Lemma 7, when $\overline{F}$ is not $-(k+1)$-concave everywhere, there will exist an inefficient equilibrium with inefficiency confined to an interval $[y,z]$ where $\overline{F}$ is strictly $-(k+1)$-convex. From the marginal-revenue analysis (Lemma 3, Section 4), the last expected price $E\mathbf{P}_k$ along this equilibrium's path will be below average. However, from Lemma 7 again, all prices from the second auction on will be equal in expectation, that is, $E\mathbf{P}_2 = ... = E\mathbf{P}_k$. Consequently, the expected price sequence must be such that $E\mathbf{P}_1 > E\mathbf{P}_2 = ... = E\mathbf{P}_k$. The converse of Lemma 4 (Section 4) for $k>2$ and Lemma 8 below follows.

Lemma 8: *For $k>2$, there exist equilibria of SSPA(k,$k+1$) where the expected-price sequence is decreasing if and only if $\overline{F}$ is not $-(k+1)$-concave everywhere, in which case there also exist such equilibria of SSPA(k,n) for all $n \geq k+1$.*

7. The main result

Gathering the results, Lemma 2 in Section 3, Lemma 6 in Section 5, and Lemma 8 in the previous section imply the following theorem.

Theorem: *Assume $k>1$. The three following statements are equivalent:*

(i) $\sup_{w\in(0,1)} \frac{d\ln f}{d\ln \overline{F}}(w) > k+2$, *that is, $\overline{F}$ is not $-(k+1)$-concave everywhere.*

(ii) *There exist equilibria of SSPA(k,$k+1$) with decreasing expected-price sequences.*

(iii) *For all $n \geq k+1$, equilibria of SSPA(k,n) exist with decreasing expected-price sequences.*

Thus, with k units and any larger number of bidders, equilibria with decreasing expected prices exist as soon as the value ccdf is not $-(k+1)$-concave everywhere. With one more bidder than units, only under this condition on the value distribution can such equilibria exist.

8. Conclusion

In the Milgrom and Weber's IPV model of the sequential second-price auction with k units, equilibria with decreasing expected prices exist for all larger numbers of bidders if and only if the value complementary cumulative function is not $-(k+1)$-concave everywhere. While this result holds for sequentially rational equilibria where no player follows a dominated strategy, further research should investigate its robustness to the imposition of additional requirements on the equilibria. Also left for future research is describing all equilibria, satisfying the same or stronger definitions, and the price movements along their paths.

APPENDICES

Appendix A

In Figure 1 (Section 2), bidders 2 and 3 follow the same bidding function $\beta(v) = E(v \wedge \mathbf{V})$. Bidder 1 bids 0 for all values.

As is easily checked, the bid $\beta(v)$ satisfies the first-order condition for optimality. A slight bid increase from $\beta(v)$ by bidder 2 with value v matters only when it makes him win the first auction by avoiding a near tie with bidder 3. In this case, he receives the payoff $v - \beta(v)$, rather than the payoff $v - E(v \wedge \mathbf{V}_1)$, where $\mathbf{V}_1$ is bidder 1's value, he would have received by competing with bidder 1 at the second auction. As both payoffs are equal, the first-order condition is satisfied. Global optimality of $\beta(v)$, for any value v, then follows from the supermodularity of a bidder's total expected payoff in his value and first-auction bid (by Lemma 1 in Section 3).[17]

[17] It also follows from Milgrom's sufficiency theorem (2004, pp. 103-104). Of course, under differentiability, supermodularity implies the smooth single crossing differences property.

With value $v = 1$, bidder 1's payoff would decrease if he increased his bid since the first-order effect of an increase from any $\beta(w)$ would be negative, as the difference between $1 - \beta(w)$ and $1 - E(\mathbf{V}_i | \mathbf{V}_i \leq w)$ is negative, where $\mathbf{V}_i$ is bidder i's value for $i = 2$ or 3. It follows from supermodularity that bidding 0 is optimal for bidder 1 no matter what his value is.

In the equilibrium displayed in Figure 2 (Section 2), the bidders follow MW's equilibrium bidding function, which I denote $\beta_{MW}^{[2,3]}(v)$, outside an interval $[0, \eta]$ and follow the bidding functions $\beta_1(v) = E(v \wedge \mathbf{V} | \mathbf{V} \leq \varphi(v))$ and $\beta_2(v) = \beta_3(v) = E(\varphi^{-1}(v) \wedge \mathbf{V} | \mathbf{V} \leq v)$ over $[0, \eta]$. Here, the "matching function" $\varphi = \beta_i^{-1} \circ \beta_1$, $i = 2, 3$, is the unique solution to the differential equation below with initial condition $\varphi(\eta) = \eta$:

$$\frac{d \ln F(\varphi(v))}{d \ln F(v)} = \frac{E(\mathbf{V} | \mathbf{V} \leq \varphi(v)) - E(v \wedge \mathbf{V} | \mathbf{V} \leq \varphi(v))}{E(v \wedge \mathbf{V} | \mathbf{V} \leq \varphi(v)) - E(\mathbf{V} | \mathbf{V} \leq v)}.$$

From its definition, the bidding function β_1 already satisfies the first-order condition. The differential equation above comes from the first-order condition the bidding function $\beta_2 = \beta_3$ should satisfy. The RHS of this equation can also be written as $\frac{E(\mathbf{V} - v | v \leq \mathbf{V} \leq \varphi(v))}{E(v - \mathbf{V} | \mathbf{V} \leq v)}$, which extends as a continuously differentiable function with value 0 at $(v, \varphi(v))$ where $v = \varphi(v) > 0$.

With the singularity at the initial condition so removed, the standard theorems on differential equations apply, and a solution φ indeed exists and is unique. As the derivative of this solution φ vanishes at η and would vanish at any value $v > 0$ where $\varphi(v)$ and v were equal, it can be continued to the left of η over the interval $(0, \eta]$, while being strictly increasing and strictly above the identity function. If its limit $\varphi(0)$ for v approaching 0 was strictly positive, $\frac{d \ln F(\varphi(v))}{d \ln F(v)}$ above would become infinite, hence larger than 1, and $F(\varphi(v)) / F(v)$ would be increasing with a finite limit at $v = 0$, which would contradict $\varphi(0) > 0$. Consequently, $\varphi(0) = 0$.

By construction, the bidding functions satisfy the first-order optimality conditions. From supermodularity,[18] they form an equilibrium.

The uniform distribution is an example where the equation for this equilibrium is easily solved. In this case, $\varphi(v) = \frac{2v}{1+(v/\eta)^2}$, $\beta_1(v) = v\frac{3-(v/\eta)^2}{4}$, and $\beta_2(v) = \beta_3(v) = \beta_1(\varphi^{-1}(v))$ over $[0, \eta]$, and all bidding functions are equal to $\beta_{MW}^{[2,3]}(v) = v/2$ outside of $[0, \eta]$.

Appendix B

Proof of Lemma 1 (Section 3): Increasing his first-auction bid from b' to b'' would only change the payoff of a bidder i with value v_i if it allowed him to win against the highest bid b from the other participating bidders. In this case, bidder i's payoff would be $v_i - b$, instead of the expected payoff $\pi_i(v_i|b, b', b'')$ he would have obtained by acting optimally after losing the first auction. In this latter case, let $q_i(v_i|b, b', b'')$ be his probability of receiving a unit. From standard incentive-compatibility arguments, $\pi_i(v_i|b, b', b'')$ is not larger than $\pi_i(w_i|b, b', b'') + (v_i - w_i) q_i(v_i|b, b', b'')$ and hence than $\pi_i(w_i|b, b', b'') + v_i - w_i$, for any other possible value w_i smaller than v_i. Therefore, $(v_i - b) - \pi_i(v_i|b, b', b'') \geq (w_i - b) - \pi_i(w_i|b, b', b'')$, and supermodularity, here equivalent to the nondecreasing differences property, follows. ||

Proof of Lemma 3 (Section 4): The expected difference $\Delta = \sum_{t=1}^{k-1} E(\mathbf{P}_t) - (k-1) E\mathbf{P}_k$ is equal to $E\left(\sum_{t=1}^{k} \mathbf{P}_t\right) - kE\mathbf{P}_k$. As for the two-unit examples in Section 2, $E\left(\sum_{t=1}^{k} \mathbf{P}_t\right)$ is the expected sum of the marginal revenues at the winners' values, due to the revenue-equivalence theorem and the fact that no payoff goes to any bidder with value 0. These winners' values are the k

[18] Or, through Milgrom (2004)'s sufficiency theorem.

highest when the allocation is efficient. Therefore, Δ is equal to:

$$E\left(\begin{array}{c}\sum_{i=1}^{\mathbf{T}} MR_{(1)}\left(\mathbf{V}_{(\mathbf{W}_i,n)}\right)+\\ \sum_{\substack{i=1\\ i\neq \mathbf{L}_1,\ldots,\ \mathbf{L}_\mathbf{T}}}^{k} MR_{(1)}\left(\mathbf{V}_{(i,n)}\right)-k\mathbf{V}_{(\mathbf{L}_1,n)}|INEF\left(\boldsymbol{\sigma}\right)\end{array}\right)\Pr\left(INEF\left(\boldsymbol{\sigma}\right)\right)$$
$$+E\left(\sum_{i=1}^{k} MR_{(1)}\left(\mathbf{V}_{(i,n)}\right)-k\mathbf{V}_{(k+1,n)}|EFF\left(\boldsymbol{\sigma}\right)\right)\Pr\left(EFF\left(\boldsymbol{\sigma}\right)\right),$$

where $EFF\left(\boldsymbol{\sigma}\right)$ is the complementary event of $INEF\left(\boldsymbol{\sigma}\right)$ and, conditional on $INEF\left(\boldsymbol{\sigma}\right)$, $\mathbf{V}_{(\mathbf{W}_1,n)},\ldots,\mathbf{V}_{(\mathbf{W}_\mathbf{T},n)}$ are the inefficient winners' values and $\mathbf{V}_{(\mathbf{L}_1,n)},\ldots,\mathbf{V}_{(\mathbf{L}_\mathbf{T},n)}$ the inefficient losers' values, with $\mathbf{W}_T>\ldots>\mathbf{W}_1>k\geq \mathbf{L}_\mathbf{T}>\ldots>\mathbf{L}_1$. It follows from the uniform $k+1^{th}$-price auction's being the Vickrey-Groves-Clarke mechanism, that $0=E\left(\sum_{i=1}^{k} MR_{(1)}\left(\mathbf{V}_{(i,n)}\right)-k\mathbf{V}_{(k+1,n)}\right)$. Subtracting this equality from the previous expression for Δ gives the new expression:

$$E\left(\begin{array}{c}\sum_{i=1}^{\mathbf{T}} MR_{(1)}\left(\mathbf{V}_{(\mathbf{W}_i,n)}\right)+k\mathbf{V}_{(k+1,n)}\\ -\left(\sum_{i=1}^{\mathbf{T}} MR_{(1)}\left(\mathbf{V}_{(\mathbf{L}_i,n)}\right)+k\mathbf{V}_{(\mathbf{L}_1,n)}\right)|INEF\left(\boldsymbol{\sigma}\right)\end{array}\right)\Pr\left(INEF\left(\boldsymbol{\sigma}\right)\right).\text{(B.1)}$$

If $k=n-1$, the only possible inefficient winner is the bidder with the lowest value $\mathbf{V}_{(k+1,k+1)}$, that is, $\mathbf{T}=1$ and $\mathbf{W}_1=k+1$. The equality in the statement of the lemma follows from Lemma 2 (i) (Section 2).

Conditional on $INEF\left(\boldsymbol{\sigma}\right)$ in the general case $k\leq n-1$, the inequalities $\mathbf{V}_{(\mathbf{W}_i,n)}\leq \mathbf{V}_{(k+1,n)}\leq \mathbf{V}_{(k,n)}\leq \mathbf{V}_{(\mathbf{L}_i,n)}$ hold for all $1\leq i\leq \mathbf{T}$. As $\mathbf{V}_{(\mathbf{W}_i,n)}$ and $\mathbf{V}_{(\mathbf{L}_i,n)}$ (almost surely) belong to I, so do $\mathbf{V}_{(k+1,n)}$ and $\mathbf{V}_{(k,n)}$. Under -1-convexity of $\overline{F}$ over I, $MR_{(1)}$ is nonincreasing (from Lemma 2) and, therefore, $MR_{(1)}\left(\mathbf{V}_{(\mathbf{W}_i,n)}\right)\geq MR_{(1)}\left(\mathbf{V}_{(k+1,n)}\right)\geq MR_{(1)}\left(\mathbf{V}_{(k,n)}\right)\geq MR_{(1)}\left(\mathbf{V}_{(\mathbf{L}_i,n)}\right)$, for all $1\leq i\leq \mathbf{T}$. The expression (B.1) then implies the following lower bound for Δ:

$$E\left(\begin{array}{c}(\mathbf{T}-1)\left(MR_{(1)}\left(\mathbf{V}_{(k+1,n)}\right)-MR_{(1)}\left(\mathbf{V}_{(k,n)}\right)\right)\\+\left(MR_{(1)}\left(\mathbf{V}_{(k+1,n)}\right)+k\mathbf{V}_{(k+1,n)}\right)\\-\left(MR_{(1)}\left(\mathbf{V}_{(\mathbf{L}_1,n)}\right)+k\mathbf{V}_{(\mathbf{L}_1,n)}\right)|INEF\left(\boldsymbol{\sigma}\right)\end{array}\right)\Pr\left(INEF\left(\boldsymbol{\sigma}\right)\right)$$

$$\geq\ E\left(\begin{array}{c}MR_{(1)}\left(\mathbf{V}_{(k+1,n)}\right)+k\mathbf{V}_{(k+1,n)}\\-\left(MR_{(1)}\left(\mathbf{V}_{(\mathbf{L}_1,n)}\right)+k\mathbf{V}_{(\mathbf{L}_1,n)}\right)|INEF\left(\boldsymbol{\sigma}\right)\end{array}\right)\Pr\left(INEF\left(\boldsymbol{\sigma}\right)\right);$$

and the inequality in the statement of the lemma follows from Lemma 2 (i). ||

Proof of Lemma 5 (Section 5): Let $D(y,u)$ be the integrand in (1). It is equal to the following difference:

$$E\left(y\wedge\mathbf{V}_{(k,n-1)}|\mathbf{V}_{(1,n-1)}{=}u\right)-E_{\widetilde{\mathbf{V}}\sim F}\left(\widetilde{\mathbf{V}}_{(k,n-1)}|\mathbf{V}_{(1,n-2)}=u;\widetilde{\mathbf{V}}\leq y\right), \quad \text{(B.2)}$$

which is strictly negative for $u<y$. (i) is proved. Differentiating[19] $D(u,u)=0$ gives $\left(\frac{\partial}{\partial y}D(y,u)\right)_{y=u}+\left(\frac{\partial}{\partial u}D(y,u)\right)_{y=u}=0$, for all u in $(0,1)$. It is easily verified that $\left(\frac{\partial}{\partial y}D(y,u)\right)_{y=u}<0$. In fact, the derivative with respect to y of the first term in (B.2) vanishes at $y=u$. The second term is equal to $E_{\widetilde{\mathbf{V}}\sim F}\left(h\left(\widetilde{\mathbf{V}},u\right)|\widetilde{\mathbf{V}}\leq y\right)$, where $h(v,u)=E\left(\widetilde{\mathbf{V}}_{(k,n-1)}|\mathbf{V}_{(1,n-2)}=u;\widetilde{\mathbf{V}}=v\right)$. Its derivative with respect to y at $y=u$ is strictly positive, as $h(v,u)$ is a nondecreasing function of v in $(0,1)$ that is strictly increasing over $(0,u)$. Therefore, the inequality $\left(\frac{\partial}{\partial u}D(y,u)\right)_{y=u}>0$ must hold, for all $u=y$ in $(0,1)$.

Let y be in $(0,1)$. From $D(y,y)=0$ and the last inequality in the previous paragraph, $D(y,u)$ is strictly positive for all $u>y$ sufficiently close to y. (ii) then follows. Indeed, there exists $z'>y$ such that $D(y,u)>0$ for all u in (y,z'). For all $x<y$ sufficiently close to y: $0<-\int_x^y D(y,u)\,dF^{n-1}(u)<$

[19] Joint differentiability in (y,u) follows from the joint continuity of the partial derivatives.

$\int_y^{z'} D(y,u)\, dF^{n-1}(u)$. For each such x, a unique z in (y, z') exists such that $-\int_x^y D(y,u)\, dF^{n-1}(u) = \int_y^z D(y,u)\, dF^{n-1}(u)$, that is, such that (1) holds. The last equality implies that this z tends towards y if x does. $||$

Appendix C

Lemma D1:[20] *Let $\beta_1, \beta_2, \beta_3$ be the first-auction bidding functions under full participation in an equilibrium of SSPA(2,3) as in the example* $k = 2$ of *Section 5. Then:*

(i) $E\left(\mathbf{P}_2|\mathbf{P}_1 = \beta_{MW}^{[2,3]}(x)\right) = \beta_{MW}^{[2,3]}(x)$ *and* $E(\mathbf{P}_2|\mathbf{P}_1 = p) = p$*, for almost all* p *outside* $\left[\beta_{MW}^{[2,3]}(x), \beta_{MW}^{[2,3]}(z)\right]$*;*

(ii) $E\left(\mathbf{P}_2|\mathbf{P}_1 = \beta_{MW}^{[2,3]}(z)\right) < \beta_{MW}^{[2,3]}(z)$ *(Arrow 1 in Figure 4);*

(iii) $E\left(\mathbf{P}_2|\mathbf{P}_1 = \beta_{MW}^{[2,3]}(y)\right) > \beta_{MW}^{[2,3]}(y)$ *(Arrow 2 in Figure 4);*

(iv) $E(\mathbf{P}_2|\mathbf{P}_1 = p) < p$*, for almost all* p *in* $\left(\beta^{[2,3;y]}(x), \beta_{MW}^{[2,3]}(y)\right)$ *(Arrow 3 in Figure 4).*

Proof:: Proof of (i): The statement can be proved in the same way the martingale property of MW's equilibrium is. Note that conditional on $\mathbf{P}_1 = \beta_{MW}^{[2,3]}(x)$, bidder 1 is almost surely the second highest bidder at the first auction.

Proof of (ii): Conditional on $\mathbf{P}_1 = \beta_{MW}^{[2,3]}(z)$, bidder 1 with value in $[y, z]$ is the almost-sure second highest bidder at the first auction. Therefore, the expected price at the second auction will be $E(\mathbf{V}_i \wedge \mathbf{V}_1 | \mathbf{V}_1 \in [y, z]; \mathbf{V}_i < z)$, where $\mathbf{V}_i$ is the value of the other bidder, bidder 2 or 3, who remains at the second auction. As it is strictly smaller than $E(\mathbf{V}|\mathbf{V} < z) = \beta_{MW}^{[2,3]}(z)$, (ii) follows.

Proof of (iii): If bidder 1 wins the first auction and both bidders 2 and 3 have submitted $\beta_{MW}^{[2,3]}(y)$, bidders 2 and 3's values belong to $[y, z]$ and the expected second-auction price will be $E(\mathbf{V}_2 \wedge \mathbf{V}_3 | \mathbf{V}_2, \mathbf{V}_3 \in [y, z])$, which

[20] Because the derivatives of $\beta_{MW}^{[2,3]}$ and $\beta^{[2,3;y]}$ are continuous and different from zero over $(0, 1)$ and (x, y), respectively, "almost all" in (i) and (iv) below means "Lebesgue-measure almost all."

is strictly larger than the first-auction price $\beta_{MW}^{[2,3]}(y) = E(\mathbf{V}|\mathbf{V} < y)$. The expected price does not change from the first-auction price $\beta_{MW}^{[2,3]}(y)$ in the other possible cases.

Proof of (iv): The second-auction price only differs in expectation from the first-auction price $\beta^{[2,3;y]}(v)$ with v in (x, y) if bidder 1 wins the first auction. In this case, the second highest bidder at the first auction will win the second auction and pay $E(\mathbf{V}|\mathbf{V} < v)$ in expectation, which is smaller than $\beta^{[2,3;y]}(v) = E(v \wedge \mathbf{V}|\mathbf{V} < y)$. ||

Appendix D

I denote $\succeq$ and $\succ$ the weak and strict relations of first-order stochastic dominance.[21] I call a distribution G acceptable if it is absolutely continuous and has an interval $[0, \overline{v}_G] \subseteq [0, 1]$ as support.

Lemma D1: *For all* $1 < l < m$, u *in* $(0, 1]$, *and probability distributions* G *and* H *over* $[0, 1]$:

(i) *if* $G \succeq H$, *then* $\widetilde{\beta}^{[l,m;G]} \geq \widetilde{\beta}^{[l,m;H]}$.

(ii) $\beta^{[l,m;u]}(u) = \beta_{MW}^{[l,m]}(u)$

(iii) *if* $G \succ F|_{[0,u]}$, *then* $\widetilde{\beta}^{[l,m;G]}(u) > \beta_{MW}^{[l,m]}(u)$.

(iv) *if* $l > 2$, $\widetilde{\beta}^{[l,m;G]}(v) = E\left(\widetilde{\beta}^{[l-1,m-1;G]}\left(\mathbf{V}_{(2,m-2)}\right)|\mathbf{V}_{(1,m-2)} = v\right)$.

(v) *if* G *is an acceptable distribution:* $\widetilde{\beta}^{[l,m;G]}$ *is continuous;* $\widetilde{\beta}^{[l,m;G]}$ *is strictly increasing everywhere if* $(l, m) \neq (2, 3)$*; and* $\widetilde{\beta}^{[2,3;G]}$ *is strictly increasing over* $[0, \overline{v}_G]$ *and constant over* $[\overline{v}_G, 1]$.

Proof: From its definition, $\beta^{[l,m;u]}(u)$ is $E_{\widetilde{\mathbf{V}} \sim F}\left(\widetilde{\mathbf{V}}_{(l,m-1)}|\mathbf{V}_{(1,m-2)} = u; \widetilde{\mathbf{V}} \leq u\right)$, that is, $E\left(\mathbf{V}_{(l,m-1)}|\mathbf{V}_{(1,m-1)} = u\right)$, which is $\beta_{MW}^{[l,m]}(u)$. (ii) follows, and (iii) is then an immediate consequence of the definition of $\widetilde{\beta}^{[l,m;G]}$. (i) and (v) also follow immediately from this definition.

If $\mathbf{U}_{(1,m-2)}, \mathbf{U}_{(2,m-2)}$ are order statistics of the same random sample of $m-2$ values drawn according to F independently of all other random variables,

[21] $G \succ H$ if and only if $G \succeq H$ and $G \neq H$.

the RHS of the equality in (iv) is:

$$\begin{aligned} & E\left(\widetilde{\beta}^{[l-1,m-1;G]}\left(\mathbf{U}_{2,m-2}\right)|\mathbf{U}_{(1,m-2)}=v\right) \\ = \;& E\left(E_{\widetilde{\mathbf{V}}\sim G}\left(\widetilde{\mathbf{V}}_{(l-1,m-2)}|\mathbf{V}_{(1,m-3)}=\mathbf{U}_{(2,m-2)}\right)|\mathbf{U}_{(1,m-2)}=v\right) \\ = \;& E_{\widetilde{\mathbf{V}}\sim G}\left(\widetilde{\mathbf{V}}_{(l-1,m-2)}|\mathbf{V}_{(1,m-3)}=\mathbf{U}_{(2,m-2)}\leq\mathbf{U}_{(1,m-2)}=v\right). \end{aligned}$$

As $l-1\geq 2$, the inequality $\widetilde{\mathbf{V}}_{(l-1,m-2)}\leq\mathbf{V}_{(1,m-3)}$ holds in the last conditional expectation above. Consequently, this expectation is also $E_{\widetilde{\mathbf{V}}\sim G}\left(\widetilde{\mathbf{V}}_{(l,m-1)}|\mathbf{V}_{(1,m-2)}=v\right)$, which is $\widetilde{\beta}^{[l,m;G]}(v)$. (iv) is proved. ||

The definitions below of the strategies μ_1 and $\mu_i^{[G]}$, $i\neq 1$, simplify the statements and proofs of the next lemmas. In case (b.1) (Section 6) of their definitions, $\tau_1^{[x,y,z]}$ and $\tau_i^{[x,y,z]}$, $i\neq 1$, reduce to such strategies after the first auction.

Definitions: *Let G be a probability distribution over $[0,1]$ and k, n be such that $1<k<n$.*

(i) *$\mu_i^{[G]}$ is the undominated strategy in SSPA(k,n) that, when $l>1$ units remain and $m>l$ bidders participate, recommends bidder $i\neq 1$ to follow $\widetilde{\beta}^{[l,m;G]}$ if bidder 1 is among the participants and $\beta_{MW}^{[l,m]}$ if he is not.*

(ii) *μ_1 is the undominated strategy in SSPA(k,n) that recommends bidder 1 to bid 0 at any auction before the last.*

In Lemma D2 below, the requirement that the other participants in an auction are believed to have previously followed their strategies still applies (see Section 3).

Lemma D2: *Let y be in $(0,1)$ and k, n be such that $1<k<n$. Then:*

(i) *For all w, v in $[0,1]$ and acceptable distribution G, following $\mu_i^{[G]}$ is sequentially rational in SSPA(k,n) for bidder $i\neq 1$ with value v when he believes that:*

. all bidders $j \neq 1, i$ have their values initially iid according to $F|_{[0,w]}$ and follow $\mu_j^{[G]}$ after entering the first auction; and

. bidder 1 has his value initially distributed according to G and follows μ_1 after entering the first auction.

(ii) *There exists $\delta > 0$ such that, for all w, v in $[0,1]$ and acceptable distribution G, if (ii.1) $G \succeq F|_{[0,w]}$ or (ii.2) $w \in [y, y+\delta]$ and $G \succeq F|_{[0,y]}$, then: following μ_1 is sequentially rational in SSPA(k,n) for bidder 1 with value v when he believes that all bidders $i \neq 1$ have their values initially iid according to $F|_{[0,w]}$ and follow $\mu_i^{[G]}$ after entering the first auction.*

Proof: Proof of (i): I prove that if the statement (i) holds true for all k' such that $1 < k' < k$, then it also holds true for k. The result will follow by induction on k, as the induction hypothesis is trivially satisfied for $k = 2$.

Assume $k < m \leq n$, where m is the number of participants in the first auction. As the sequential auction goes on, bidder i's beliefs about the other participating bidders' values will still satisfy the assumptions in (i): the distribution of bidder 1's value will remain G and the distribution of any other bidder $j \neq i$'s value will be the restriction of F to some common interval $[0, w']$, with $[0, w'] \subseteq [0, w]$.

I may assume bidder 1 participates at the first auction, as the optimality of $\mu_i^{[G]}$ follows from MW otherwise. Under this assumption, I only need to prove the optimality of $\widetilde{\beta}^{[k,m;G]}(v)$ for bidder i at the first auction when he follows $\mu_i^{[G]}$ in the subsequent auctions. However, for $v \leq w$, if bidder i lost a tie for highest bidder at $\widetilde{\beta}^{[k,m;G]}(v)$, the expected price he would effectively pay at the next auction would also be $\widetilde{\beta}^{[k,m;G]}(v)$. This follows from Lemma D1 (v), the definition of $\widetilde{\beta}^{[k,m;G]}(v)$ when $k = 2$, and Lemma D1 (iv) when $k > 2$. In the latter case, $\widetilde{\beta}^{[k,m;G]}(v)$ is the expected price $E\left(\widetilde{\beta}^{[k-1,m-1;G]}\left(\mathbf{V}_{(2,m-2)}\right) | \mathbf{V}_{(1,m-2)} = v\right)$ bidder i would actually pay at the next auction. When $k = 2$, $m = 3$, bidder i with value v in the interval $[\overline{v}_G, 1]$ is indifferent between winning and losing the tie at the constant value of $\widetilde{\beta}^{[2,3;G]}$ over this interval. Supermodularity implies the optimality of

$\widetilde{\beta}^{[k,m;G]}(v)$ for any $v \leq w$, as well as the optimality of any bid at or above $\widetilde{\beta}^{[k,m;G]}(w)$, in particular $\widetilde{\beta}^{[k,m;G]}(v)$, for any value $v \geq w$.

Proof of (ii): As for (i), I prove (ii) by induction on k and first assume it holds true for all k' such that $1 < k' < k$. Let $\delta_{k'}$ be the δ whose existence (ii) assures for k', y, and n,[22] and let δ' be the strictly positive minimum of these $\delta_{k'}$'s over all k' such that $1 < k' < k$.

Let $m > k$ be the number of participants. As long as there remain at least two units, bidder 1's beliefs about the other bidders' values can only become the product of the same restriction of F to an interval $[0, w']$ smaller than $[0, w]$. Consequently, assumption (ii.1) or (ii.2) will be satisfied. If $\delta < \delta'$, I may assume bidder 1 bids 0 after the first auction.

From Lemma D1 (v), slightly increasing his bid from $\widetilde{\beta}^{[k,m;G]}(u)$, with $u \in (0, w \wedge \overline{v}_G)$, would only change the payoff of bidder 1 with value 1 if it made him win, in which case he would save the difference between $E\left(\mathbf{V}_{(k,m-1)} | \mathbf{V}_{(1,m-1)} = u\right) = \beta_{MW}^{[k,m]}(u)$, which he would have paid at the last auction, and $\widetilde{\beta}^{[k,m;G]}(u)$ he would pay at the first. But, by Lemma D1 (iii), $\beta_{MW}^{[k,m]}(u) - \widetilde{\beta}^{[k,m;G]}(u)$ is strictly negative for all $u < w$ under (ii.1), where $\overline{v}_G \geq w$, and for all $u < y$ under (ii.2), where $\overline{v}_G \geq y$. By supermodularity, bidding 0 as μ_1 recommends is better than submitting any bid under (ii.1) and any bid in $\left(0, \widetilde{\beta}^{[k,m;G]}(y)\right)$ under (ii.2). In particular, optimality under (ii.1) is proved.

Under assumption (ii.2), Lemma D1 (i) implies $\widetilde{\beta}^{[k,m;G]}(u) \geq \widetilde{\beta}^{\left[k,m;F|_{[0,y]}\right]}(u) = \beta^{[k,m;y]}(u)$, for all u in $(0, y)$, and the change in payoff due to increasing the bid from 0 to any bid at or above $\widetilde{\beta}^{[k,m;G]}(y)$ is not larger than:

$$\left\{ \int_0^y \left(\beta_{MW}^{[k,m]}(u) - \beta^{[k,m;y]}(u) \right) dF(u)^{m-1} + \left(F(w)^{m-1} - F(y)^{m-1} \right) \right\} / F(w)^{m-1}.$$

[22]It is not necessary to decrease n to the maximal number $n - (k - k')$ of participants when k' units will remain.

Up to the factor $F(w)^{-(m-1)}$, the first term between braces is an upper bound of the change from a bid increase to $\widetilde{\beta}^{[k,m;G]}(u)$ for $u \rightarrow_{<} y$. A further increase of the bid is consequential only if it allows bidder 1 to win the first auction, in which case the change in payoff is the difference between the current and future prices. As this difference does not exceed 1 and the probability of this event does not exceed $1-(F(y)/F(w))^{m-1}$, the product of the second term and $F(w)^{-(m-1)}$ bounds the expected change due to such a further bid increase.

By Lemma D1 (iii), the first term is strictly negative. The second term vanishes at the limit $w \rightarrow_{>} y$. As a consequence, there exists $0 < \delta_m < \delta'$ such that bidding 0 is optimal for bidder 1 with value 1 if $w - y < \delta_m$. Optimality of bidding 0 for bidder 1 with any value follows from supermodularity.

Statement (ii) for k follows by taking δ equal to $\delta = \wedge \{\delta_m | k < m \leq n\}$, which is strictly positive. $||$

Lemma D3: *For $2 < k < n$, consider SSPA(k,n) where the bidders' values are distributed independently: each bidder $i \neq 1$'s according to $F|_{[0,w]}$, with w in $[0,1]$, and bidder 1's according to an acceptable distribution G. If the bidders follow the strategies $\mu_1, \mu_2^{[G]}, ..., \mu_n^{[G]}$, the allocation is efficient almost surely and the price process is a martingale.*

Proof: Almost surely, the lowest value belongs to $(0, \overline{v}_G)$. When at least two units remain, bidder 1 bids 0 and the other bidders follow the functions $\widetilde{\beta}^{[l,m;G]}$, $1 < l < m$, which increase strictly over $(0, \overline{v}_G)$ (by Lemma D1 (v)). Consequently, the bidder with the lowest value loses all auctions before the last auction. As he also loses the last auction, where bidders bid their values, he receives no unit. Efficiency when $n = k+1$ is proved. When $n > k+1$, the bidders with $k-1$ highest values among the bidders in $\mathcal{N} \setminus \{1\}$ win the $k-1$ first auctions, as their bidding functions are strictly increasing (by Lemma D1 (v)). The bidder with the highest value among all remaining bidders wins the k^{th} and last auction. Therefore, the allocation is efficient.

If a bidder wins at price p the auction where $l > 2$ units and hence $m = n - k + l > 3$ bidders remain, he must be the bidder $i \neq 1$ with the highest value among the $m - 1$ remaining bidders different from bidder 1. Almost surely, the price p is the bid $\widetilde{\beta}^{[l,m;G]}(v)$ of the bidder $j \neq 1$ with the second highest value v among these bidders. This bidder j will win the next auction and pay the expected price $E\left(\widetilde{\beta}^{[l-1,m-1;G]}\left(\mathbf{V}_{(2,m-2)}\right)|\mathbf{V}_{(1,m-2)} = v\right)$, which, by Lemma D1 (iv), is equal to the current price $\widetilde{\beta}^{[l,m;G]}(v) = p$.

If $l = 2$ and $p = \widetilde{\beta}^{[2,m;G]}(v)$, where v is the price setter's value, the expected price at the last auction will be the expectation $E_{\widetilde{\mathbf{V}}\sim G}\left(\widetilde{\mathbf{V}}_{(2,m-1)}|\mathbf{V}_{(1,m-2)} = v\right)$ of the second highest among the values of all bidders who will remain, which, by definition of $\widetilde{\beta}^{[2,m;G]}$, is equal to $\widetilde{\beta}^{[2,m;G]}(v)$. $||$

Proof of Lemma 7 (Section 6): For y, k, and n, let $\delta > 0$ be as in Lemma D2 (ii). Let x, z be as in Lemma 5 (ii) and such that $x < y < z < y + \delta \wedge \varepsilon$ and $\beta_{MW}^{[k,n]}(z) \leq y$. I may assume full participation at the first auction, as the strategies become MW's equilibrium strategy otherwise.

Sequential rationality after the first auction: After losing the first auction, bidder 1 assumes all other participants at the second auction have their values iid according to the restriction of F to some interval $[0, w]$, share the same beliefs G about his own value, and will follow the same strategy $\tau_i^{[x,y,z]}$, $i \neq 1$, under (b.1) (Section 6), which reduces to $\mu_i^{[G]}$. The distribution G can only be: the same restriction $F|_{[0,w]}$; the restriction $F|_{[0,y]}$, in which case $w \in [x, z]$; the combination $\gamma F|_{[0,x]} + (1-\gamma) F|_{[x,y]}$ with $\gamma = 2F(x)/(F(x) + F(y))$, in which case $w = x$; or $\gamma' F|_{[0,y]} + (1-\gamma') F|_{[y,z]}$ with $\gamma' = 2F(y)/(F(y) + F(z))$, in which case $w = z$. Each possible case satisfies the assumptions in Lemma D2 (ii). From this lemma, bidder 1 has no strict incentive to deviate from μ_1, which $\tau_1^{[x,y,z]}$ reduces to.

After bidder $i \neq 1$ loses the first auction, he assumes all other bidders $j \neq i$ at the second auction have their values iid according to some restriction $F|_{[0,w]}$. Furthermore, when bidder 1 does not participate, bidder i assumes that these bidders j will follow MW's equilibrium strategy from the second auction on. Therefore, he has no incentive to deviate from this same strategy, which $\tau_i^{[x,y,z]}$ recommends in this case.

Under the assumption in (b.1) when bidder 1 participates at the second auction, bidder i assumes that all other participating bidders $j \neq 1$ will share his beliefs G about bidder 1's value. Bidder i therefore assumes that all these bidders will follow the same strategy $\mu_j^{[G]}$, which $\tau_j^{[x,y,z]}$ reduces to. From Lemma D2 (i), bidder i does not gain by deviating from $\mu_i^{[G]}$, hence, nor from $\tau_i^{[x,y,z]}$.

If the assumption in (b.1) does not hold, the assumption in (b.2) must. Bidder i assumes all other participating bidders $j \neq 1$ have their values iid over an interval $[0, w]$ and share common beliefs G about bidder 1's value. Obviously, it is not in bidder i's best interest to deviate from $\tau_i^{[x,y,z]}$, as it recommends he respond optimally. This optimal response is a solution to a dynamic programming problem with a finite horizon, and its existence follows from the overall continuity of the problem. As $[0, w]$ is included in the support of G under (b.2), not even at the auction with $l = 2$ units and only bidders 1, i, and j (hence $m = 3$) can the probability distribution of a bidder j's bid have a mass point.

No profitable deviation at the first auction: The proof can proceed as in the two-unit case. The first-order condition for any bidder with value $v \notin [x, z]$ and any bidder $i \neq 1$ with value v in (x, z) is satisfied. Any bidder $i \neq 1$ with value x and z is indifferent between losing and winning a tie with bidder 1 at $\beta_{MW}^{[k,n]}(x)$ and $\beta_{MW}^{[k,n]}(z)$, respectively. From the condition (1) and Lemma 5, bidder 1 with value y reaches at the two extremities the maximum of his payoff over β_1's discontinuity jump. That $\tau_1^{[x,y,z]}, ..., \tau_n^{[x,y,z]}$ define an equilibrium follows by supermodularity.

Localized inefficiency at the first auction and martingale property after it: Along the equilibrium path, the strategies after the first auction become MW's equilibrium strategy or are as in Lemma D3. In both cases, the $k - 1$

remaining units are allocated efficiently among the remaining bidders. Therefore, inefficiency only occurs at the first auction, when all k highest values and bidder 1's strictly lower value belong to $[y, z]$, where bidder 1 follows a higher bidding function. (i) is proved. (ii) is another immediate consequence of Lemma D3. ||

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